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Chomsky-Like Neural Network Hierarchy

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References

This lecture surveys selected computability results of the project

FoNeCo: Analytical Foundations of Neurocomputing

(Czech Science Foundation, GA22-02067S, 2019-2021),

published in the following papers (two won the Best ICS Paper Award):

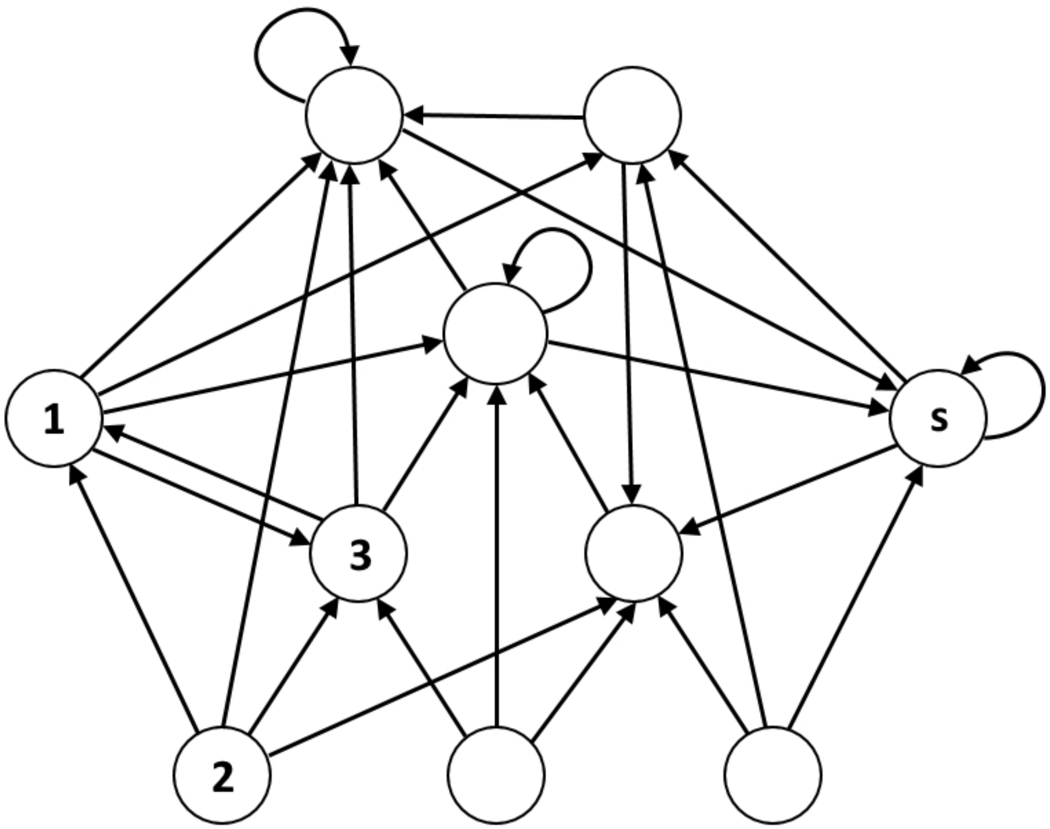
- J. Šíma: Subrecursive neural networks. *Neural Networks* 116:208-223, 2019.
- J. Šíma: Analog neuron hierarchy. *Neural Networks* 128:199-218, 2020.
- J. Šíma, P. Savický: Quasi-periodic β-expansions and cut languages. *Theoretical Computer Science* 720:1-23, 2018.
- P. Jančar, J. Šíma: The simplest non-regular deterministic context-free language. *Proceedings of the MFCS 2021*, LIPIcs 202, pp. 63:1-63:18, Dagstuhl, 2021.
- J. Šíma: Stronger separation of analog neuron hierarchy by deterministic context-free languages. *Neurocomputing* 493:605-612, 2022.

Outline of Talk

1. The Neural Network Model
2. The Computational Power of Neural Networks
3. A Chomsky-Like Neural Network Hierarchy
4. Periodic Numbers in Positional Systems with Non-Integer Base
5. $\mathcal{C}$-Simple Problems

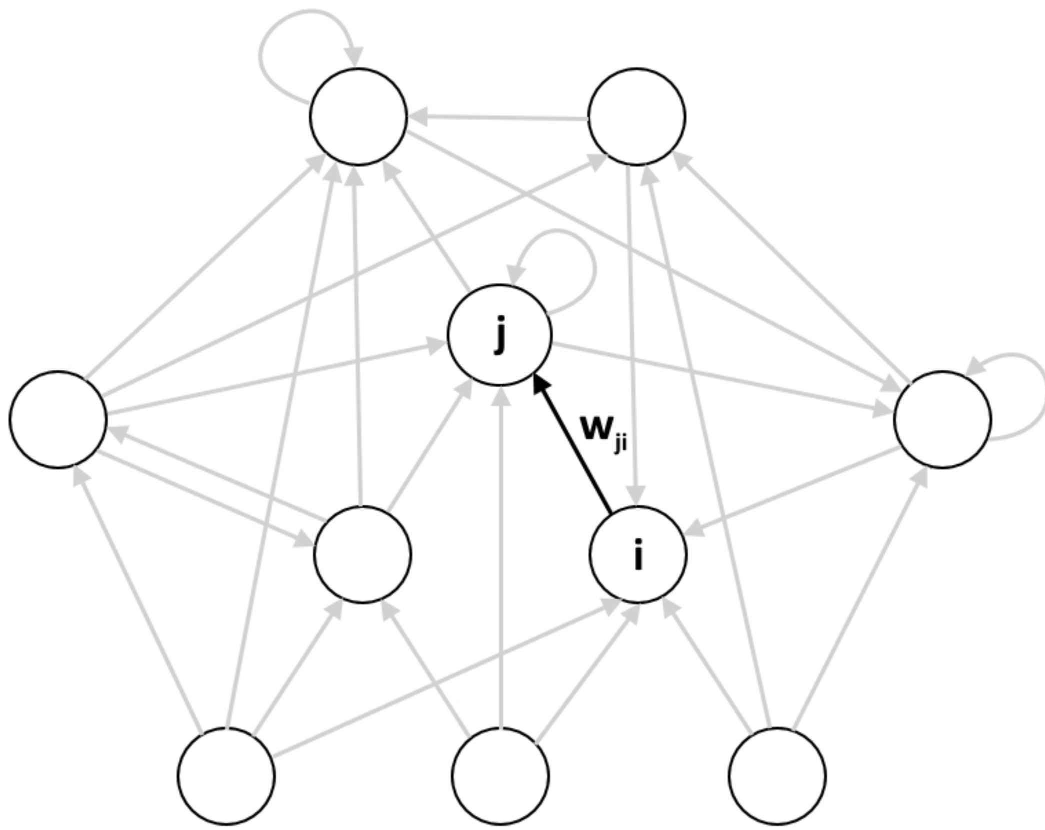
The Neural Network Model – Architecture

s computational units (neurons), indexed as $V = \{1, \ldots, s\}$, connected into a directed graph (V, E) where $E \subseteq V \times V$



