# Exponential Transients in Continuous-Time Liapunov Systems

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**Abstract**

We consider the convergence behavior of a class of continuous-time dynamical systems corresponding to so called symmetric Hopfield nets studied in neural networks theory. We prove that such systems may have transient times that are exponential in the system dimension (i.e. number of "neurons"), despite the fact that their dynamics are controlled by Liapunov functions. This result stands in contrast to many proposed uses of such systems in e.g. combinatorial optimization applications, in which it is often implicitly assumed that their convergence is rapid. An additional interesting observation is that our example of an exponential-transient continuous-time system (a simulated binary counter) in fact converges more slowly than any discrete-time Hopfield system of the same representation size. This suggests that continuous-time systems may be worth investigating for gains in descriptional efficiency as compared to their discrete-time counterparts.

*Key words:* Dynamical system, Continuous time, Hopfield network, Liapunov function, Convergence time

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## 1 Introduction

In recent years, a number of authors have sought to understand the computational characteristics of "natural" dynamical systems. One much studied class of systems are those that are defined by various *neural network* models [3,6,12]. This interest is motivated partly by the quest to understand the fundamental limits and possibilities of practical neurocomputing, and partly by the realization that despite their formal simplicity, neural networks are computationally quite powerful, and thus may serve as a useful reference model for investigating more complicated systems. In general, the computational properties of discrete-time systems are by now fairly well understood [12], but in the area of continuous-time systems much work remains to be done [9,11].

Probably the best-known, and most widely-used continuous-time neural network model is that popularized by John Hopfield in 1984 [7], and known as the "continuous-time Hopfield model".[3] As practical neural networks, proposed uses of Hopfield-type systems include associative memory [7] and fast approximate solution of combinatorial optimization problems [8], and designs exist for implementing them in analog electrical [7] and optical [16] hardware. It is well known [2,7] that the dynamics of any Hopfield-type system with a *symmetric* interconnection matrix is governed by a *Liapunov*, or *energy function*. At first sight, the existence of a Liapunov function for a system would appear to severely limit its dynamical capabilities, because it implies that any trajectory converges towards some stable equilibrium state. For instance, non-damping oscillations of the system state obviously cannot be created under this constraint, whereas such oscillations are easily obtained in systems with *asymmetric* coupling weights.

Because of the apparent simplicity of Hopfield system dynamics, one might also assume that they always converge rapidly—an assumption that seems to often be implicitly made in e.g. discussing the potential of such networks as "fast analog solvers" for optimization problems. Contrary to this expectation, we shall construct for every integer $n \geq 0$ a continuous-time Hopfield system $C = C_n$ of $6n + 1$ variables with a symmetric interconnection weight matrix and a saturated-linear "activation function" that simulates an $(n+1)$-bit binary counter and thus produces a sequence of $2^n - 1$ well-controlled oscillations before it converges. The original idea for a corresponding discrete-time symmetric counter network stems from [4]. Besides suggesting some caution in applying neural networks to optimization problems, this provides to our knowledge the first known example of a continuous-time, Liapunov-function controlled dynamical system with an exponential transient period. Such an

[3] Although in fact the dynamics of this model were already analyzed earlier by Cohen and Grossberg in a more general setting [2].

exponential-transient oscillator can also be used to support a general Turing machine simulation by continuous-time symmetric Hopfield systems [13].

In terms of bit representations, this bounds translates to a convergence time of $2^{\Omega(g(M))}$ time units for Hopfield systems whose interconnection weight matrices can be encoded within $M$ bits, where $g(M)$ is an arbitrary continuous function such that $g(M) = o(M)$, $g(M) = \Omega(M^{2/3})$, and $M/g(M)$ is increasing. This convergence time lower bound can be compared to a general upper bound of $2^{O(\sqrt{N})}$ for discrete Hopfield networks [15]. It turns out that the continuous-time system $C$ converges later than any discrete Hopfield network of the same description length, assuming that the time interval between two subsequent discrete updates corresponds to a continuous time unit. This suggests that continuous-time analog models of computation may be worth investigating more for their gains in representational efficiency than for their (theoretical) capability for arbitrary-precision real number computation [12].

This article is organized as follows. After a brief review of the basic definitions in Section 2, our main construction of the continuous-time symmetric Hopfield system $C_n$ simulating an $(n+1)$-bit binary counter is described in Section 3 where its dynamics is also informally explained. The formal verification of this construction, which has the form of a rather tedious case analysis, is given in Section 4. In Section 5 a numerical simulation example witnessing the validity of the construction is presented. Section 6 concludes with some open problems.

An extended abstract of this paper appeared in [14].

## 2 Preliminaries

We first recall the basic definition of a *continuous-time (symmetric) Hopfield system* [4] of dimension $m$. This system consists of a set of $m$ symmetrically coupled ordinary differential equations in real variables $y_1, \ldots, y_m$:

$$\frac{dy_p}{dt}(t) = -y_p(t) + \sigma(\xi_p(t)), \quad p = 1, \ldots, m, \tag{1}$$

where $\sigma : \Re \longrightarrow \Re$ is some nonlinear *activation function*, and

$$\xi_p(t) = \sum_{q=0}^{m} v(q,p) y_q(t) \tag{2}$$

[4] We shall henceforth discuss only symmetric systems.

is the real-valued *excitation* for site $p = 1, \ldots, m$. The excitation (2) includes the real *weights* $v(p,q) = v(q,p)$ for all $1 \leq p, q \leq m$ forming a symmetrical coupling matrix whereas $v(0,p)$ is a local *bias*, associated with a formal constant variable $y_0(t) \equiv 1$. Further, we fix the activation function $\sigma$ in (1) to be the *saturated linear* map:

$$\sigma(\xi) = \begin{cases} 1 & \text{for } \xi \geq 1 \\ \xi & \text{for } 0 < \xi < 1 \\ 0 & \text{for } \xi \leq 0 \end{cases} \tag{3}$$

which implies $y_1, \ldots, y_m \in [0,1]$.

A convenient intuitive way of representing a system of type (1), which we shall adopt, is to interpret each of the variables $y_p$ as the real-valued *state (output)* of a computational *unit (neuron)* $p$ in a *Hopfield net* evolving in continuous time, and to represent the symmetric coupling coefficient $v(p,q)$ as the weight on an undirected edge connecting unit $p$ to unit $q$. The absence of an edge in the graph indicates a zero weight between the respective units, and vice versa. The *initial network state* $\vec{y}(0) \in [0,1]^m$ determines the boundary condition for system (1).

The dynamics of a continuous-time *symmetric* Hopfield system of type (1) is controlled by the following *Liapunov* or *energy* function, introduced in [2,7]:

$$E(\vec{y}) = -\frac{1}{2}\sum_{p=1}^{m}\sum_{q=1}^{m} v(q,p) y_q y_p - \sum_{p=1}^{m} v(0,p) y_p + \sum_{p=1}^{m} \int_0^{y_p} \sigma^{-1}(y)\,dy\,. \tag{4}$$

The characteristic properties of function $E(\vec{y})$ are that it is bounded on the system's state space $[0,1]^m$, and that it is properly decreasing (i.e. $dE/dt < 0$) along any nonconstant trajectory of the system's dynamics. It then follows that the system (1) always converges, from any initial condition, towards some stable equilibrium state with $dy_p/dt = 0$ for all $p = 1, \ldots, m$.

## 3 A Simulated Binary Counter

A continuous-time Hopfield system $C_n$ of dimension $m = 6n + 1$ will now be constructed which simulates an $(n+1)$-bit binary counter, and thus has a transient period that is exponential in the parameter $m$. The original idea for a corresponding discrete-time counter network stems from [4]. In our simulation, the binary states 0 and 1 of the counter will be represented by excitations

(2) of the corresponding real-valued units in $C$ that are either below the lower saturation threshold of 0 or above the upper saturation threshold of 1, respectively, for activation function (3). For brevity, we shall simply say that a unit $p$ is *saturated* at 0 or 1 at time $t \geq 0$ if its excitation satisfies $\xi_p(t) \leq 0$ or $\xi_p(t) \geq 1$, respectively. We also say that $p$ is *unsaturated* when $0 < \xi_p(t) < 1$.

