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TABLE ROUNDING PROBLEM

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Abstract. From time to time, people dealing with accounting are faced with the following table rounding problem. Consider a $m \times n$ table with numerical values (e. g., amount of money) in which the last column contains check sums of numbers in particular rows. Similarly, the last row consists of column check sums. A new table is to be produced in which the original numbers, and check sums are rounded off (e. g., to integers). However, the classical rounding procedure (i. e., rounding fractions smaller than 0.5 down, otherwise up) can generally violate the validity of sums. Therefore, the possibility to round off the non-integer numbers in the table to adjacent integers (i. e., either up or down independently on their fractions) is explored in order to preserve the check sums. We formulate a necessary and sufficient condition stating when rounding, which is consistent with prescribed (integer) check sums, exists. We also prove that such rounding always exists providing that the rounding of check sums is not given as a part of the input but these sums are allowed to be rounded to adjacent integers as well. Using maximum flow algorithm the desired rounding can be found in both cases in polynomial time $O((m+n)^3)$ (provided that it exists). Moreover, rounding with the minimal sum of absolute round-errors is computed within $O(mn(m+n)^2)$ time.

Keywords. table, rounding, adjacent integer, check sum, maximum flow, minimum cut, minimum cost flow problem

1 INTRODUCTION

One day I met with my aunt who asked me to help her with a computational problem because she knows I work at the institute of computer science. She is employed by an accounting department of one company where she faces the following table

[1] This research was supported by GA ČR Grant No. 201/98/0717.

rounding problem. They process tables with numerical values corresponding to some economical factors. These tables contain the check sums of rows and columns. For a very simple example of such scheme see table 1.

Table 1. Example of a table with check sums.

1.5	5.5	1.6	8.6
2.6	0.6	3.6	6.8
4.1	6.1	5.2	15.4

It is required that all non-integer numbers in the resulting arrangement of the table should be rounded off[2]. However, the usage of the *classical rounding procedure* [10] in which the absolute value of a number is rounded down if its fraction is less than 0.5 and it is rounded up otherwise, can generally violate the validity of check sums. This is the case of table 1 in which the above-mentioned rounding rule does not preserve almost any row or column sum as it is shown in table 2. For large tables the error in the check sums can even be arbitrarily large. Clearly, if sums are required to agree after rounding, we must generally not insist on the classical rounding approach.

Table 2. Table 1 after rounding.

2	6	2	9
3	1	4	7
4	6	5	15

The same problem arises in common alternative rounding rules such as rounding always up (towards plus infinity), truncating, etc. (see the IEEE 754 floating point standard which is ubiquitous in current CPUs). For example, the prominent position among these procedures is occupied by round-to-even rule [8] which makes an effort to avoid the apparent bias in the classical rounding procedure for fractions equal to 0.5. This approach suggests to round to the nearest even integer if the fraction 0.5 occurs which reduces the respective bias for many probability distributions on input data.

The above-described situation led my aunt to round several numbers in the table by hand inconsistently with respect to the classical rounding so that the check sums agree. She wants me to find a way to round the table automatically. When I was sharing this problem with several people from a commercial sphere, they confirmed

[2] For notation simplicity we consider rounding to integers but all ideas in this paper can be straightforwardly generalized for rounding to a given number of valid digits.

that they had encountered the same problem in their accounting. That is why I found this issue to be important from a practical point of view.

As my aunt has mentioned, it is possible to weaken the rounding condition in order to preserve the validity of check sums. Instead of rounding to the nearest integer as the classical approach suggests, she generally rounded off to an adjacent integer either up or down independently on the fraction. There is a natural question whether this modified rounding procedure can always preserve the row and column sums.

Sometimes the rounding of check sums is even prescribed beforehand because, for example, the respective sums can represent certain important economical results which should not be distorted too much. However, in this case the rounding of numbers in the table which is consistent with the given check sums may not exist. In Section 4 a necessary and sufficient condition stating when such rounding exists will be formulated. For this purpose the table rounding problem is reduced to a maximum flow problem and its correspondence with a minimum cut is exploited. Therefore, in Section 3 some basic notions and properties concerning the maximum flow problem are mentioned. On the other hand, it is proved in Section 5 that the underlying rounding always exists when the check sums are not prescribed at the input but are allowed to be rounded off to adjacent integers as well.

By using the maximum flow algorithm, the table rounding can be found in polynomial time $O((m+n)^3)$ where m is the number of rows and n is the number of columns, provided that it exists. Moreover, it is shown that the adjacent integer rounding with minimal sum of absolute round-errors can be computed within $O(mn(m+n)^2)$ time using an algorithm for the so-called minimum cost flow problem.

In fact, the connection between the table rounding problem and the flows in networks has actually been re-discovered in this paper as it was pointed out to me by an anonymous referee. Namely, in [1] a similar proof technique, attributed here to L. Lovász, appeared in a more general graph-theoretic context. The proof of the fact that the table rounding problem without prescribed check sums is always solvable (see Theorem 5.2), is sketched as an auxiliary lemma by referring to the integrity theorem of flows in networks [3]. However, the purpose of the paper [1] is different and the rounding problem is not further discussed in detail here.