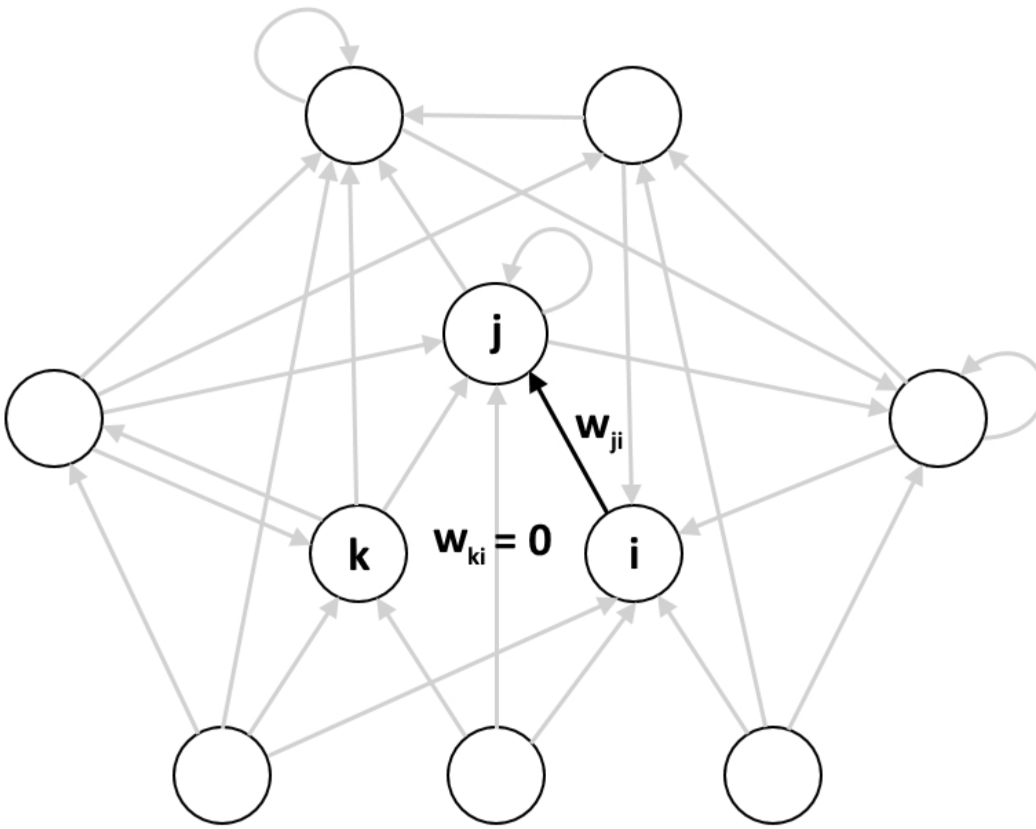
The Neural Network Model – Weights

each edge $(i, j) \in E$ from unit i to j is labeled with a real weight $w_{ji} \in \mathbb{R}$



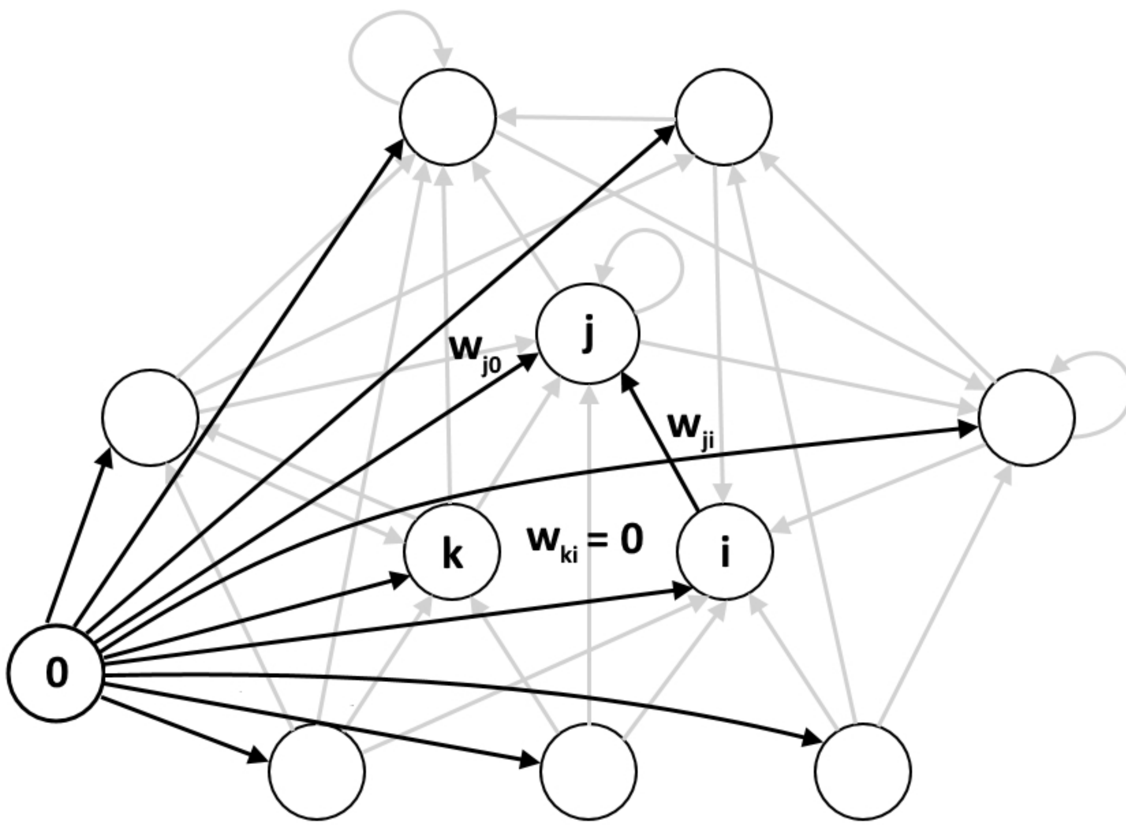
The Neural Network Model – Zero Weights

each edge $(i, j) \in E$ from unit i to j is labeled with a real weight $w_{ji} \in \mathbb{R}$ $(w_{ki} = 0$ iff $(i, k) \notin E)$