Note that we use the *excitations* $\xi_p(t)$ of continuous-time units $p \in C$ rather than their actual *states* $y_p(t)$ to represent the binary values since starting at any point within the open interval $(0, 1)$, the outputs $y_p(t)$ of the units saturated at 0 or 1 only converge to limit values 0 or 1, respectively, and never reach these boundaries (see Lemma 3.1 below). The following theorem summarizes the result:

**Theorem 1** *For every integer $n \geq 0$ there exists a continuous-time symmetric Hopfield system $C = C_n$ of dimension $m = 6n+1$ whose global state transition from the zero initial network state to saturation at 1 requires continuous time $\Omega(2^{m/6}/\varepsilon)$, for any $0 < \varepsilon < 0.05$ such that*

$$2^{m/2} < \varepsilon 2^{1/\varepsilon} . \tag{5}$$

*This convergence bound translates to $2^{\Omega(g(M))}$ time units, where $M$ represents the number of bits that are sufficient for encoding the weights in $C$ and $g(M)$ is an arbitrary continuous function such that $g(M) = o(M)$, $g(M) = \Omega(M^{2/3})$, and $M/g(M)$ is increasing.*

**PROOF.** The construction of Hopfield system $C_n$ with $m = 6n + 1$ units (variables) and zero initial conditions $\vec{y}(0) = 0^m$ simulating an $(n + 1)$-bit binary counter will be described by induction on $n$. The operation of the corresponding Hopfield net will first be discussed intuitively, and its correctness will then be formally verified in Section 4.

The induction starts with a system $C_0$ that contains only a single unit $c_0 \in C_0$, with feedback coupling $v(c_0, c_0) = 1 + \varepsilon > 1$ and bias $v(0, c_0) = \varepsilon > 0$ corresponding to its initial positive excitation. This represents the least significant counter bit of "order 0". Because of its feedback greater than 1 the state of $c_0$ gradually grows from initial 0 towards 1. Eventually $c_0$ saturates at 1, at which point we say that the unit $c_0$ becomes *active* or *fires*. Recall that we associate the simulated discrete counter behavior to the excitations of the units rather than their outputs. The external state of $c_0$ of course evolves smoothly converging to 1, and exhibits no abrupt "firing" transitions. Thus, $c_0$ simply implements counting from 0 to 1 as required. This trick of gradual transition from 0 to 1, formally described in Lemma 4 below, is used repeatedly throughout our construction of $C$.

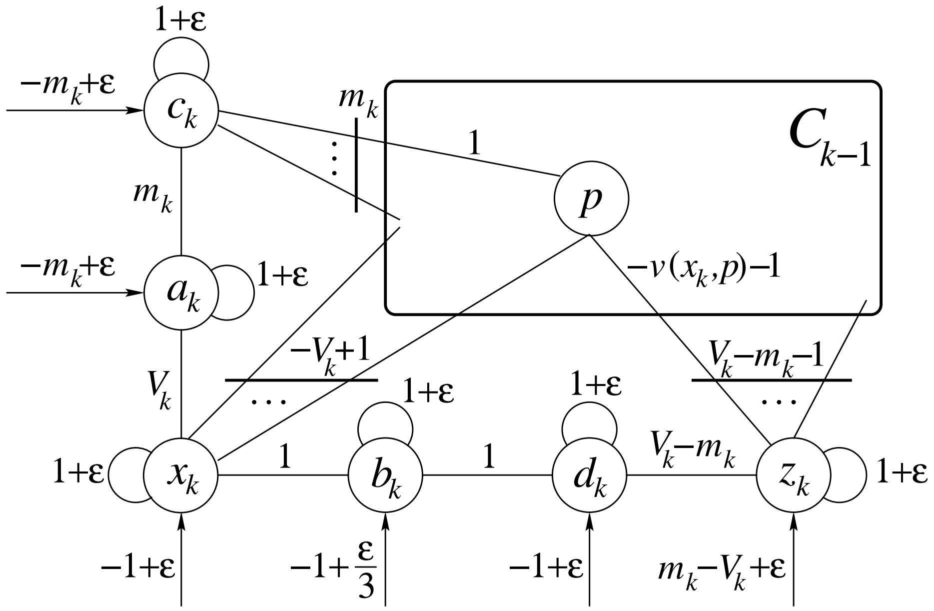



Fig. 1. Inductive construction of $C_k$.

For the induction step depicted in Fig. 1 (the undirected edges connecting units in this graph are labeled with the corresponding symmetric weights whereas the oriented edges drawn without an originating unit correspond to the biases), assume that an "order $(k-1)$" counter network $C_{k-1}$ $(1 \leq k \leq n)$ has been constructed, containing the first $k$ counter units $c_0, \ldots, c_{k-1} \in C_{k-1}$, together with auxiliary units $a_\ell, x_\ell, b_\ell, d_\ell, z_\ell \in C_{k-1}$ $(\ell = 1, \ldots, k-1)$, for a total of

$$m_k = |C_{k-1}| = k + 5(k-1) = 6k - 5 \tag{6}$$

units. Then the next counter unit $c_k$ is connected to all the $m_k$ units $p \in C_{k-1}$ via unit weights $v(p, c_k) = 1$, which, together with its bias $v(0, c_k) = -m_k + \varepsilon$ and feedback weight $v(c_k, c_k) = 1 + \varepsilon$, cause $c_k$ to fire shortly after all these units are active (Lemma 4). This includes the first $k$ active counter bits $c_0, \ldots, c_{k-1}$, which means that the simulated counting from 0 to $2^k - 1$ has been accomplished, and hence, the next counter bit $c_k$ must fire. In addition, unit $c_k$ is connected to a sequence of five auxiliary units $a_k, x_k, b_k, d_k, z_k$ with feedback weights $v(a_k, a_k) = v(x_k, x_k) = v(b_k, b_k) = v(d_k, d_k) = v(z_k, z_k) = 1 + \varepsilon$, which are being, one by one, activated after $c_k$ fires (Lemma 4). This is implemented by the weights $v(c_k, a_k) = m_k$, $v(a_k, x_k) = V_k$, $v(x_k, b_k) = v(b_k, d_k) = 1$, $v(d_k, z_k) = V_k - m_k$, and biases $v(0, a_k) = -m_k + \varepsilon$, $v(0, x_k) = v(0, d_k) = -1 + \varepsilon$, $v(0, b_k) = -1 + \varepsilon/3$, $v(0, z_k) = m_k - V_k + \varepsilon$, where $V_k > 0$ is a sufficiently large, positive parameter, whose value will be determined in (8) so that units $x_k, z_k$ are not directly influenced by the computations occurring in units from $C_{k-1}$ except via $c_k$.

The purpose of the auxiliary units $a_k, b_k, d_k$ is only to slow down the continuous-time state flow in order to synchronize the computation.

The unit $x_k$ is used to reset all the lower-order units in $C_{k-1}$ back to values near 0 by saturating them at 0 (Lemma 3.2b) after $c_k$ fires, which is consistent with the correct counter computation. To achieve this effect, $x_k$ is linked with each $p \in C_{k-1}$ via a large negative weight

$$v(x_k, p) = -\left\lceil v(c_k, p) + \sum_{q \in C_{k-1};\, v(q,p)>0} v(q, p) \right\rceil < 0 \tag{7}$$

that exceeds the positive influence of units in $C_{k-1} \cup \{c_k\}$ on $p$. The value of parameter

$$V_k = 1 - \sum_{p \in C_{k-1}} v(x_k, p) > 0 \tag{8}$$

is determined so that unit $x_k$ fires after $a_k$ is activated in spite of the negative contributions through weights (7) from $p \in C_{k-1}$ to the excitation of $x_k$.