Moreover, the table rounding problem is related to "integer-making" theorems that have been proved for more general structures including two-dimensional tables. Particularly, in [2] the rounding errors of given sums containing each number (which is allowed to be rounded to an adjacent integer) from a given set at most k times, are upper bounded by $k-1$. In the case of two-dimensional tables without prescribed check sums, k equals 2 since each element in the table is contained in at most two sums, namely, in the respective row and column check sums. Hence, here the rounding error of the check sums is less than 1 and these sums can always be rounded to adjacent integers as well. Again, this is exactly the proposition of Theorem 5.2 which provides an alternative proof based on the correspondence between maximum flow and minimum cut. In addition, the respective geometrical argument from [2] leads to a more complex rounding procedure that requires solving a linear system of inequalities in order to round one number in the table.

Furthermore, in [5] the result from [2] has been generalized for weighted sums in which, in addition, each number may be multiplied by a given real constant. However, for the case of table rounding without prescribed check sums, the time complexity of the proposed rounding algorithm can be expressed as

$$O\left((m+n)^3 \log\left(1 + \frac{mn}{m+n}\right) + d^2(m+n)\right)$$

where d is the number of distinct numbers in the table, and this is again worse than the efficiency $O(m+n)^3$ of the rounding procedure described in this paper.

A preliminary version of this paper has appeared in [9].

2 TABLE ROUNDING PROBLEM

In this section two types of the table rounding problem are formulated. For this purpose the following notation is introduced. Let c_{ij} for $i = 1, \dots, m$, $j = 1, \dots, n$ be the *common numbers* in the table and $a_i = \sum_{j=1}^{n} c_{ij}$ $(i = 1, \dots, m)$, $b_j = \sum_{i=1}^{m} c_{ij}$ $(j = 1, \dots, n)$ are the corresponding *row* and *column check sums*, respectively. Furthermore, denote by $\sigma = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}$ the *total sum* of all common numbers. We will also use the notation $\lfloor x \rfloor$ for the greatest integer less or equal to x and similarly, $\lceil x \rceil$ denotes the least integer greater or equal to x. This means $\lceil x \rceil - \lfloor x \rfloor = 1$ for non-integer x and $\lceil x \rceil = \lfloor x \rfloor = x$ for integer x.

In spite of that the original motivation of accounting has implicitly assumed non-negative real common numbers, the table rounding problem may be considered in a more general setting including negative reals. In this general case, a sufficiently large positive integer (namely, the least integer greater than the maximum absolute value over the negative common numbers) is added to each common number so that all numbers in the table are positive. After a rounding procedure is performed the rounded table is transformed back by subtracting the respective integer from all common numbers to achieve the rounding of the original table. In addition, the adjacent integer rounding of a common number in the table can cause a change of at most 1. Therefore, $0 \leq c_{ij} < 1$ may be assumed without loss of generality since the tables with general values can easily be *normalized* to satisfy this condition as it is shown for table 1 in table 3. It is clear that the resulting *adjacent integer rounding* (i. e., whether round-down or round-up) for the normalized form which preserves the check sums is also a consistent rounding for the original table.

At first the total sum σ of all common numbers is rounded off to integer $\bar{\sigma} \in \mathcal{Z}$ (e. g., by using the classical rounding procedure) since this number has the greatest significance. Then we will distinguish two variants of the table rounding problem. First the *table rounding problem with prescribed sums* (TRPS) can be formulated in the following way. Given the prescribed integer rounding $\bar{a}_i \in \mathcal{Z}$, $\bar{b}_j \in \mathcal{Z}$ of the row and column check sums a_i $(i = 1, \dots, m)$, b_j $(j = 1, \dots, n)$, respectively, such that their sums $\sum_{i=1}^{m} \bar{a}_i = \sum_{j=1}^{n} \bar{b}_j = \bar{\sigma}$ equal to the rounded total sum, find the

Table 3. Table 1 after normalization.

0.5	0.5	0.6	1.6
0.6	0.6	0.6	1.8
1.1	1.1	1.2	3.4

rounding $\bar{c}_{ij} \in \{0, 1\}$ of common numbers c_{ij} $(i = 1, \ldots, m,\ j = 1, \ldots, n)$, so that $\bar{a}_i = \sum_{j=1}^{n} \bar{c}_{ij}$ $(i = 1, \ldots, m)$, $\bar{b}_j = \sum_{i=1}^{m} \bar{c}_{ij}$ $(j = 1, \ldots, n)$. In addition, the integer common numbers have to remain the same, i. e. $\bar{c}_{ij} = 0$ for $c_{ij} = 0$. Table 4 is an example input of this problem for table 3.

Table 4. Table 3 as an example of input for TRPS.