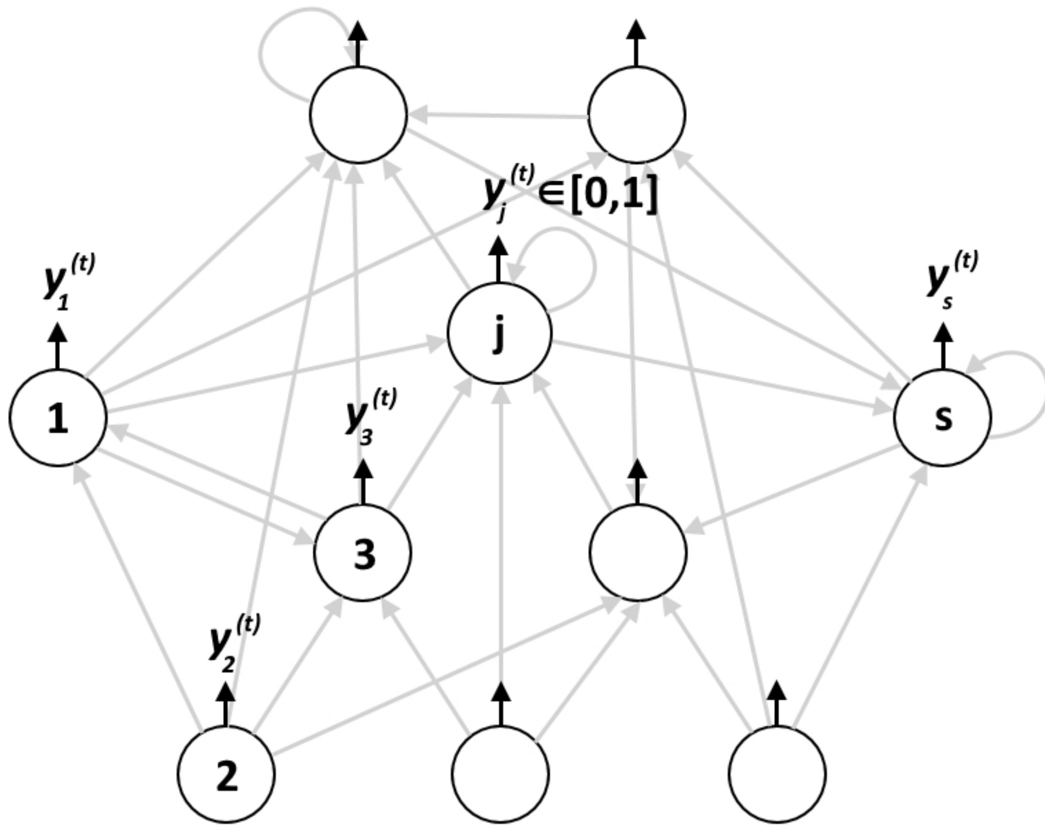
The Neural Network Model – Biases

each neuron $\boldsymbol{j} \in \boldsymbol{V}$ is associated with a real bias $\boldsymbol{w_{j0}} \in \mathbb{R}$
(i.e. a weight of $(\boldsymbol{0}, \boldsymbol{j}) \in \boldsymbol{E}$ from an additional formal neuron $\boldsymbol{0} \in \boldsymbol{V}$)



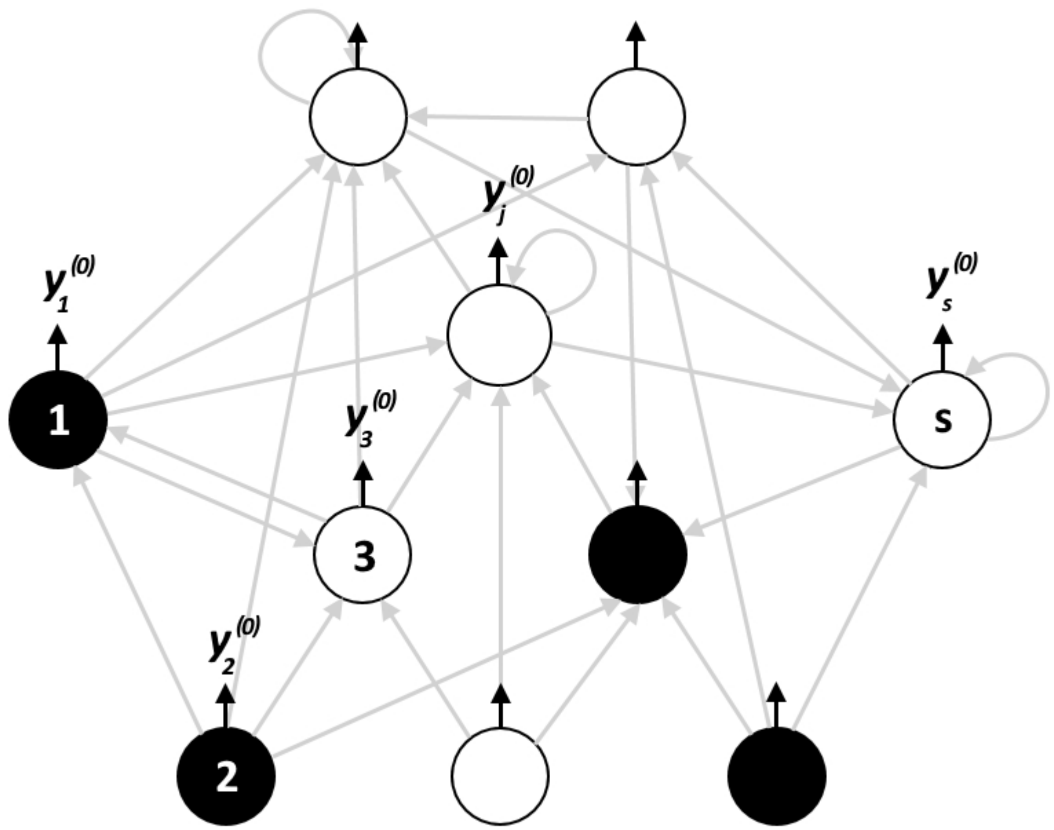
Discrete-Time Computational Dynamics – Network State

the evolution of global network state (output) $\mathbf{y}^{(t)} = (y_1^{(t)}, \ldots, y_s^{(t)}) \in [0,1]^s$
at discrete time instant $t = 0, 1, 2, \ldots$