Finally, unit $z_k$ balances the negative influence of $x_k$ on $C_{k-1}$ so that the first $k$ counter bits can again count from 0 to $2^k - 1$ but now with $c_k$ being active. This is achieved by exact positive weights

$$v(z_k, p) = -v(x_k, p) - 1 > 0 \tag{9}$$

for $p \in C_{k-1}$ in which $-v(x_k, p)$ eliminates the influence of $x_k$ on $p$ whereas $-1$ compensates for $v(c_k, p) = 1$. Clearly, units $p \in C_{k-1}$ cannot reversely activate $z_k$ since their maximal contribution to the excitation of $z_k$,

$$\sum_{p \in C_{k-1}} v(p, z_k) = -m_k - \sum_{p \in C_{k-1}} v(x_k, p) = V_k - m_k - 1 \tag{10}$$

according to (8), (9), cannot overcome its bias $v(0, z_k) = m_k - V_k + \varepsilon$. This completes the inductive step of the counter network construction.

## 4 Formal Verification

Now the correct state evolution of the continuous-time Hopfield system $C$ described in Section 3 needs to be verified. This is achieved by a sequence of lemmas analyzing the behavior of the corresponding system of differential equations (1). Lemma 2 first upper bounds the maximum sum of absolute values of weights incident on any unit in $C$. Lemma 3 then describes explicitly the continuous-time state evolution for saturated units. An analysis of how the decreasing *defects*, i.e. distances from limit values in the states of saturated units, affect the excitation of any other unit reveals that the units in $C$ actually approximate the discrete update rule of corresponding threshold gates after a certain transient time, provided that the incident saturated units stay saturated. Furthermore, the transfer of the activity in the counter $C$ from a unit to a subsequent one, when all the incident units are saturated, will be analyzed explicitly in Lemma 4. (But note that the dynamics of unit $c_0$ at time $t = 0$ slightly differs from this analysis.) A crucial fact for the proof of Theorem 1 is that the duration time of this transfer turns out to be *constant* and not affected by any initial defect. This introduces a "discrete time" into the counter operation. In Lemma 4.2 the result is also partially generalized to the case when some of the incident units may become unsaturated.

**Lemma 2** *For any unit $p \in C$ in the Hopfield system $C$ constructed in Section 3, the sum of absolute values of its incident weights (excluding its local bias) is upper bounded by*

$$\Xi_p = \sum_{q=1}^{m} |v(q,p)| < \varepsilon 2^{1/\varepsilon}\,. \tag{11}$$

**PROOF.** Clearly, for $n = 0$ the only unit $c_0$ satisfies (11) since $\Xi_{c_0} = v(c_0, c_0) = 1 + \varepsilon < \varepsilon 2^{1/\varepsilon}$ due to $0 < \varepsilon < 0.05$. For $n > 0$ the maximum value of $\Xi_p$ among $p \in C$ is reached by unit $x_n$ of the highest order $n$, that is

$$\begin{aligned}\Xi_{x_n} &= v(x_n, x_n) + v(a_n, x_n) + v(b_n, x_n) + \sum_{q \in C_{n-1}} |v(q, x_n)| \\ &= 2V_n + 1 + \varepsilon \geq \Xi_p \end{aligned} \tag{12}$$

according to (8) (e.g. compare to $\Xi_{z_n} = 2V_n - 2m_n + \varepsilon < \Xi_{x_n}$).

Parameter $V_n$ in (12) will be computed by induction on $n \geq 1$ starting with obvious

$$V_1 = 1 - v(x_1, c_0) = 1 + \lceil v(c_1, c_0) + v(c_0, c_0) \rceil = 4 \tag{13}$$

from (8), (7). For $k > 1$ the definition (8) of $V_k$ can be rewritten as follows:

$$V_k = 1 - \sum_{p \in C_{k-2}} v(x_k, p) - \sum_{p \in C_{k-1} \setminus C_{k-2}} v(x_k, p) \tag{14}$$

where $v(x_k, p)$ for $p \in C_{k-2}$ $(k > 1)$ defined by (7) can be expressed recursively:

$$\begin{aligned} v(x_k, p) &= v(x_{k-1}, p) - \sum_{q \in C_{k-1} \setminus C_{k-2};\, v(q,p)>0} v(q, p) \\ &= v(x_{k-1}, p) - v(c_{k-1}, p) - v(z_{k-1}, p) = 2v(x_{k-1}, p) \end{aligned} \tag{15}$$

according to Fig. 1 and equation (9). By introducing formula (15) into equation (14) and using definition (8) the recursive formula for $V_k$ is obtained:

$$V_k = 2V_{k-1} - 1 - \sum_{p \in C_{k-1} \setminus C_{k-2}} v(x_k, p)\,. \tag{16}$$

The weights $v(x_k, p)$ for $p \in C_{k-1} \setminus C_{k-2} = \{c_{k-1}, a_{k-1}, x_{k-1}, b_{k-1}, d_{k-1}, z_{k-1}\}$ in (16) can be calculated from Fig. 1 and by definition (7) in which $\lceil v(c_k, p) + v(p, p) \rceil = 3$ as follows:

$$\begin{aligned} -v(x_k, c_{k-1}) &= 3 + v(a_{k-1}, c_{k-1}) + \sum_{q \in C_{k-2};\, v(q, c_{k-1})>0} v(q, c_{k-1}) \\ &= 2m_{k-1} + 3 && (17) \\ -v(x_k, a_{k-1}) &= 3 + v(c_{k-1}, a_{k-1}) + v(x_{k-1}, a_{k-1}) = V_{k-1} + m_{k-1} + 3 && (18) \\ -v(x_k, x_{k-1}) &= 3 + v(a_{k-1}, x_{k-1}) + v(b_{k-1}, x_{k-1}) = V_{k-1} + 4 && (19) \\ -v(x_k, b_{k-1}) &= 3 + v(x_{k-1}, b_{k-1}) + v(d_{k-1}, b_{k-1}) = 5 && (20) \\ -v(x_k, d_{k-1}) &= 3 + v(b_{k-1}, d_{k-1}) + v(z_{k-1}, d_{k-1}) = V_{k-1} - m_{k-1} + 4 && (21) \\ -v(x_k, z_{k-1}) &= 3 + v(d_{k-1}, z_{k-1}) + \sum_{q \in C_{k-2};\, v(q, z_{k-1})>0} v(q, z_{k-1}) \\ &= 2V_{k-1} - 2m_{k-1} + 2 && (22) \end{aligned}$$

where formula (10) has been employed in equation (22). The weights (17)–(22) are summed up as

$$- \sum_{p \in C_{k-1} \setminus C_{k-2}} v(x_k, p) = 5V_{k-1} + 21 \tag{23}$$

which is plugged in formula (16):

$$V_k = 7V_{k-1} + 20\,. \tag{24}$$

It follows from (13) and (24) that

$$V_n = \frac{2}{3}\left(11 \cdot 7^{n-1} - 5\right)\,. \tag{25}$$

Hence for any $p \in C$,

$$\Xi_p \leq \frac{4}{3}\left(11 \cdot 7^{n-1} - 5\right) + 1 + \varepsilon \tag{26}$$

according to (12) and (25). For $1 \leq n \leq 5$, the right side of equation (26) is clearly less than $\varepsilon 2^{1/\varepsilon}$ from $0 < \varepsilon < 0.05$ whereas for $n > 5$ it is less than $2^{3n}$ implying

$$\Xi_p < 2^{3n} < 2^{m/2} < \varepsilon 2^{1/\varepsilon} \tag{27}$$

by assumption (5). □

**Lemma 3**
***1.*** *Let $p \in C$ be a unit saturated at $b \in \{0,1\}$ with a* defect

$$\delta_p(t) = |b - y_p(t)|\,, \tag{28}$$

*for the duration of a continuous time interval $\tau = [t_0, t_f]$ for some $t_0 \geq 0$. Then the state dynamics of $p$ converging towards value $b$ can be explicitly solved as*

$$y_p(t) = \left|b - \delta_p e^{-(t-t_0)}\right| \tag{29}$$

*for $t \in \tau$, where $\delta_p = \delta_p(t_0)$ is $p$'s initial defect.*

***2a.*** *Let $Q \subseteq C$ be a subset of units saturated for the duration of time interval $\tau = [t_0, t_f]$. Then the dynamics of the excitation $\xi_p(t)$ for any unit $p \in C$ can be described as*

$$\xi_p(t) = v(0,p) + \sum_{q \in Q;\, \xi_q(t) \geq 1} v(q,p) + \sum_{q \in C \setminus Q} v(q,p) y_q(t) + \Delta_{pQ} e^{-(t-t_0)} \tag{30}$$