0.5	0.5	0.6	1
0.6	0.6	0.6	2
1	1	1	3

The second type of the *table rounding problem* (TRP) does not have prescribed rounding of check sums and can be formulated as follows. Given the table of common numbers $0 \leq c_{ij} < 1$ $(i = 1, \ldots, m,\ j = 1, \ldots, n)$ and the rounded total sum $\bar{\sigma} \in \{\lfloor \sigma \rfloor, \lceil \sigma \rceil\}$ find the rounding $\bar{c}_{ij} \in \{0, 1\}$ and the rounding of the row and column check sums a_i, b_j to the adjacent integer $\bar{a}_i \in \{\lfloor a_i \rfloor, \lceil a_i \rceil\}$, $\bar{b}_j \in \{\lfloor b_j \rfloor, \lceil b_j \rceil\}$ such that $\bar{a}_i = \sum_{j=1}^{n} \bar{c}_{ij}$ $(i = 1, \ldots, m)$, $\bar{b}_j = \sum_{i=1}^{m} \bar{c}_{ij}$ $(j = 1, \ldots, n)$, and $\bar{\sigma} = \sum_{i=1}^{m} \bar{a}_i = \sum_{j=1}^{n} \bar{b}_j$. Again, the integers must be preserved, i. e. $\bar{c}_{ij} = 0$ for $c_{ij} = 0$, $\bar{a}_i = a_i$ for $\lfloor a_i \rfloor = a_i$, and $\bar{b}_j = b_j$ for $\lfloor b_j \rfloor = b_j$.

Considering both types of table rounding problems we will look for the rounding with a minimal error. A *round-error* $\varepsilon(x)$ of the number x from the table (i. e. c_{ij}, a_i or b_j) is defined as the absolute error $\varepsilon(x) = |x - \bar{x}|$. The *total rounding error* $E_S = \sum_{i=1}^{m} \sum_{j=1}^{n} \varepsilon(c_{ij})$ for the TRPS and $E = E_S + \sum_{i=1}^{m} \varepsilon(a_i) + \sum_{j=1}^{n} \varepsilon(b_j)$ for the TRP is a sum of round-errors of the relevant numbers in the table.

3 MAXIMUM FLOW PROBLEM

We recall the definition of maximum flow problem [6]. A *network* is a 4-tuple $S = (G, z, s, c)$, where $G = (V, E)$ is an oriented graph with the set of vertices V and the set of edges $E \subseteq V \times V$, $z \in V$ is a *source*, $s \in V$ is a *sink* and $c : E \longrightarrow \Re_0^+$ is a *capacity* which associates each edge $(u, v) \in E$ with a non-negative real number $c(u, v)$. A function $t : E \longrightarrow \Re_0^+$ is said to be a *flow* of the network S, if $0 \leq t(u, v) \leq c(u, v)$ for any edge $(u, v) \in E$ and for each vertex $v \in V - \{z, s\}$ the *conservation* *law* holds:

$$\sum_{(u,v)\in E} t(u,v) = \sum_{(v,u)\in E} t(v,u)\,. \tag{3.1}$$

In addition the *flow size* is defined:

$$|t| = \sum_{(z,v)\in E} t(z,v) - \sum_{(v,z)\in E} t(v,z) \tag{3.2}$$

and t is said to be the *maximum flow* if $|t_1| \leq |t|$ holds for any flow t_1 of S. The *maximum flow problem* (MFP) is to determine the maximum flow for a given network. We will confine ourselves to a special case in which the edge capacity of the network is an integer function. Then it can be shown [6] that there exists a maximum flow which assigns each edge with an integer and can be found in polynomial time:

Theorem 3.1. Given a network with integer edge capacity then the integer maximum flow can be found within $O(|V|^3)$ time, in which $|V|$ is the number of network vertices.

Furthermore, the *cut* $z \in C \subseteq V - \{s\}$ of the network S is defined as a subset of vertices including the source and excluding the sink and the *cut size* is determined:

$$|C| = \sum_{\substack{(u,v)\,\in\, E \\ u\,\in\, C,\, v\,\notin\, C}} c(u,v)\,. \tag{3.3}$$

The following theorem recalls the well-known relationship between the maximum flow and the minimum cut:

Theorem 3.2. In any network the size of a flow is smaller or equal to the size of any cut. Especially, the maximum flow size equals the minimum cut size.

Finally, a so-called *edge cost* [4, 7] is introduced which is a function $h : E \longrightarrow \Re$ which associates each edge $(u,v) \in E$ with a real number $h(u,v)$. Using this notion the *cost* $h(t)$ of the flow t is defined as

$$h(t) = \sum_{(u,v)\in E} h(u,v)t(u,v)\,. \tag{3.4}$$

Then given the flow size T the *minimum cost flow problem* (MCFP) is to determine the flow t of the size $|t| \geq T$ with minimal $h(t)$. This problem can be solved in a polynomial time [7] as well. We will use the following theorem:

Theorem 3.3. Given an integer T and a network with integer edge capacity and with non-negative edge cost h, then the integer flow t of the size $|t| \geq T$ with minimal $h(t)$ can be found within $O(T \cdot |V|^2)$ time, where $|V|$ is the number of network vertices.