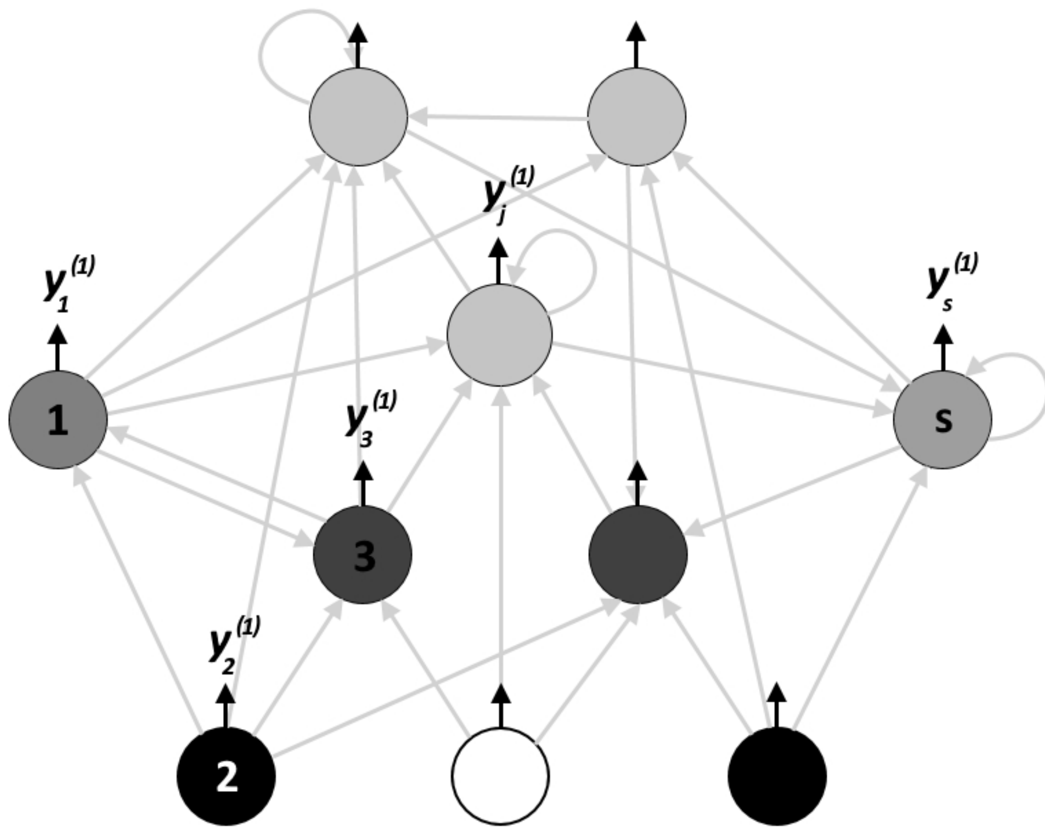
Discrete-Time Computational Dynamics – Initial State

$t = 0$: initial network state $\mathbf{y}^{(0)} \in \{0,1\}^s$



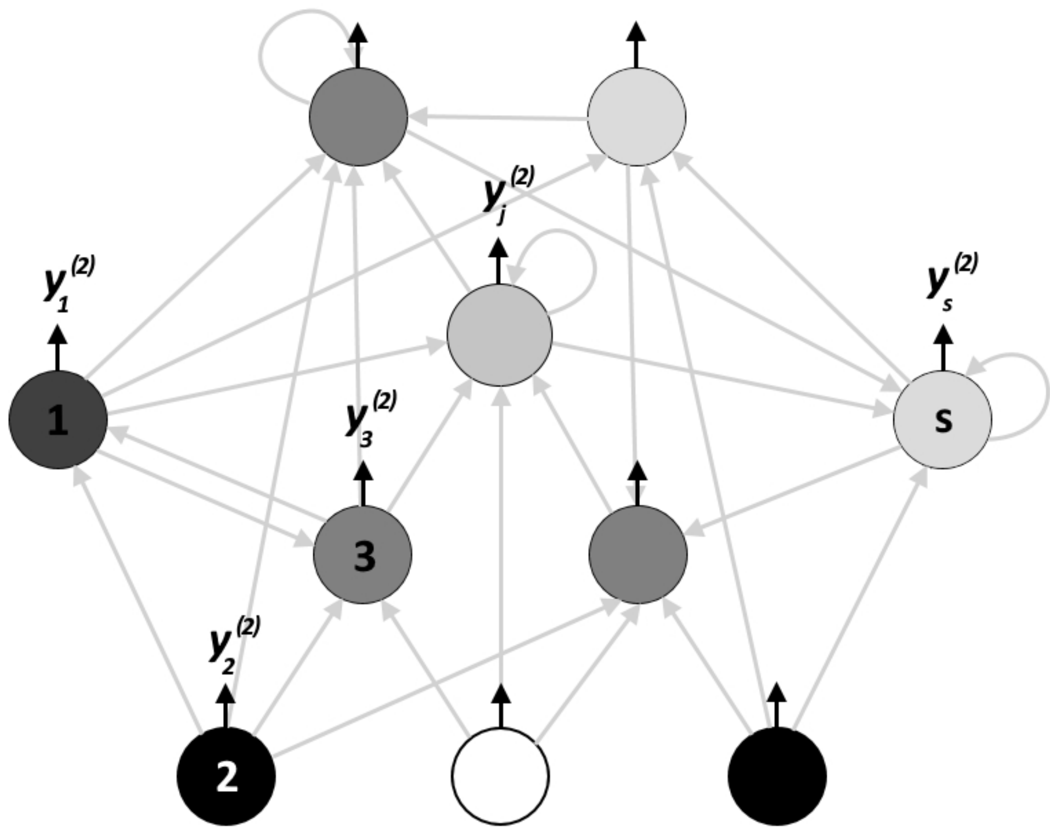
Discrete-Time Computational Dynamics: $t = 1$

$t = 1$: network state $\mathbf{y}^{(1)} \in [\mathbf{0}, \mathbf{1}]^s$



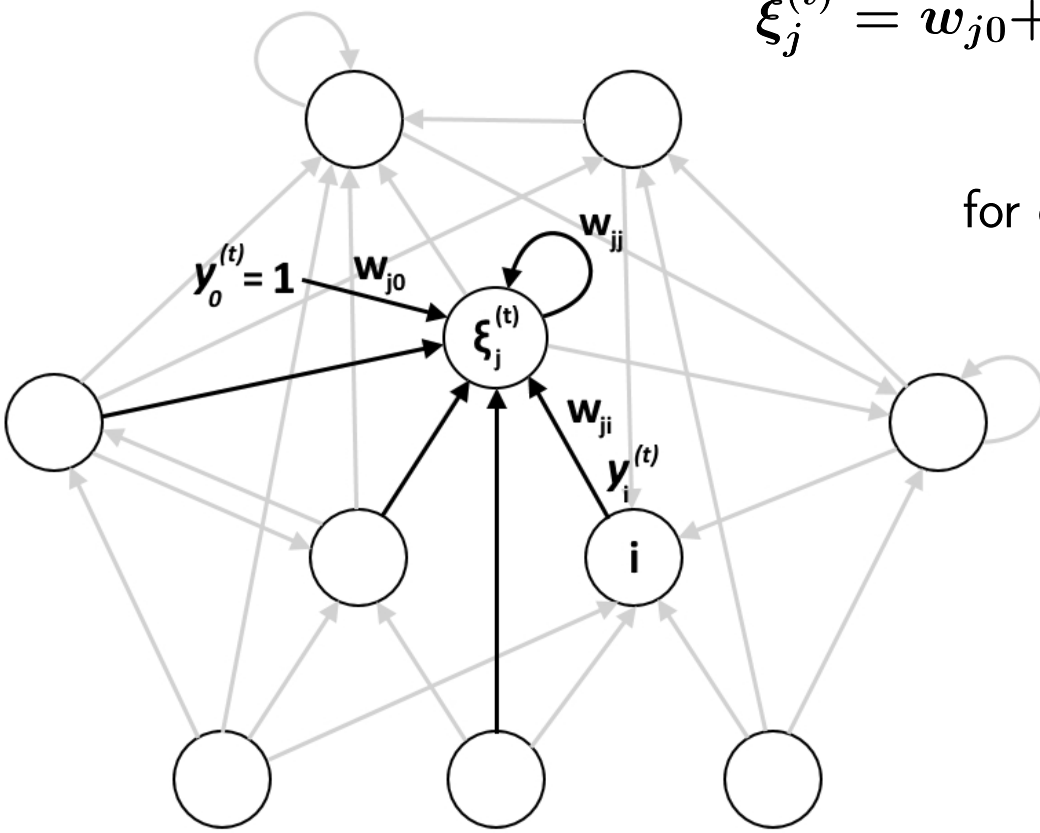
Discrete-Time Computational Dynamics: $t = 2$