*for $t \in \tau$, where*

$$\Delta_{pQ} = \sum_{q \in Q;\, \xi_q(t_0) \leq 0} v(q,p)\delta_q - \sum_{q \in Q;\, \xi_q(t_0) \geq 1} v(q,p)\delta_q \tag{31}$$

*is the initial* total weighted defect *of $Q$ affecting $\xi_p(t_0)$.*

***2b.*** *In addition, let $t_f > t_0 + t_1$ where*

$$t_1 = \frac{\ln 2}{\varepsilon}\,, \tag{32}$$

*and assume that the respective weights in $C$ satisfy either*

$$v(0,p) + \sum_{q\in Q;\,\xi_q(t_0)\geq 1} v(q,p) + \sum_{q\in C\setminus Q;\,v(q,p)>0} v(q,p) < -\varepsilon \tag{33}$$

*or*

$$v(0,p) + \sum_{q\in Q;\,\xi_q(t_0)\geq 1} v(q,p) + \sum_{q\in C\setminus Q;\,v(q,p)<0} v(q,p) > 1+\varepsilon\,. \tag{34}$$

*Then $p$ is saturated at either $0$ or $1$, respectively, for the duration of time interval $[t_0+t_1, t_f]$.*

**PROOF.**

**1.** According to (1) and (3) the state $y_p(t)$ of unit $p$ saturated at $b \in \{0,1\}$ for $t \in \tau$ is independent of outputs from the remaining units and its continuous-time dynamics is described by a differential equation

$$\frac{dy_p}{dt}(t) = -y_p(t) + b \tag{35}$$

with a boundary condition

$$y_p(t_0) = |b - \delta_p| \tag{36}$$

obtained from (28) for initial defect $\delta_p$. Hence, its explicit solution (29) follows.

**2a.** The excitation $\xi_p(t)$ of unit $p$ defined by (2) is split among the contributions from units outside $Q$ and from those in $Q$ saturated at 0 and 1 whose dynamics for $t \in \tau$ is given by (29):

$$\begin{aligned}\xi_p(t) = v(0,p) &+ \sum_{q\in C\setminus Q} v(q,p)y_q(t) + \sum_{q\in Q;\,\xi_q(t)\leq 0} v(q,p)\delta_q e^{-(t-t_0)} \\ &+ \sum_{q\in Q;\,\xi_q(t)\geq 1} v(q,p)\left(1 - \delta_q e^{-(t-t_0)}\right)\,.\end{aligned} \tag{37}$$

By introducing the initial total weighted defect (31) into formula (37) the dynamics (30) follows.

**2b.** The defect term $\Delta_{pQ}e^{-(t-t_0)}$ in (30) vanishes quickly as time proceeds and its absolute value can be bounded for $t \in [t_0+t_1, t_f]$ as follows

$$\left|\Delta_{pQ}e^{-(t-t_0)}\right| \leq \Xi_p e^{-t_1} < \varepsilon \tag{38}$$

by using Lemma 2 and equation (32). Hence, unit $p$ is saturated at either 0 or 1 for the duration of time interval $[t_0 + t_1, t_f]$ when condition (33) or (34), respectively, is assumed in (30). □

**Lemma 4**
***1.*** *Consider a situation as depicted in Fig. 2, where a unit $p \in C$ with fractional part of bias $\varepsilon' \in \{\varepsilon, \varepsilon/3\}$ and feedback weight $v(p,p) = 1+\varepsilon$ is supposed to receive a signal from preceding unit $o \in C$, activate itself, and further transfer the signal to a subsequent unit $r \in C$ with bias fraction $\varepsilon$ and $v(r,r) = 1+\varepsilon$ via weight $v(p,r) \geq 1$. Let all the units incident on $p, r$ excluding $p, r$ be saturated for the duration of some sufficiently large time interval $\tau = [t_0, t_f]$ (e.g. $t_f > t_0 + t_2$ where $t_2$ is defined in (44) below), starting at a time $t_0 > 0$ when $\xi_p(t_0) = 0$. Assume that the initial defects meet*

$$\delta_p + \Delta_{rQ} < \varepsilon \tag{39}$$

*for $Q = C \setminus \{p\}$. Further assume that the respective weights satisfy*

$$v(0,p) + \sum_{q \in Q;\, \xi_q(t_0) \geq 1} v(q,p) = \varepsilon' \tag{40}$$

$$v(0,r) + \sum_{q \in Q;\, \xi_q(t_0) \geq 1} v(q,r) = \varepsilon - v(p,r)\,. \tag{41}$$

*Then $p$ is unsaturated with the state dynamics*

$$y_p(t) = \frac{\varepsilon'\left(e^{\varepsilon(t-t_0)} - 1\right)}{\varepsilon(1+\varepsilon)} - \frac{\varepsilon' + \Delta_{pQ} e^{-(t-t_0)}}{1+\varepsilon} \tag{42}$$

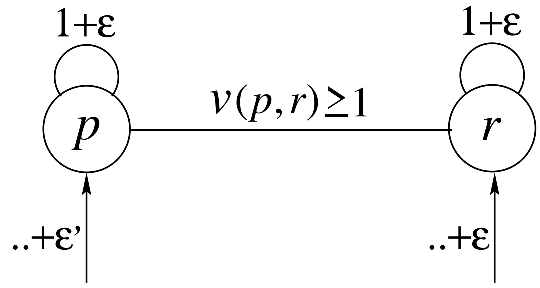
*exactly for the duration of time interval $(t_0, t_0 + t_1')$, where*

$$t_1' = \frac{\ln\left(1 + \frac{\varepsilon}{\varepsilon'}\right)}{\varepsilon} \tag{43}$$

*(note $t_1' = t_1$ for $\varepsilon' = \varepsilon$ and $t_1' = 2t_1$ for $\varepsilon' = \varepsilon/3$), and $r$ is saturated at 0. In addition, $p$ is saturated at 1 for the duration of $[t_0 + t_1', t_0 + t_2]$ and remains further saturated independently of the output from $r$, while $r$ unsaturates from 0 at time $t_0 + t_2$ where*

$$t_2 = \ln \frac{v(p,r)\left((\varepsilon + \varepsilon')\left(1 + \frac{\varepsilon}{\varepsilon'}\right)^{1/\varepsilon} + \Delta_{pQ}\right) - (1+\varepsilon)\Delta_{rQ}}{\varepsilon(1+\varepsilon)} \geq t_1'\,. \tag{44}$$

***2.*** *Consider a situation (see Fig. 1) where unit $x_k \in C$ ($1 \leq k \leq n$) is supposed to receive a signal from preceding unit $a_k$, activate itself, and further transfer*




| $o$ | $p$ | $r$ | unit order |
|---|---|---|---|
| $z_1$ | $c_0$ | $c_k$ | $2 \le k \le n$ |
| $z_k$ | $c_0$ | $c_1$ | $2 \le k \le n$ |
| $c_0$ | $c_k$ | $a_k$ | $1 \le k \le n$ |
| $c_k$ | $a_k$ | $x_k$ | $1 \le k \le n$ |
| $x_k$ | $b_k$ | $d_k$ | $1 \le k \le n$ |
| $b_k$ | $d_k$ | $z_k$ | $1 \le k \le n$ |
| $d_k$ | $z_k$ | $c_0$ | $1 \le k \le n$ |

Fig. 2. Activity transfer from unit $p$ to unit $r$ in $C$.