4 TABLE ROUNDING WITH PRESCRIBED SUMS

In this section we will deal with the TRPS. At first the TRPS is reduced to the MFP:

Definition 4.1. Let $\bar{a}_i$ $(i = 1, \ldots, m)$, $\bar{b}_j$ $(j = 1, \ldots, n)$ be the prescribed rounding of check sums from the TRPS. We define the network $S(\bar{a}_i, \bar{b}_j) = (G = (V, E), z, s, c)$ of the corresponding MFP as follows:

$$\begin{aligned} V &= \{A_i \mid i = 1, \ldots, m\} \cup \{B_j \mid j = 1, \ldots, n\} \cup \{z, s\}, \\ E &= \{(A_i, B_j) \mid i = 1, \ldots, m;\ j = 1, \ldots, n\} \cup \\ &\quad \cup \{(z, A_i) \mid i = 1, \ldots, m\} \cup \{(B_j, s) \mid j = 1, \ldots, n\}, \\ c(A_i, B_j) &= \lceil c_{ij} \rceil - \lfloor c_{ij} \rfloor \quad i = 1, \ldots, m;\ j = 1, \ldots, n, \\ c(z, A_i) &= \bar{a}_i \quad i = 1, \ldots, m, \\ c(B_j, s) &= \bar{b}_j \quad j = 1, \ldots, n\,. \end{aligned}$$

An example of such reduction for table 4 is in picture 1, in which the edges of the network are associated with the relevant capacities. The correctness of the previous reduction is expressed in the following theorem:

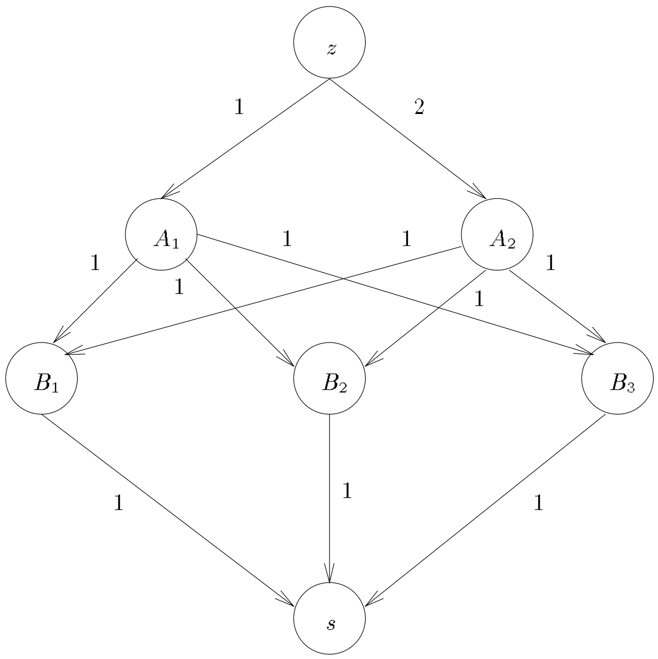

Fig. 1. An example of TRPS reduction of table 4 to MFP.

Theorem 4.1. A solution of the TRPS $\bar{a}_i$, $\bar{b}_j$ exists iff there is an integer flow t in the network $S(\bar{a}_i, \bar{b}_j)$ of the size $|t| = \sum_{i=1}^m \bar{a}_i = \sum_{j=1}^n \bar{b}_j$.

Proof. Let t be the integer flow of the desired size $|t| = \sum_{i=1}^m \bar{a}_i = \sum_{j=1}^n \bar{b}_j$. Consider the cut $C_z = \{z\}$ of the size $|C_z| = \sum_{i=1}^m c(z, A_i) = \sum_{i=1}^m \bar{a}_i$ which equals $|t| = \sum_{i=1}^m t(z, A_i)$ and thus, t represents maximum flow due to Theorem 3.2. This also implies $t(z, A_i) = c(z, A_i)$ for $i = 1, \ldots, m$, since $t(z, A_i) \leq c(z, A_i)$. Further, it follows from conservation law (3.1) that $|t| = \sum_{i=1}^m t(z, A_i) = \sum_{i=1}^m \sum_{j=1}^n t(A_i, B_j) = \sum_{j=1}^n \sum_{i=1}^m t(A_i, B_j) = \sum_{j=1}^n t(B_j, s)$. Then similarly, consider the cut $C_s = V - \{s\}$ of the size $|C_s| = \sum_{j=1}^n c(B_j, s) = \sum_{j=1}^n \bar{b}_j$ which equals $|t| = \sum_{j=1}^n t(B_j, s)$ and this implies $t(B_j, s) = c(B_j, s)$ for $j = 1, \ldots, n$ since $t(B_j, s) \leq c(B_j, s)$.