$t = 2:$ network state $\mathbf{y}^{(2)} \in [0, 1]^s$



Discrete-Time Computational Dynamics – Excitations

at discrete time instant $t \geq 0$, an excitation is computed as



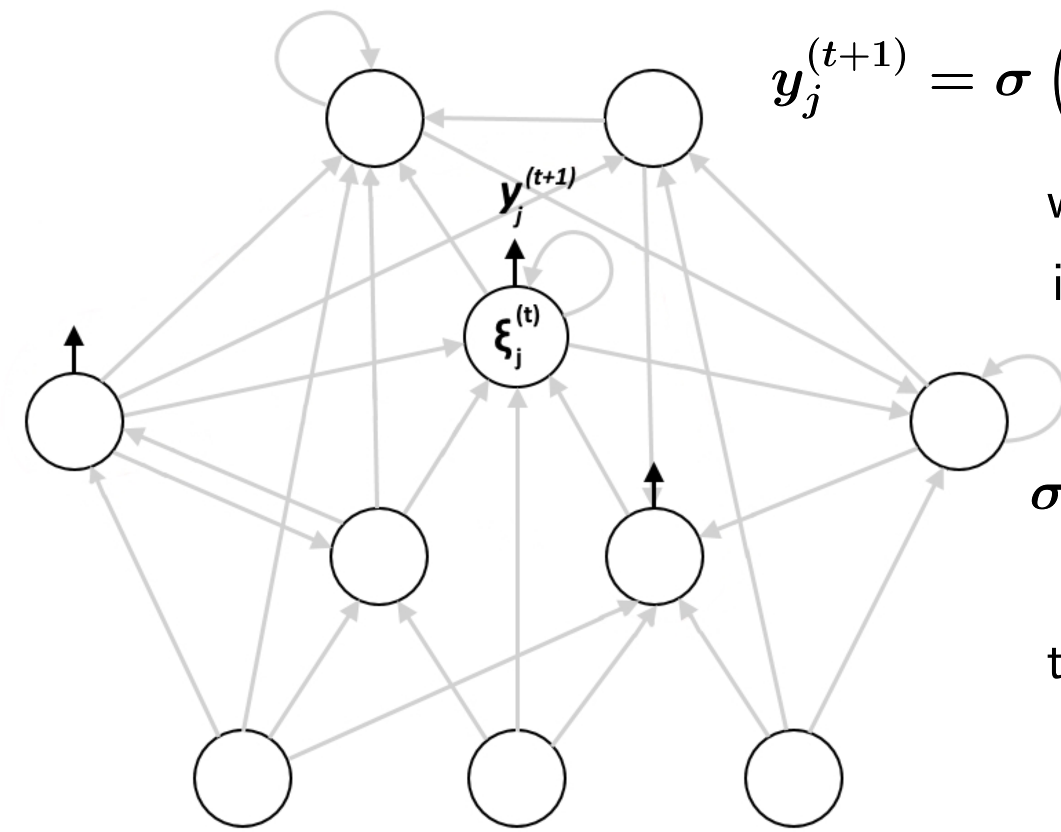
$$\xi_j^{(t)} = w_{j0} + \sum_{i=1}^{s} w_{ji} y_i^{(t)} = \sum_{i=0}^{s} w_{ji} y_i^{(t)}$$

for every $j \in \{1, \ldots, s\}$

where unit $0 \in V$ has constant output $y_0^{(t)} \equiv 1$ for every $t \geq 0$

Discrete-Time Computational Dynamics – Outputs

at the next time instant $t+1$, every neuron $j \in V$ updates its state in parallel (a so-called fully parallel mode):



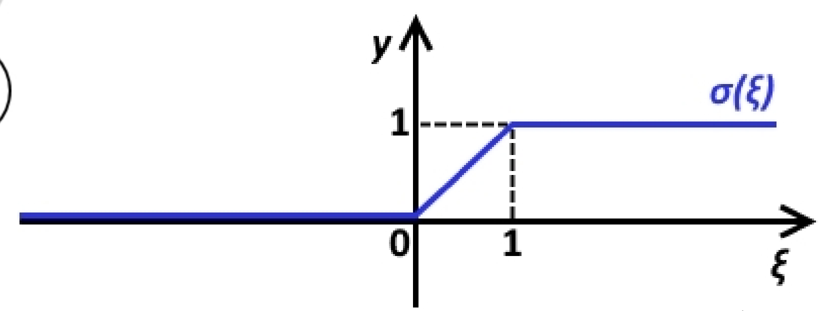
$$y_j^{(t+1)} = \sigma\left(\xi_j^{(t)}\right) \quad \text{for every } j = 1, \ldots, s$$

where $\sigma : \mathbb{R} \longrightarrow [0,1]$

is an activation function, e.g.

$$\sigma(\xi) = \begin{cases} 1 & \text{for } \xi \geq 1 \\ \xi & \text{for } 0 < \xi < 1 \\ 0 & \text{for } \xi \leq 0 \end{cases}$$

the saturated-linear function



The Computational Power of NNs – Motivations

- the potential and limits of general-purpose computation with NNs:

  What is ultimately or efficiently computable by particular NN models?

- idealized mathematical models of practical NNs which abstract away from implementation issues, e.g. analog numerical parameters are true real numbers

- methodology: the computational power and efficiency of NNs is investigated by comparing formal NNs to traditional computational models such as finite automata, Turing machines, Boolean circuits, etc.

- NNs may serve as reference models for analyzing alternative computational resources (other than time or memory space) such as analog state, continuous time, energy, temporal coding, etc.