*the signal to subsequent unit $b_k$ while the units in $C_{k-1}$ incident on $x_k$ may unsaturate from 1 after $x_k$ unsaturates from 0. Let all the units incident on $x_k, b_k$ excluding $x_k, b_k$ and $C_{k-1}$ be saturated for the duration of a sufficiently large time interval $\tau = [t_0, t_f]$ (at least until $b_k$ unsaturates from 0) starting at a time $t_0 > 0$ when $\xi_{x_k}(t_0) = 0$. Assume that the initial defects meet*

$$\delta_{b_k} < \varepsilon 2^{-1/\varepsilon} \tag{45}$$

$$\Delta_{b_k Q'} < \varepsilon 2^{-1/\varepsilon} \tag{46}$$

*for $Q' = C \setminus (C_{k-1} \cup \{x_k\})$, and*

$$(1+\varepsilon)\delta_{x_k} - \sum_{p \in C_{k-1}} v(p, x_k)\delta_p \le \varepsilon 2^{-1/\varepsilon} \tag{47}$$

*outside $Q'$. Further, assume that the respective weights satisfy*

$$v(0, x_k) + \sum_{q \in Q';\, \xi_q(t_0) \ge 1} v(q, x_k) + \sum_{p \in C_{k-1}} v(p, x_k) = \varepsilon \tag{48}$$

$$\sum_{q \in Q';\, \xi_q(t_0) \ge 1} v(q, b_k) = 0\,. \tag{49}$$

*Then $x_k$ saturates at 1 in time at most $t_0 + 2t_1$, remaining then saturated until time at least $t_f$, and $b_k$ unsaturates from 0 only after $x_k$ is saturated at 1.*

**PROOF.**
**1.** A summary of the dynamics of units $p, r$ under discussion here is presented in Table 1, which is verified step by step below.

<table>
<tr><th>$t$</th><th>$\xi_p(t)$</th><th>$y_p(t)$</th><th>$\frac{dy_p}{dt}(t)$</th><th>$\xi_r(t)$</th><th>$y_r(t)$</th></tr>
<tr><td>$= t_0$</td><td>$= 0$</td><td>$= \delta_p$</td><td rowspan="2">$< 0$</td><td rowspan="6">$< 0$</td><td>$= \delta_r$</td></tr>
<tr><td>$\in (t_0, t_0 + t_g)$</td><td rowspan="3">$> 0$</td><td rowspan="3">eq. (42)</td><td rowspan="6">$= \delta_r e^{-(t-t_0)}$</td></tr>
<tr><td>$= t_0 + t_g$</td><td>$= 0$</td></tr>
<tr><td>$\in (t_0 + t_g, t_0 + t_1')$</td><td rowspan="5">$> 0$</td></tr>
<tr><td>$= t_0 + t_1'$</td><td>$= 1$</td><td>eq. (58)</td></tr>
<tr><td>$\in (t_0 + t_1', t_0 + t_2)$</td><td rowspan="3">$> 1$</td><td rowspan="3">eq. (64)</td></tr>
<tr><td>$= t_0 + t_2$</td><td>$= 0$</td></tr>
<tr><td>$\in (t_0 + t_2, t_f]$</td><td>$> 0$</td><td>—</td></tr>
</table>

Table 1
The chronology of the dynamics of units $p, r$.

Excitation

$$\xi_p(t) = \varepsilon' + (1+\varepsilon)y_p(t) + \Delta_{pQ}e^{-(t-t_0)} \tag{50}$$

of unit $p$ in time $t \in [t_0, t_0 + t_2]$ is derived from (30) by assumption (40) and determines $p$'s state dynamics (1) by differential equation

$$\frac{dy_p}{dt}(t) = -y_p(t) + \varepsilon' + (1+\varepsilon)y_p(t) + \Delta_{pQ}e^{-(t-t_0)} \tag{51}$$

when $p$ is unsaturated. The corresponding boundary condition

$$y_p(t_0) = \frac{-\varepsilon' - \Delta_{pQ}}{1+\varepsilon} = \delta_p \tag{52}$$

comes from $\xi_p(t_0) = 0$ that is applied to (50) and determines also the initial defect

$$-\Delta_{pQ} = \varepsilon' + (1+\varepsilon)\delta_p \tag{53}$$

which can be bounded as

$$-1 - \varepsilon - \varepsilon' \le \Delta_{pQ} \le -\varepsilon' < 0 \tag{54}$$

due to $1 \ge \delta_p \ge 0$. The explicit solution (42) for differential equation (51) follows when initial condition (52) is provided.

By plugging solution (42) in equation (51) an explicit formula for $p$'s state derivative can be calculated:

$$\frac{dy_p}{dt}(t) = \frac{e^{-(t-t_0)}\left(\varepsilon' e^{(1+\varepsilon)(t-t_0)} + \Delta_{pQ}\right)}{1+\varepsilon}\,. \tag{55}$$

It follows that before the state $y_p(t)$ of unit $p$ starts to grow it is initially nonincreasing within time $t \in [t_0, t_0 + t_g]$ where

$$t_g = \frac{\ln \frac{-\Delta_{pQ}}{\varepsilon'}}{1+\varepsilon} \tag{56}$$

since $p$'s state derivative (55) is nonpositive for $t \in [t_0, t_0 + t_g]$ and $t_g \geq 0$ according to (54).

Further, by introducing solution (42) into equation (50) the dynamics of $p$'s excitation can also be expressed explicitly:

$$\xi_p(t) = \frac{\varepsilon'\left(e^{\varepsilon(t-t_0)} - 1\right)}{\varepsilon} > 0\,. \tag{57}$$

This ensures that unit $p$ is unsaturated for the duration of the whole time interval $(t_0, t_0 + t_1')$ even though its state $y_p(t)$ is initially decreasing for $t \in (t_0, t_0 + t_g)$. Eventually unit $p$ saturates at 1 exactly at time instant $t_0 + t_1'$ where $t_1'$ is given by formula (43) that is derived from equation (57) for $\xi_p(t_0 + t_1') = 1$. The state of unit $p$ at $t_0 + t_1'$ can be computed by substituting (43) into (42):

$$y_p(t_0 + t_1') = \frac{1 - \varepsilon' - \Delta_{pQ}\left(1 + \frac{\varepsilon}{\varepsilon'}\right)^{-1/\varepsilon}}{1+\varepsilon} \tag{58}$$

Clearly, $t_1' > t_g$ for $\varepsilon < 0.05$ which confirms the actual growth of $p$'s state. Notice that the length $t_1'$ of the period when $p$ is unsaturated is *constant and independent of the initial defects*. This introduces a notion of "discrete time" into the counter operation based on $t_1$. Recall that the detailed chronology of the dynamics of units $p, r$ during the activity transfer is shown in Table 1.