Now define the adjacent integer rounding $\bar{c}_{ij} = t(A_i, B_j) \in \{0, 1\}$ of common numbers in the table. From the conservation law for A_i, B_j it follows that this rounding preserves the check sums:

$$\bar{a}_i = c(z, A_i) = t(z, A_i) = \sum_{j=1}^n t(A_i, B_j) = \sum_{j=1}^n \bar{c}_{ij} \quad i = 1, \ldots, m \tag{4.1}$$

$$\bar{b}_j = c(B_j, s) = t(B_j, s) = \sum_{i=1}^m t(A_i, B_j) = \sum_{i=1}^m \bar{c}_{ij} \quad j = 1, \ldots, n\,. \tag{4.2}$$

The opposite implication is similar. □

Using the reduction to the MFP and its relationship with the minimum cut, we formulate a necessary and sufficient condition when the adjacent integer table rounding, preserving the prescribed check sums, exists.

Theorem 4.2. A solution for the TRPS $\bar{a}_i$ $(i = 1, \ldots, m)$, $\bar{b}_j$ $(j = 1, \ldots, n)$ exists iff for any choice of p rows $P \subseteq \{1, \ldots, m\}$ and of q columns $Q \subseteq \{1, \ldots, n\}$ it holds

$$\sum_{i \in P} \bar{a}_i - \sum_{i \in Q} \bar{b}_j \leq p(n - q) - z_{PQ} \tag{4.3}$$

where $z_{PQ} = |\{c_{ij} = 0 \,|\, i \in P, j \notin Q\}|$ is the number of zero elements situated in the rows from P which are not in the columns from Q at the same time.

Proof. Reduce the TRPS $\bar{a}_i$, $\bar{b}_j$ to the MFP $S(\bar{a}_i, \bar{b}_j)$. All possible cuts C_{PQ} $(P \subseteq \{1, \ldots, m\}$, $Q \subseteq \{1, \ldots, n\})$ in the network $S(\bar{a}_i, \bar{b}_j)$ have the following form:

$$C_{PQ} = \{z\} \cup \{A_i \,|\, i \in P\} \cup \{B_j \,|\, j \in Q\}\,. \tag{4.4}$$

Thus, the size of any cut C_{PQ} can be expressed:

$$\begin{aligned} |C_{PQ}| &= \sum_{i \notin P} c(z, A_i) + \sum_{i \in P} \sum_{j \notin Q} c(A_i, B_j) + \sum_{j \in Q} c(B_j, s) = \\ &= \sum_{i=1}^m \bar{a}_i - \sum_{i \in P} \bar{a}_i + p(n - q) - z_{PQ} + \sum_{j \in Q} \bar{b}_j\,. \end{aligned} \tag{4.5}$$

From Theorem 4.1 a solution of the TRPS exists iff there is an integer flow t in the network $S(\bar{a}_i, \bar{b}_j)$ of the size $|t| = \sum_{i=1}^{m} \bar{a}_i$, i. e. iff the size of any cut in this network is greater or equal to this flow, due to Theorem 3.2. This means $|C_{PQ}| \geq \sum_{i=1}^{m} \bar{a}_i$ for any $P \subseteq \{1, \ldots, m\}$, $Q \subseteq \{1, \ldots, n\}$. Substituting for $|C_{PQ}|$ from (4.5) we obtain

$$\sum_{i=1}^{m} \bar{a}_i - \sum_{i \in P} \bar{a}_i + p(n-q) - z_{PQ} + \sum_{j \in Q} \bar{b}_j \geq \sum_{i=1}^{m} \bar{a}_i \tag{4.6}$$

from which the proposition follows:

$$\sum_{i \in P} \bar{a}_i - \sum_{i \in Q} \bar{b}_j \leq p(n-q) - z_{PQ} \,. \tag{4.7}$$

□

An example of the TRPS instance for which the condition (4.3) is not satisfied is in table 5 (consider the first two rows and the first column), which means that the desired rounding does not exist, due to Theorem 4.2.

Table 5. An example of TRPS which does not have a solution.

0.7	0.9	0.9	3
0.6	0.9	0.9	3
0.1	0.7	0.6	1
1	3	3	7

On the other hand the table adjacent integer rounding, consistent with the prescribed check sums, can be found in a polynomial time provided that it exists.

Corollary 4.1. The TRPS for the table which consists of m rows and of n columns can be solved in $O((m+n)^3)$ time and the respective rounding with minimal total rounding error can be computed within $O(mn(m+n)^2)$ time.

Proof. Using Definition 4.1, reduce the TRPS $\bar{a}_i$, $\bar{b}_j$ to the MFP with the network $S(\bar{a}_i, \bar{b}_j)$ having $m+n+2$ vertices. Then employ the algorithm from Theorem 3.1 to find the integer maximum flow t within $O((m+n)^3)$ time. Check whether the flow size achieves $|t| = \sum_{i=1}^{m} \bar{a}_i = \sum_{j=1}^{n} \bar{b}_j$. If it is the case, then the relevant table rounding exists due to Theorem 4.1 and can be determined as $\bar{c}_{ij} = t(A_i, B_j) \in \{0, 1\}$ from the proof of this theorem.