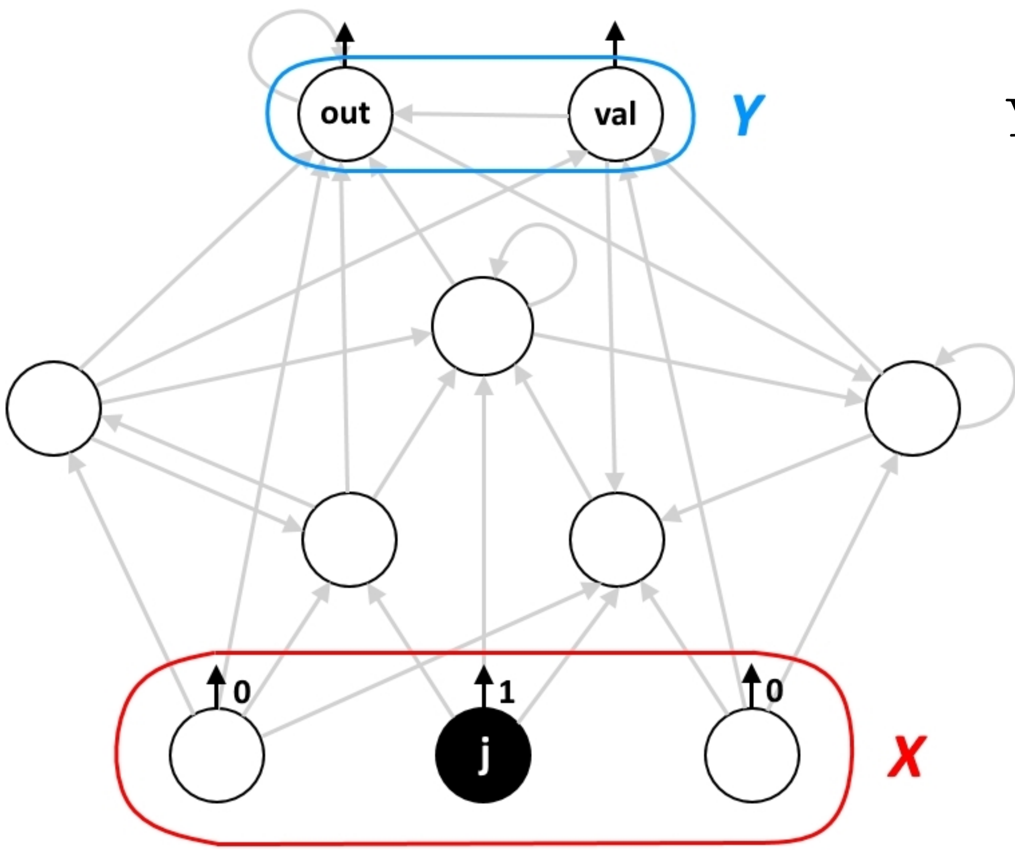
- NNs capture basic characteristics of biological nervous systems (plenty of densely interconnected simple unreliable computational units)

  $\longrightarrow$ computational principles of mental processes

Neural Networks As Formal Language Acceptors

a language $L \subseteq \Sigma^*$ over finite alphabet Σ represents a decision problem

$$y_{\text{out}}^{(T(n))} = \begin{cases} 1 & \text{if } \mathbf{x} \in L \\ 0 & \text{if } \mathbf{x} \notin L \end{cases} \qquad y_{\text{val}}^{(t)} = \begin{cases} 1 & \text{if } t = T(n) \\ 0 & \text{if } t \neq T(n) \end{cases}$$



$Y = \{\text{out}, \text{val}\}$ output neurons

$T(n)$ is the computation time in terms of input length $n \geq 0$

$d \geq 1$ is the time overhead for processing a single input symbol

$X = \text{enum}(\Sigma)$ input neurons
one-hot encoding

$$\uparrow \; y_j^{(d(i-1)+k)} = 1 \ \text{ iff } \ j = \text{enum}(x_i)$$

$$\mathbf{x} = x_1 \ldots x_{i-1} \longleftarrow x_i \longleftarrow x_{i+1} \ldots x_n \in \Sigma^*$$ input word

The Computational Power of NNs – Integer Weights

depends on the information content of weight parameters:

1. **integer** weights: finite automaton (FA) (Minsky, 1967)

$$\boldsymbol{w_{ji}} \in \mathbb{Z} \quad \longrightarrow \quad \text{excitations} \quad \boldsymbol{\xi_j} \in \mathbb{Z} \quad \longrightarrow \quad \text{states} \quad \boldsymbol{y_j} \in \{0, 1\}$$

$$\longrightarrow \quad \boldsymbol{2^s} \text{ global NN states } \mathbf{y} \in \{0,1\}^s \quad \sim \quad \text{FA states}$$

size-optimal implementations:

- $\boldsymbol{\Theta}\left(\sqrt{\boldsymbol{m}}\right)$ neurons for a deterministic FA with $\boldsymbol{m}$ states (Indyk, 1995; Horne, Hush, 1995)

- $\boldsymbol{\Theta}(\boldsymbol{m})$ neurons for a regular expression of length $\boldsymbol{m}$ (Šíma, Wiedermann 1998)

The Computational Power of NNs – Rational Weights

depends on the information content of weight parameters:

2. **rational** weights: Turing machine (Siegelmann, Sontag, 1995)

- $\boldsymbol{w_{ji}} \in \mathbb{Q}$ are fractions $\frac{\boldsymbol{p}}{\boldsymbol{q}}$ where $\boldsymbol{p} \in \mathbb{Z}$, $\boldsymbol{q} \in \mathbb{N}$
- NNs compute algorithmically solvable problems
- real-time simulation of TMs $\longrightarrow$ polynomial time $\equiv$ complexity class P
- a universal NN with 25 neurons (Indyk, 1995)

  $\longrightarrow$ the halting problem of whether a small NN terminates its computation, is algorithmically undecidable

The Computational Power of NNs – Real Weights

depends on the information content of weight parameters:

3. arbitrary **real** weights: "super-Turing" computation (Siegelmann, Sontag, 1994)

- $w_{ji} \in \mathbb{R}$, e.g. irrational weights $\sqrt{2}$, π
- infinite precision of **ONE** real weight (vs. an algorithm has a finite description)
  can encode any function f: $\mathbf{0\,.\,code(}C_1\mathbf{)\,code(}C_2\mathbf{)\,code(}C_3\mathbf{)\ldots}$
  (code(C_n) encodes the circuit C_n computing f for inputs of length n)

  $\longrightarrow$ exponential time $\equiv$ any I/O mapping
  (including algorithmically undecidable problems)

- polynomial time $\equiv$ nonuniform complexity class P/poly:

  problems solvable by a polynomial-time (P) algorithm that for input $x \in \Sigma^*$ of length $n = |x|$, receives an external advise: a string $s(n) \in \Sigma^*$ of polynomial length $|s(n)| = O(n^c)$ (poly), which depends only on n

The Computational Power of NNs – A Summary

depends on the information content of weight parameters:

1. **integer** weights: finite automaton
2. **rational** weights: Turing machine
   polynomial time $\equiv$ complexity class P
3. arbitrary **real** weights: "super-Turing" computation
   polynomial time $\equiv$ nonuniform complexity class P/poly
   exponential time $\equiv$ any I/O mapping