Similarly, excitation

$$\xi_r(t) = \varepsilon - v(p,r) + v(p,r)y_p(t) + \Delta_{rQ}e^{-(t-t_0)} \tag{59}$$

of unit $r$ in time $t \in [t_0, t_0 + t_2]$ is derived from (30) by assumption (41). In order to verify that the state dynamics of unit $p$ is indeed controlled by equation (51) for the duration of time interval $(t_0, t_0 + t_1')$ it must also be checked that unit $r$ remains saturated at 0 in this period, that is $\xi_r(t) \leq 0$. Since $v(p,r) \geq 1$, according to (59) it suffices to show

$$\xi_r(t) \leq \varepsilon + y_p(t) - 1 + \Delta_{rQ} e^{-(t-t_0)} \leq 0 \tag{60}$$

for all $t \in [t_0, t_0 + t_1']$ which can be rewritten as

$$\varepsilon\left(\varepsilon' + (1+\varepsilon)(\delta_p + \Delta_{rQ})\right) e^{-(t-t_0)} + \varepsilon'\left(e^{\varepsilon(t-t_0)} - 1\right) - \varepsilon \leq \varepsilon\left(\varepsilon' - \varepsilon^2\right) \tag{61}$$

by substituting solution (42) for $y_p(t)$ in which $-\Delta_{pQ}$ is replaced by (53). Inequality (61) further reduces to

$$\varepsilon\left(\varepsilon' + \varepsilon(1+\varepsilon)\right) e^{-(t-t_0)} + \varepsilon'\left(e^{\varepsilon(t-t_0)} - 1\right) - \varepsilon \leq \varepsilon\left(\varepsilon' - \varepsilon^2\right) \tag{62}$$

due to assumption (39). For $t \in [t_0, t_0 + t_\varepsilon]$ where

$$t_\varepsilon = \ln \frac{\varepsilon' + \varepsilon(1+\varepsilon)}{\varepsilon' - \varepsilon^2}\,, \tag{63}$$

term $e^{\varepsilon(t-t_0)}$ reaches its maximum at time instant $t_0 + t_\varepsilon$ whereas $e^{-(t-t_0)} \leq 1$, which implies (62) for $\varepsilon < 0.05$. For $t \in [t_0 + t_\varepsilon, t_0 + t_1']$, on the other hand, term $\varepsilon(\varepsilon' + \varepsilon(1+\varepsilon))e^{-(t-t_0)}$ in inequality (62) achieves its maximum $\varepsilon(\varepsilon' - \varepsilon^2)$ at time instant $t_0 + t_\varepsilon$ while $\varepsilon'(e^{\varepsilon(t-t_0)} - 1) - \varepsilon \leq 0$ reaches 0 at time instant $t_0 + t_1'$. Hence, unit $r$ is saturated at 0 within the period $(t_0, t_0 + t_1')$ when unit $p$ is unsaturated.

The state $y_p(t)$ of unit $p$ saturated at 1 follows further the dynamics equation (29), that is

$$\begin{aligned} y_p(t) &= 1 - \delta_p(t_0 + t_1')e^{-(t-t_0-t_1')} \\ &= 1 - \frac{\left(\varepsilon + \varepsilon' + \Delta_{pQ}\left(1 + \frac{\varepsilon}{\varepsilon'}\right)^{-1/\varepsilon}\right) e^{-(t-t_0-t_1')}}{1+\varepsilon} \end{aligned} \tag{64}$$

for $t \in [t_0 + t_1', t_f]$ where the corresponding defect $\delta_p(t_0 + t_1') = 1 - y_p(t_0 + t_1')$ is calculated from (58). By substituting formula (64) into equation (50) the dynamics of $p$'s excitation is obtained:

$$\xi_p(t) = 1 + (\varepsilon + \varepsilon')\left(1 - e^{-(t-t_0-t_1')}\right) \geq 1 \tag{65}$$

for $t \in [t_0 + t_1', t_0 + t_2]$ which confirms that unit $p$ remains saturated at 1 at least until unit $r$ becomes unsaturated from 0. Also excitation $\xi_r(t)$ of unit $r$

saturated at 0 after $p$ saturates at 1 can be expressed by introducing formula (64) into equation (59) as follows:

$$\xi_r(t) = \varepsilon - v(p,r)\delta_p(t_0 + t_1')e^{-(t-t_0-t_1')} + \Delta_{rQ}e^{-(t-t_0)} \tag{66}$$

that reaches 0 at time instant $t_0 + t_2$, i.e. $\xi_r(t_0 + t_2) = 0$ which gives formula (44) for $t_2$ by using equation (43). Hence, unit $r$ is unsaturated from 0 after time $t_0 + t_2$ and the dynamics of $r$'s state is again described by a differential equation of the form (51). In this way, the activity of unit $p$ is transferred to $r$.

Finally, it must be checked that unit $p$ remains still saturated at 1 when $r$ is unsaturated from 0. For this purpose, excitation (50) of unit $p$ can be rewritten as

$$\begin{aligned}\xi_p(t) = {}& \varepsilon' + 1 + \varepsilon + v(p,r)y_r(t) \\ & -(1+\varepsilon)\delta_p(t_0+t_2)e^{-(t-t_0-t_2)} + (\Delta_{pQ} - v(p,r)\delta_r)\, e^{-(t-t_0)}\end{aligned} \tag{67}$$

for $t \in [t_0 + t_2, t_f]$ according to (30) in which the subset of saturated units $Q$ is now replaced with $Q_1 = C \setminus \{r\}$ while the initial defects $\delta_r, \Delta_{pQ}$ in equation (67) are still related to time instant $t_0$ and $Q = C \setminus \{p\}$. The defect $\delta_p(t_0+t_2)$ at time instant $t_0 + t_2$ of unit $p$ saturated at 1 can be calculated by equation (64):

$$\delta_p(t_0 + t_2) = 1 - y_p(t_0 + t_2) = \frac{\varepsilon}{v(p,r) - \frac{(1+\varepsilon)\Delta_{rQ}}{(\varepsilon+\varepsilon')\left(1+\frac{\varepsilon}{\varepsilon'}\right)^{1/\varepsilon} + \Delta_{pQ}}} \tag{68}$$

where formula (44) used. In order to prove that $\xi_p(t) \geq 1$ for all $t \in [t_0+t_2, t_f]$ the underlying negative defect terms in equation (67) having the least value for $t = t_0 + t_2$ will be lower bounded by $-\varepsilon' - \varepsilon$ whereas $v(p,r)y_r(t) \geq 0$ is neglected assuring that unit $p$ remains saturated at 1 independently of the output from $r$. Thus, it is sufficient to prove

$$\begin{aligned}& -(1+\varepsilon)\delta_p(t_0+t_2) + (\Delta_{pQ} - v(p,r)\delta_r)\, e^{-t_2} \\ & = \frac{-\varepsilon(1+\varepsilon)\left((\varepsilon+\varepsilon')\left(1+\frac{\varepsilon}{\varepsilon'}\right)^{1/\varepsilon} + v(p,r)\delta_r\right)}{v(p,r)\left((\varepsilon+\varepsilon')\left(1+\frac{\varepsilon}{\varepsilon'}\right)^{1/\varepsilon} + \Delta_{pQ}\right) - (1+\varepsilon)\Delta_{rQ}} > -\varepsilon' - \varepsilon\end{aligned} \tag{69}$$

where formulas (68), (44) have been employed, which further reduces to

$$(1+\varepsilon)(\varepsilon+\varepsilon')\left(\varepsilon\left(1+\frac{\varepsilon}{\varepsilon'}\right)^{1/\varepsilon} + \Delta_{rQ}\right)$$

$$< v(p,r)\left((\varepsilon+\varepsilon')^2\left(1+\frac{\varepsilon}{\varepsilon'}\right)^{1/\varepsilon}+(\varepsilon+\varepsilon')\Delta_{pQ}-\varepsilon(1+\varepsilon)\delta_r\right). \tag{70}$$

According to (39), (53) it suffices to show inequality (70) with $\Delta_{rQ}$ and $\Delta_{pQ}$ replaced by $\varepsilon$ and $-\varepsilon'-(1+\varepsilon)\varepsilon$, respectively. In addition, $v(p,r)\geq 1$ and $\delta_r\leq 1$, which leads to

$$\begin{aligned}&(1+\varepsilon)(\varepsilon+\varepsilon')\left(\varepsilon\left(1+\frac{\varepsilon}{\varepsilon'}\right)^{1/\varepsilon}+\varepsilon\right)\\ &<(\varepsilon+\varepsilon')^2\left(1+\frac{\varepsilon}{\varepsilon'}\right)^{1/\varepsilon}-\varepsilon'(\varepsilon+\varepsilon')-\varepsilon(1+\varepsilon)(1+\varepsilon+\varepsilon').\end{aligned} \tag{71}$$

that holds for $\varepsilon<0.05$. This completes the argument for unit $p$ to be saturated at 1 after $r$ becomes unsaturated from 0.