Moreover, the rounding with minimal total rounding error E_S can be computed by using the algorithm for the minimum cost flow problem from Theorem 3.3. Here, the prescribed size of the integer flow t is set to $T = \sum_{i=1}^{m} \bar{a}_i$ and the edge cost is defined as follows: $h(A_i, B_j) = \lceil c_{ij} \rceil - c_{ij}$, $h(z, A_i) = h(B_j, s) = 0$ ($i = 1, \ldots, m$, $j = 1, \ldots, n$). The inequality $T \leq \sum_{i=1}^{m} \sum_{j=1}^{n} \lceil c_{ij} \rceil \leq mn$ implies that this algorithm terminates in polynomial time $O(mn(m+n)^2)$ with the minimal flow cost which can be expressed as follows:

$$h(t) = \sum_{i=1}^{m} \sum_{j=1}^{n} h(A_i, B_j) t(A_i, B_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} (\lceil c_{ij} \rceil - c_{ij}) \bar{c}_{ij} \,. \tag{4.8}$$

Denote by $U = \{c_{ij} | \bar{c}_{ij} = \lceil c_{ij} \rceil = 1,\, i = 1, \ldots, m,\, j = 1, \ldots, n\}$ the set of non-integer common numbers which are rounded up and $\bar{U}$ is the complement of U excluding integers, i. e. $\bar{U}$ is the set of non-integer common numbers which are rounded down. Then the minimal flow cost can further be rewritten as

$$h(t) = \sum_{U} (\lceil c_{ij} \rceil - c_{ij}) \,. \tag{4.9}$$

We prove that the minimal flow cost $h(t)$ minimizes the total rounding error E_S. Let U correspond to the minimal flow cost which means that

$$\sum_{U} (\lceil c_{ij} \rceil - c_{ij}) \leq \sum_{U'} (\lceil c_{ij} \rceil - c_{ij}) \tag{4.10}$$

holds for any other U'. Adding $\sum_{\bar{U} \cap \bar{U}'} (\lceil c_{ij} \rceil - c_{ij}) - \sum_{U \cap U'} (\lceil c_{ij} \rceil - c_{ij})$ to (4.10) we obtain:

$$\sum_{\bar{U}'} (\lceil c_{ij} \rceil - c_{ij}) \leq \sum_{\bar{U}} (\lceil c_{ij} \rceil - c_{ij}) \,. \tag{4.11}$$

Notice that the numbers of elements in U, U' are the same, i. e. $|U| = |U'|$ because of the prescribed total sum. Thus, inequality (4.11) multiplied by -1 is added to $|\bar{U}| = |\bar{U}'|$:

$$\sum_{\bar{U}} (1 - \lceil c_{ij} \rceil + c_{ij}) \leq \sum_{\bar{U}'} (1 - \lceil c_{ij} \rceil + c_{ij}) \tag{4.12}$$

which implies

$$\sum_{\bar{U}} (c_{ij} - \lfloor c_{ij} \rfloor) \leq \sum_{\bar{U}'} (c_{ij} - \lfloor c_{ij} \rfloor) \tag{4.13}$$

By summing (4.10) and (4.13) the total rounding error E_S for U is proved to be minimal, i. e.

$$E_S = \sum_{U} (\lceil c_{ij} \rceil - c_{ij}) + \sum_{\bar{U}} (c_{ij} - \lfloor c_{ij} \rfloor) \leq \sum_{U'} (\lceil c_{ij} \rceil - c_{ij}) + \sum_{\bar{U}'} (c_{ij} - \lfloor c_{ij} \rfloor) \tag{4.14}$$

holds for any other U'. □

Finally, it should be mentioned that the TRPS (similarly for TRP) may have more than one solution. For example, consider all common numbers in the table to be 0.5. The procedure for TRPS from Corollary 4.1 is based on the maximum flow algorithms from Theorems 3.1, 3.3. Hence, the selection among the consistent (and optimal) roundings in such an ambiguous case depends on the implementation of these algorithms which decide for themselves among the corresponding maximum flows. Thus, it is conceivable to have either a deterministic biased rounding procedure that prefers one of the possible ways of consistent rounding or a randomized choice is applied when more solutions of TRPS are available. However, in the randomized approach the resulting rounding may differ for the same input table from one report to another one and this may not be acceptable in some applications. In any case, the preference for rounding a particular common number in the table can be expressed by supplying this number with the respective additional digits (e. g. 0.51 instead of 0.5 when the rounding-up is desired) and the underlying rounding is found while minimizing the total rounding error with respect to this modified table.