Neural Networks Between Rational and Real Weights

1. **integer** weights: finite automaton

2. **rational** weights: Turing machine

   polynomial time $\equiv$ **P**

   polynomial time & increasing **Kolmogorov complexity** of real weights:

   the length of the shortest program (in a fixed programming language) that produces a real weight,

   e.g. $K\left(\text{"}\sqrt{2}\text{"}\right) = O(1)$, $K(\text{"random strings"}) = n + O(1)$

   $\equiv$ a proper hierarchy of nonuniform complexity classes **between P and P/poly**

   (Balcázar, Gavaldà, Siegelmann, 1997)

3. arbitrary **real** weights: "super-Turing" computation

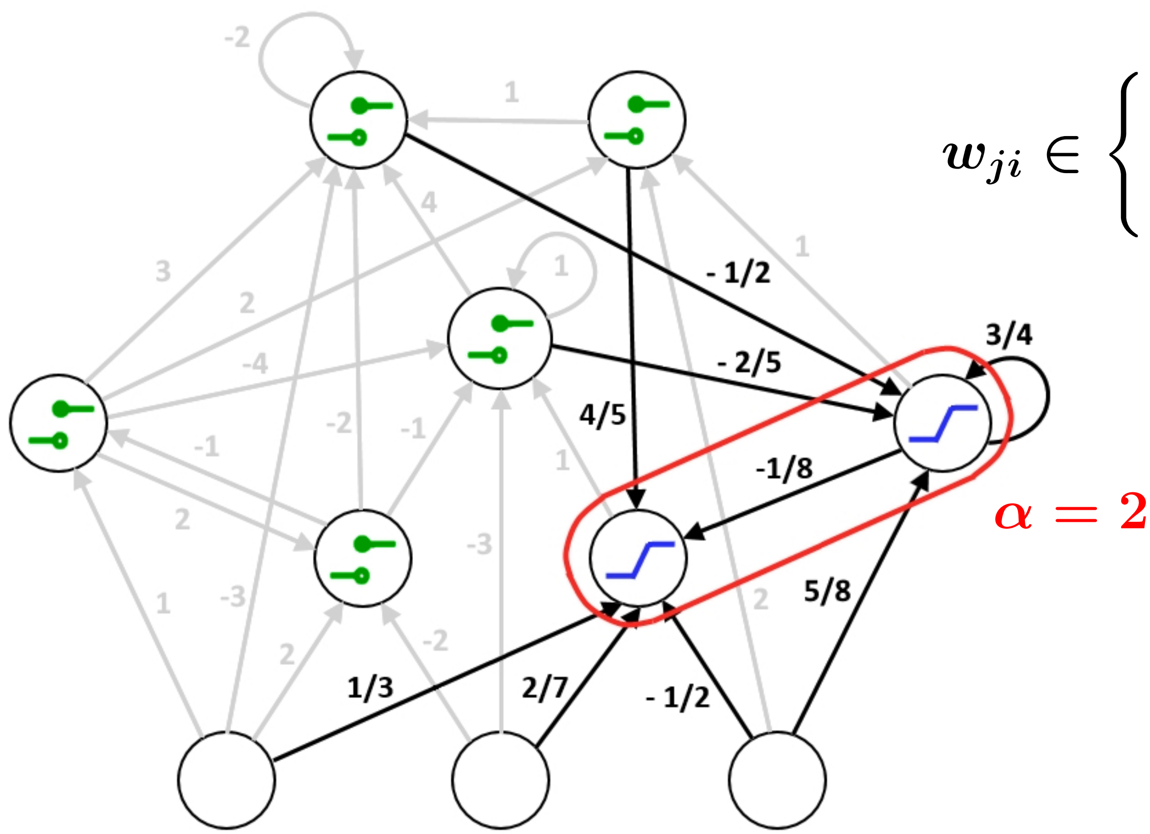
   polynomial time $\equiv$ **P/poly**

Neural Networks Between Integer and Rational Weights

from **integer** to **rational** weights

α**ANN** $=$ a **binary-state** NN with **integer** weights

$+\ \alpha$ **extra analog-state** neurons with **rational** weights



$$w_{ji} \in \begin{cases} \mathbb{Q} & j = 1, \ldots, \alpha \\ \mathbb{Z} & j = \alpha + 1, \ldots, s \end{cases}$$

$$i \in \{0, \ldots, s\}$$

Neural Networks with Increasing Analogicity

from **binary** $(\{0,1\})$ to **analog** $([0,1])$ states of neurons

α**ANN** $=$ a **binary-state** NN with **integer** weights

$+\ \alpha$ **extra** **analog-state** neurons with **rational** weights

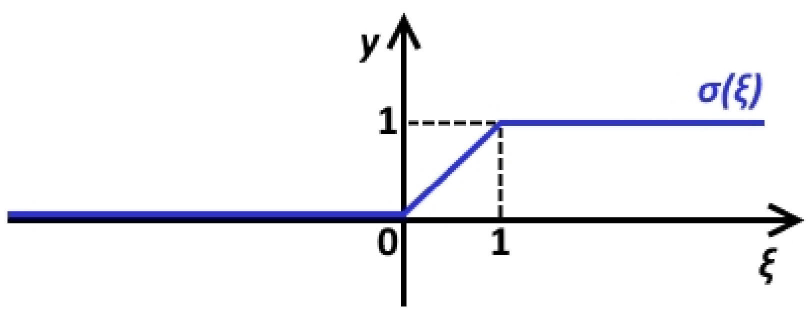
$$y_j^{(t+1)} = \sigma_j\left(\sum_{i=0}^{s} w_{ji} y_i^{(t)}\right) \qquad j = 1,\ldots,s$$

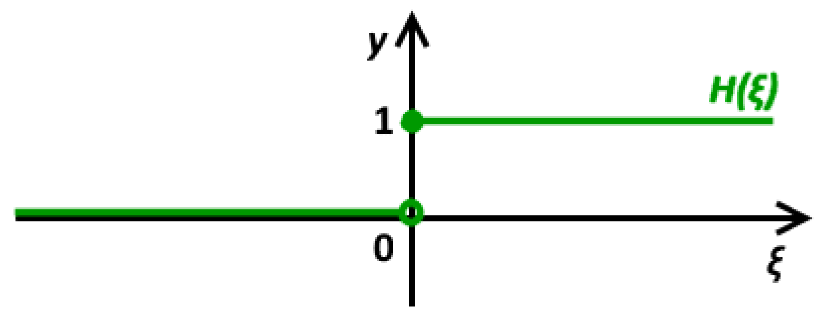
updating the states of neurons

$$\sigma_j(\xi) = \begin{cases} \sigma(\xi) = \begin{cases} 1 & \text{for } \xi \geq 1 \\ \xi & \text{for } 0 < \xi < 1 \\ 0 & \text{for } \xi \leq 0 \end{cases} & j = 1,\ldots,\alpha \\ \\ H(\xi) = \begin{cases} 1 & \text{for } \xi \geq 0 \\ 0 & \text{for } \xi < 0 \end{cases} & j = \alpha+1,\ldots,s \end{cases}$$

saturated-linear function (for $j = 1,\ldots,\alpha$)