**2.** Note that unit $a_k$ saturates at 1 according to case 1 of this lemma before $x_k$ is unsaturated from 0 at time instant $t_0$. Excitation of unit $x_k$ derived from (30) can be lower bounded as

$$\xi_{x_k}(t)\geq\varepsilon+(1+\varepsilon)y_{x_k}(t)+\Delta_{x_kQ'}\,e^{-(t-t_0)} \tag{72}$$

for $t\in\tau$ by assumption (48) because $v(p,x_k)<0$ for all $p\in C_{k-1}$ from (7). According to dynamics equation (1) this also provides the following lower bound on the state derivative of unsaturated $x_k$:

$$\frac{dy_{x_k}}{dt}(t)\geq\varepsilon y_{x_k}(t)+\varepsilon+\Delta_{x_kQ'}\,e^{-(t-t_0)}. \tag{73}$$

Moreover, in the beginning of time interval $\tau$, the state evolution of unit $x_k$ is determined by (42) before the first unit $p\in C_{k-1}$ becomes unsaturated, since assumption (40) coincides with (48) due to $\varepsilon'=\varepsilon$ and $\xi_p(t_0)\geq 1$ for all $p\in C_{k-1}$. Also the initial total weighted defect $\Delta_{x_kQ'}$ for $Q'=C\backslash(C_{k-1}\cup\{x_k\})$ can be expressed in terms of $\Delta_{x_kQ}=-\varepsilon-(1+\varepsilon)\delta_{x_k}$ for $Q=C\setminus\{x_k\}$ from (53) as follows

$$\Delta_{x_kQ'}=-\varepsilon-(1+\varepsilon)\delta_{x_k}+\sum_{p\in C_{k-1}}v(p,x_k)\delta_p \tag{74}$$

according to definition (31). Hence,

$$\Delta_{x_kQ'}\geq-\varepsilon\left(1+2^{-1/\varepsilon}\right) \tag{75}$$

by assumption (47).

By introducing inequality (75) into (73) the state derivative of unsaturated $x_k$ is further lower bounded as

$$\frac{dy_{x_k}}{dt}(t) \geq \varepsilon y_{x_k}(t) + \varepsilon - \varepsilon\left(1 + 2^{-1/\varepsilon}\right) e^{-(t-t_0)} . \tag{76}$$

Since $\varepsilon y_{x_k}(t) \geq 0$, it follows that

$$\frac{dy_{x_k}}{dt}(t) \geq \varepsilon - \varepsilon^2 > 0 \tag{77}$$

for $t \geq t_0 + t_d$ where

$$t_d = \ln \frac{1 + 2^{-1/\varepsilon}}{\varepsilon} , \tag{78}$$

provided that $x_k$ is still unsaturated. This implies that $y_{x_k}(t)$ for $t \geq t_0 + t_d$ grows at least as fast as the straight line with equation

$$\left(\varepsilon - \varepsilon^2\right)(t - t_0 - t_d) - y = 0 \tag{79}$$

until unit $x_k$ saturates at 1. Hence, $x_k$ saturates at 1 certainly before $t_0 + t_d + t_s < t_0 + 2t_1$ where

$$t_s = \frac{1}{\varepsilon - \varepsilon^2} \tag{80}$$

because $\xi_{x_k}(t) > y_{x_k}(t)$ from equation (1) due to $p$'s state derivative (77) is positive for $t \geq t_0 + t_d$.

In addition, it will be proved that the subsequent unit $b_k$ is saturated at 0 at least until $x_k$ saturates at 1. Excitation

$$\xi_{b_k}(t) = -1 + \frac{\varepsilon}{3} + y_{x_k}(t) + \Delta_{b_k Q'}\, e^{-(t-t_0)} \tag{81}$$

of unit $b_k$ saturated at 0 can be derived from (30) by assumption (49). Let $t_y > 0$ be the least local time instant at which

$$y_{x_k}(t_0 + t_y) = 1 - \frac{\varepsilon}{3} - \Delta_{b_k Q'}\, e^{-t_y} \tag{82}$$

when $b_k$ is still saturated at 0 since $\xi_{b_k}(t_0 + t_y) = 0$ follows from (81). Thus, it suffices to prove that the excitation of unit $x_k$ can be lower bounded at time instant $t_0 + t_y$ as

$$\xi_{x_k}(t_0 + t_y) \geq \varepsilon + (1+\varepsilon)\left(1 - \frac{\varepsilon}{3} - \Delta_{b_kQ'}\, e^{-t_y}\right) + \Delta_{x_kQ'}\, e^{-t_y} \geq 1 \tag{83}$$

according to (72). By substituting the error bounds (46), (75) this reduces to

$$5 - \varepsilon - 3\left(1 + (2+\varepsilon)2^{-1/\varepsilon}\right) e^{-t_y} \geq 0 \tag{84}$$

which holds for $\varepsilon < 0.05$ due to $e^{-t_y} < 1$, ensuring that unit $x_k$ is already saturated at 1 at time instant $t_0 + t_y$.

Finally, it must be checked that unit $x_k$ remains saturated at 1 after $t_0 + t_y$ when unit $b_k$ may become unsaturated from 0. Inequality (72) reads now as

$$\begin{aligned}\xi_{x_k}(t) \geq \varepsilon + (1+\varepsilon)&\left(1 - \left(\frac{\varepsilon}{3} + \Delta_{b_kQ'}\, e^{-t_y}\right) e^{-(t-t_0-t_y)}\right) + y_{b_k}(t)\\ &+ (\Delta_{x_kQ'} - \delta_{b_k})\, e^{-(t-t_0)}\end{aligned} \tag{85}$$

since the state dynamics of unit $x_k$ saturated at 1 is controlled by equation (29). In order to prove that $\xi_{x_k}(t) \geq 1$ for all $t \in (t_0 + t_y, t_f]$ it suffice to show

$$6 - (1+\varepsilon)e^{-(t-t_0-t_y)} - 3\left(1 + (3+\varepsilon)2^{-1/\varepsilon}\right) e^{-(t-t_0)} \geq 0 \tag{86}$$

according to inequality (85) in which $y_{b_k}(t) \geq 0$ and the error bounds (45),(46), and (75) have been applied. Inequality (86) follows from $e^{-(t-t_0)} \leq e^{-(t-t_0-t_y)} \leq 1$ for $\varepsilon < 0.05$. This completes the argument for unit $x_k$ to be saturated at 1 after $b_k$ becomes unsaturated from 0. □

The correct *timing* of the counter simulation still needs to be verified to ensure a sufficiently fast decrease in the defects of the continuous-time correlates of binary states, because the analysis in Lemma 4 is valid only if the defect bounds (39), (45)–(47) are satisfied. According to (38), the absolute value of the total weighted defect of saturated units affecting any unit in $C$ is bounded by $\varepsilon$ after transient time $t_1$, decreasing further to $\varepsilon 2^{-1/\varepsilon}$ by time $2t_1$. On the other hand, $t_1$ lower bounds the time necessary for activating a typical unit in $p \in C$ (see table in Fig. 2) by Lemma 4.1.

In order to validate assumption (39), consider e.g. a unit $o'$ that has also been activated according to Lemma 4.1 last before unit $p$ from (39) starts its activation. Clearly, unit $o'$ coincides with unit $o$ in table of Fig. 2 except for $o = x_k$ whose activation is analyzed in Lemma 4.2 instead, and therefore

$$o' = \begin{cases} a_k & \text{for } p = b_k\,, \quad 1 \leq k \leq n \\ o & \text{otherwise}\,. \end{cases} \tag{87}$$

It follows from Fig. 1 that $v(o', r) \geq 0$. In fact only $v(z_1, c_k) = 1$ and $v(z_k, c_1) > 0$ for $k > 1$ are positive while the remaining pairs $o'$, $r$ are actually not connected corresponding to $v(o', r) = 0$. In addition, $v(p, r) \geq 1$ and hence the defect in (39) can be upper bounded as follows:

$$\delta_p + \Delta_{rQ} \leq v(p, r)\delta_p + \Delta_{rQ} + v(o', r)\delta_{o'} = \Delta_{rQ_2} \tag{88}$$

where $Q_2 = C \setminus \{o'\}$ according to definition (31).