5 TABLE ROUNDING ALWAYS EXISTS

In this section we will generalize the results from Section 4 for the TRP using the same technique. The corresponding reduction to the MFP follows:

Definition 5.1. Let a_i $(i = 1, \ldots, m)$, b_j $(j = 1, \ldots, n)$ be the row and column check sums and $\bar{\sigma} \in \{\lfloor\sigma\rfloor, \lceil\sigma\rceil\}$ be the rounded total sum from the TRP. We define a network $S(\bar{\sigma}) = (G = (V, E), z, s, c)$ of the corresponding MFP as follows:

$$\begin{aligned}
V &= \{A_i \mid i = 1, \ldots, m\} \cup \{B_j \mid j = 1, \ldots, n\} \cup \{z, s, X, Y\}, \\
E &= \{(A_i, B_j) \mid i = 1, \ldots, m;\ j = 1, \ldots, n\} \cup \{(z, X), (Y, s)\} \cup \\
&\quad \cup \{(z, A_i) \mid i = 1, \ldots, m\} \cup \{(B_j, s) \mid j = 1, \ldots, n\} \cup \\
&\quad \cup \{(X, A_i) \mid i = 1, \ldots, m\} \cup \{(B_j, Y) \mid j = 1, \ldots, n\}, \\
c(A_i, B_j) &= \lceil c_{ij}\rceil - \lfloor c_{ij}\rfloor \quad i = 1, \ldots, m;\ j = 1, \ldots, n, \\
c(z, A_i) &= \lfloor a_i\rfloor, \quad c(X, A_i) = \lceil a_i\rceil - \lfloor a_i\rfloor \quad i = 1, \ldots, m, \\
c(B_j, s) &= \lfloor b_j\rfloor, \quad c(B_j, Y) = \lceil b_j\rceil - \lfloor b_j\rfloor \quad j = 1, \ldots, n, \\
c(z, X) &= \bar{\sigma} - \sum_{i=1}^{m} \lfloor a_i\rfloor, \qquad c(Y, s) = \bar{\sigma} - \sum_{j=1}^{n} \lfloor b_j\rfloor .
\end{aligned}$$

An example of this generalized reduction for table 5 (with the prescribed row and column sums replaced by the actual sums) is in picture 2.

Theorem 5.1. A solution of the TRP with $\bar{\sigma} \in \{\lfloor\sigma\rfloor, \lceil\sigma\rceil\}$ exists iff there is an integer flow t in the network $S(\bar{\sigma})$ of the size $|t| = \bar{\sigma}$.

Proof. Let t be the integer flow of the size $|t| = \bar{\sigma}$. Then define the rounding $\bar{c}_{ij} = t(A_i, B_j) \in \{0, 1\}$ $(i = 1, \ldots, m,\ j = 1, \ldots, n)$ of common numbers and $\bar{a}_i = t(z, A_i) + t(X, A_i)$, $\bar{b}_j = t(B_j, s) + t(B_j, Y)$ of the corresponding check sums. First we prove that this is an adjacent integer rounding, i. e. $\bar{a}_i \in \{\lfloor a_i\rfloor, \lceil a_i\rceil\}$, $\bar{b}_j \in \{\lfloor b_j\rfloor, \lceil b_j\rceil\}$. Furthermore, we will confine ourselves on the row sums $\bar{a}_i$, while the proof for $\bar{b}_j$ is similar. It is clear that $\bar{a}_i = t(z, A_i) + t(X, A_i) \leq c(z, A_i) + c(X, A_i) = \lceil a_i\rceil$. It even holds that $t(z, A_i) = c(z, A_i) = \lfloor a_i\rfloor$ from $\bar{\sigma} = |t| = \sum_{i=1}^{m} t(z, A_i) + t(z, X) \leq \sum_{i=1}^{m} c(z, A_i) + c(z, X) = \bar{\sigma}$, which implies $\bar{a}_i \geq \lfloor a_i\rfloor$.

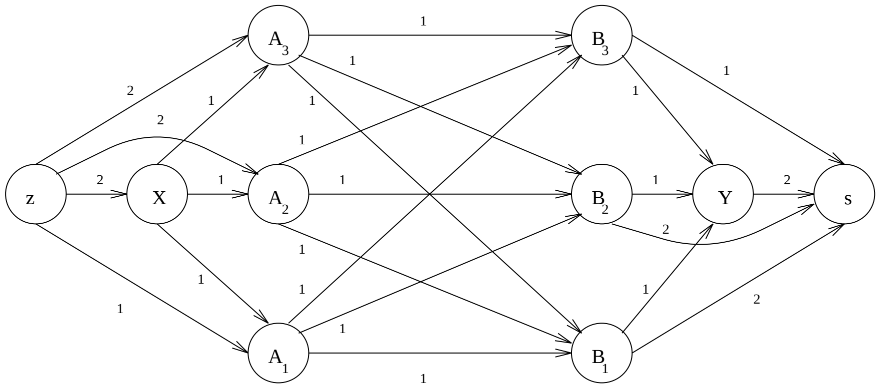

Fig. 2. An example of TRP reduction of table 5 to MFP.

Hence, $\bar{a}_i \in \{\lfloor a_i \rfloor, \lceil a_i \rceil\}$ represents an adjacent integer row rounding. Now we check whether the respective rounded sums agree. From the conservation law (3.1) for the vertex X we obtain $t(z,X) = \sum_{i=1}^{m} t(X, A_i)$. This implies

$$\bar{\sigma} = |t| = \sum_{i=1}^{m} t(z, A_i) + t(z, X) = \sum_{i=1}^{m} (t(z, A_i) + t(X, A_i)) = \sum_{i=1}^{m} \bar{a}_i \,. \tag{5.1}$$

Using (3.1) for the vertices A_i we obtain

$$\bar{a}_i = t(z, A_i) + t(X, A_i) = \sum_{j=1}^{n} t(A_i, B_j) = \sum_{j=1}^{n} \bar{c}_{ij} \quad i = 1, \ldots, m \,. \tag{5.2}$$

The opposite implication is similar. □

Now we prove that the adjacent integer table rounding always exists.