For $o' \neq b_k$ all the units in $Q_2$ are saturated when $o'$ is being activated which takes time $t_1$. On the other hand, the activation of $o' = b_k$ ($p = d_k$) takes time $2t_1$ due to its bias $v(0, b_k) = -1 + \varepsilon/3$, while simultaneously units in $C_{k-1} \subseteq Q_2$ saturate at 0 within a time period of length $t_1$ by Lemma 3.2b and then all the units in $Q_2$ are saturated for the duration of the next $t_1$-period certainly before unit $d_k$ unsaturates. Hence, $\Delta_{rQ_2} < \varepsilon$ from (38) which implies (39) according to (88).

Analogously, the stronger defect bounds (45)–(47) in Lemma 4.2 are met since according to Lemma 4.1 the transient time $2t_1$ that is sufficient to decrease the underlying defects due to (38), is guaranteed by the successive separate (all the units are saturated but one) activations of the preceding two units $c_k, a_k$ with $v(c_k, b_k) = v(a_k, b_k) = 0$ before unit $x_k$ unsaturates from 0.

Thus, unit $c_0$ representing the least significant counter bit activates altogether $2^n$ times before the Hopfield net $C$ converges and each such an activation takes time at least $t_1$ according to Lemma 4.1 which provides the lower bound $\Omega(2^n/\varepsilon) = \Omega(2^{m/6}/\varepsilon)$ on the total simulation time. From the proof of Lemma 2, the maximum integer weight parameter in $C$ is of order $2^{O(m)}$. This corresponds to $O(m)$ bits per weight that is repeated $O(m^2)$ times, and thus yields at most $O(m^3)$ bits in the representation. In addition, the biases and feedbacks of the $m$ units include fraction $\varepsilon$ (or $\varepsilon/3$), and taking this into account requires $\Theta(m \log(1/\varepsilon))$ additional bits, say at least $\kappa m \log(1/\varepsilon)$ bits for some constant $\kappa > 0$. By choosing $\varepsilon = 2^{-f(m)/(\kappa m)}$ in which $f$ is a continuous increasing function whose inverse is defined as $f^{-1}(\mu) = \mu/g(\mu)$, where $g$ satisfies $g(\mu) = \Omega(\mu^{2/3})$ (implying $f(m) = \Omega(m^3)$) and $g(\mu) = o(\mu)$, it follows that $M = \Theta(f(m))$, especially $M \geq f(m)$ from $M \geq \kappa m \log(1/\varepsilon)$. The convergence time $\Omega(2^{m/6}/\varepsilon)$ can be translated to $\Omega(2^{f(m)/(\kappa m)+m/6}) = 2^{\Omega(f(m)/m)}$ which can be rewritten as $2^{\Omega(M/f^{-1}(M))} = 2^{\Omega(g(M))}$ since $f(m) = \Omega(M)$ from

$M = \Theta(f(m))$ and $f^{-1}(M) \geq m$ from $M \geq f(m)$. This completes the proof of the theorem. $\square$

## 5 A Simulation Example

A computer program HCOUNT has been created to automate the construction from Theorem 1. For input $n \geq 0$, the program generates continuous-time Hopfield system (1) in the form of a FORTRAN subroutine corresponding to the $(n+1)$-bit binary counter to be simulated. This FORTRAN procedure is then presented to a solver from the NAG library [10] that provides a numerical solution for the system. For example, implementing a 4-bit counter on the HCOUNT generator results in a continuous-time symmetric Hopfield system $C_3$ with 19 variables. Fig. 3 shows the evolution of the states of counter units $c_0, c_1, c_2, c_3$ for a period of $2^3 - 1 = 7$ simulated discrete steps confirming the correctness of the construction. A parameter value of $\varepsilon = 0.1$ was used in this numerical simulation, showing that the theoretical estimate of $\varepsilon$ in Theorem 1 is actually quite conservative.

## 6 Conclusions and Open Problems

We have constructed a continuous-time Hopfield net simulating a binary counter whose convergence time is provably exponential in terms of the network size. To our best knowledge this provides the first known example of a continuous-time Liapunov dynamical system with an exponential transient period. The result has also negative consequences in applying the continuous-time Hopfield nets as fast heuristic analog solvers to combinatorial optimization problems since the computational time needed may in the worst case be exponential. On the other hand, it is unknown whether a matching upper bound on the convergence time of continuous-time Hopfield nets can be proved.

Also the preceding convergence time lower bound for continuous-time Hopfield nets exceeds a general upper bound on the convergence time of discrete-time symmetric networks [15] of the same representation size. This suggests that continuous-time analog models of computation may gain in descriptional efficiency.

Finally, the presented exponential-transient oscillator was used in [13] to prove that continuous-time Hopfield nets are computationally Turing universal. However, this technique is somewhat unsatisfying, since it is based on discretizing the continuous-time computation. It would be most interesting to develop

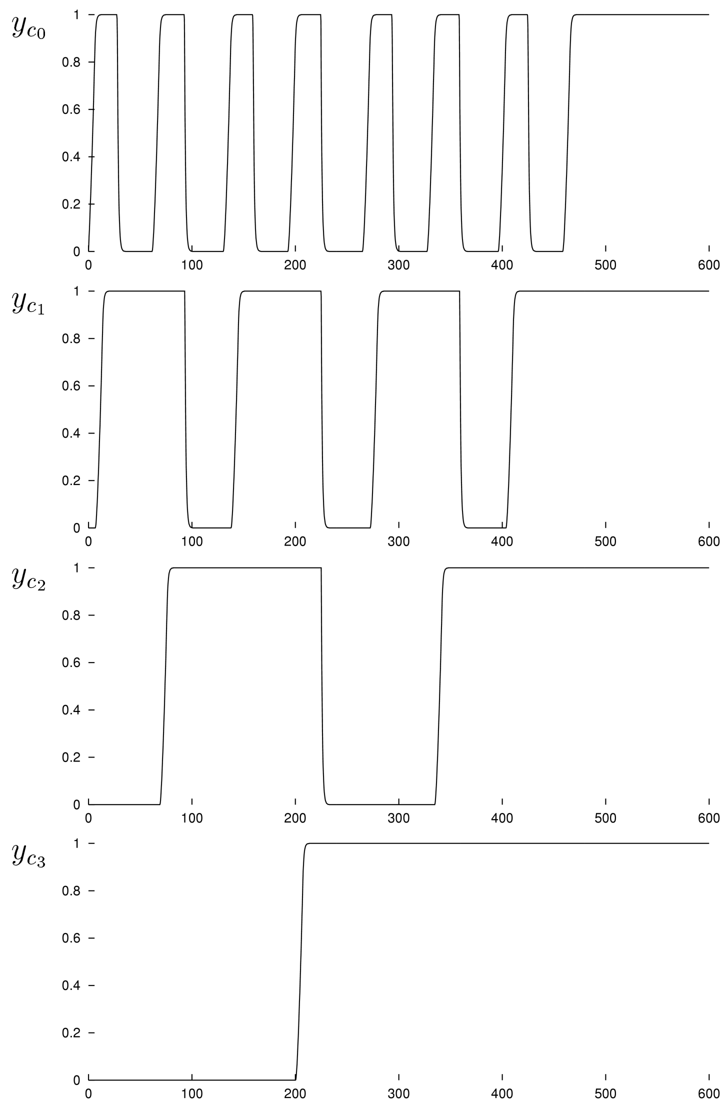



Fig. 3. Continuous-time simulation of 4-bit binary counter for $\varepsilon = 0.1$.

some theoretical tools (e.g. complexity measures, reductions, universal computation) for "naturally" continuous-time computations that exclude the use of discretizing oscillations. First steps along this direction have recently been established [1,5].

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