Theorem 5.2. A solution for the TRP with the given $\bar{\sigma} \in \{\lfloor \sigma \rfloor, \lceil \sigma \rceil\}$ always exists.

Proof. Reduce the TRP with $\bar{\sigma}$ to the MFP $S(\bar{\sigma})$. From Theorems 3.2, 5.1 it is sufficient to prove that any cut size in $S(\bar{\sigma})$ is greater or equal to the size $|C_z| = \bar{\sigma}$ of cut $C_z = \{z\}$. All possible cuts C_{PQ}^{H} in $S(\bar{\sigma})$ have the following form:

$$C_{PQ}^{H} = \{z\} \cup \{A_i \mid i \in P\} \cup \{B_j \mid j \in Q\} \cup H \,, \tag{5.3}$$

where $P \subseteq \{1, \ldots, m\}$ contains p rows, $Q \subseteq \{1, \ldots, n\}$ consists of q columns and $H \subseteq \{X, Y\}$. Remember that $z_{PQ} = |\{c_{ij} = 0 \mid i \in P, j \notin Q\}|$. We will distinguish four cases according to four possible H:

1. $X \in H$, $Y \in H$:

$$|C_{PQ}^{H}| \quad = \quad \sum_{i \notin P} \lceil a_i \rceil + p(n-q) - z_{PQ} + \sum_{j \in Q} \lfloor b_j \rfloor + \bar{\sigma} - \sum_{j=1}^{n} \lfloor b_j \rfloor = \tag{5.4}$$

$$\begin{aligned}
&= \sum_{i \notin P} \lceil a_i \rceil + p(n-q) - z_{PQ} - \sum_{j \notin Q} \lfloor b_j \rfloor + \bar{\sigma} \geq \\
&\geq \sum_{i \notin P} \sum_{j=1}^{n} c_{ij} + \sum_{i \in P} \sum_{j \notin Q} c_{ij} - \sum_{i=1}^{m} \sum_{j \notin Q} c_{ij} + \bar{\sigma} = \sum_{i \notin P} \sum_{j \in Q} c_{ij} + \bar{\sigma} \geq \bar{\sigma} \,.
\end{aligned}$$

2. $X \in H$, $Y \notin H$:

$$|C^H_{PQ}| = \sum_{i \notin P} \lceil a_i \rceil + p(n-q) - z_{PQ} + \sum_{j \in Q} \lceil b_j \rceil \geq \sum_{i=1}^{m} \bar{a}_i = \bar{\sigma} \,. \tag{5.5}$$

3. $X \notin H$, $Y \in H$:

$$\begin{aligned}
|C^H_{PQ}| &= \bar{\sigma} - \sum_{i \in P} \lfloor a_i \rfloor + p(n-q) - z_{PQ} - \sum_{j \notin Q} \lfloor b_j \rfloor + \bar{\sigma} \geq \qquad (5.6) \\
&\geq \bar{\sigma} - \sum_{i \in P} \sum_{j=1}^{n} c_{ij} + \sum_{i \in P} \sum_{j \notin Q} c_{ij} - \sum_{i=1}^{m} \sum_{j \notin Q} c_{ij} + \bar{\sigma} = \\
&= \sum_{i \notin P} \sum_{j \in Q} c_{ij} + \bar{\sigma} \geq \bar{\sigma} \,.
\end{aligned}$$

4. $X \notin H$, $Y \notin H$:

$$\begin{aligned}
|C^H_{PQ}| &= \bar{\sigma} - \sum_{i \in P} \lfloor a_i \rfloor + p(n-q) - z_{PQ} + \sum_{j \in Q} \lceil b_j \rceil \geq \qquad (5.7) \\
&\geq \bar{\sigma} - \sum_{i \in P} \sum_{j=1}^{n} c_{ij} + \sum_{i \in P} \sum_{j \notin Q} c_{ij} + \sum_{i=1}^{m} \sum_{j \in Q} c_{ij} = \bar{\sigma} + \sum_{i \notin P} \sum_{j \in Q} c_{ij} \geq \bar{\sigma} \,.
\end{aligned}$$

□

Corollary 5.1. The TRP for the table containing m rows and n columns can be solved in $O((m+n)^3)$ time and the respective rounding with minimal total rounding error can be computed within $O(mn(m+n)^2)$ time.

Proof. The proof is analogous to Corollary 4.1. In addition, the total rounding error E takes into account rounding errors of check sums which are represented as the edge costs $h(X, A_i) = \lceil a_i \rceil - a_i$ $(i = 1, \ldots, m)$, $h(B_j, Y) = \lceil b_j \rceil - b_j$ $(j = 1, \ldots, n)$. □

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