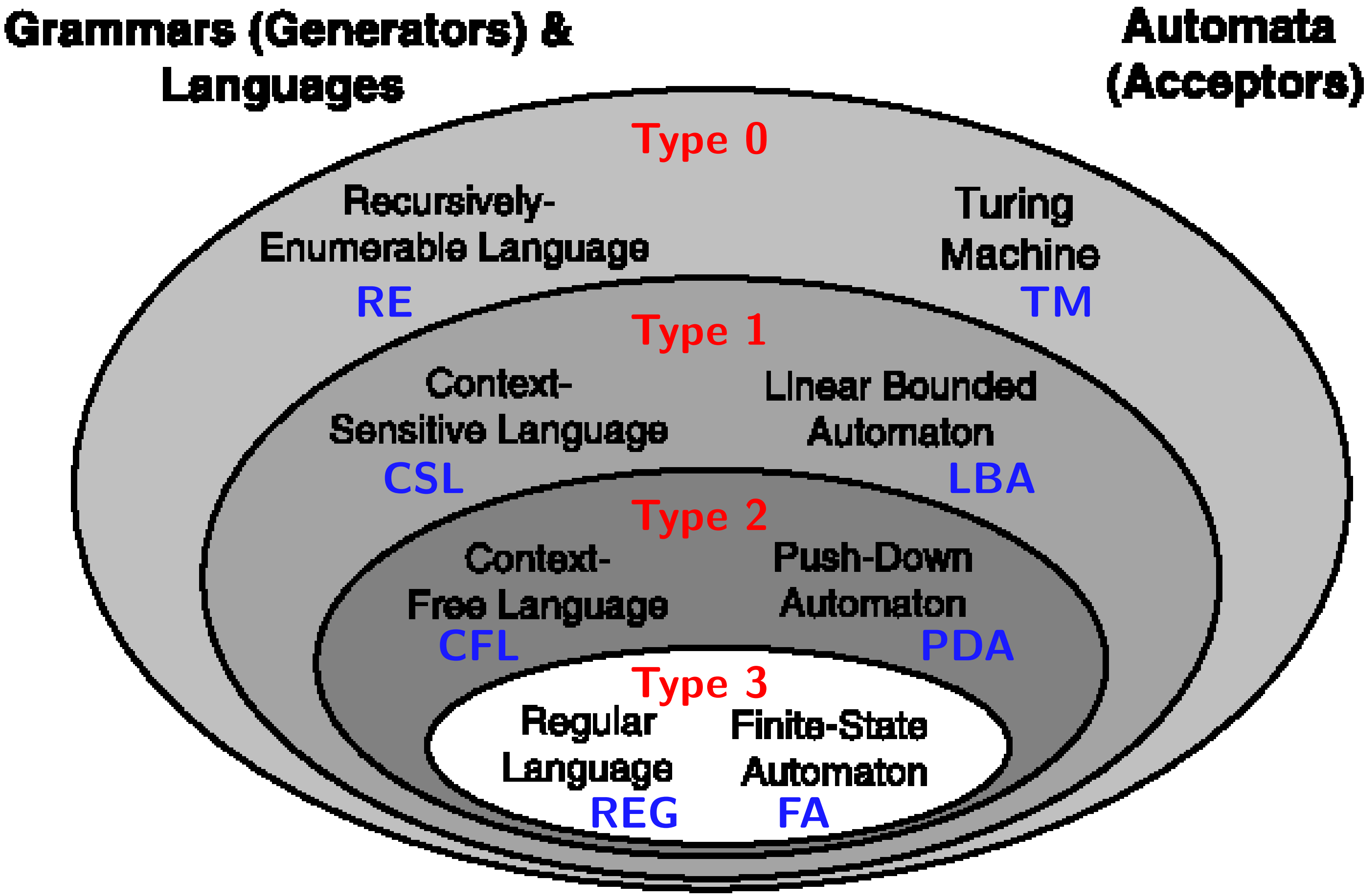
Heaviside function (for $j = \alpha+1,\ldots,s$)





The Chomsky Formal Language Hierarchy

from finite automata to Turing machines



The Analog Neuron Hierarchy (ANH)

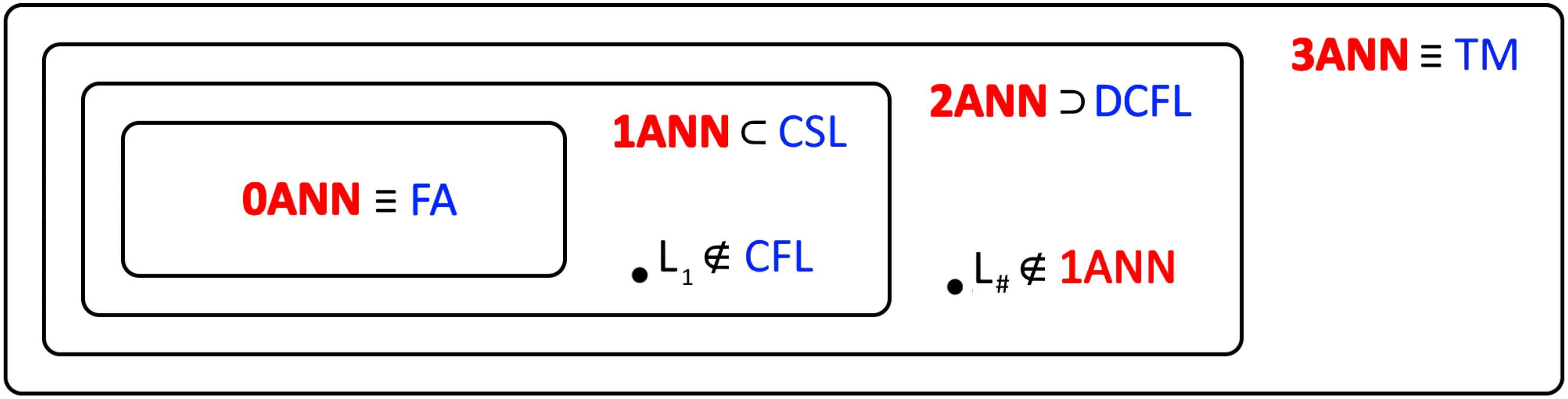
the computational power of α**ANNs**

increases with the number α of extra analog-state neurons:

$$
\begin{array}{ccccccccccccccc}
 & & \textbf{integer}\text{ weights} & & & & & & & & & & \textbf{rational}\text{ weights} & & \\
 & & \downarrow & & & & & & & & & & \downarrow & & \\
\textbf{FA} & \equiv & \textbf{REG} & = & \textbf{0ANN} & \subseteq & \textbf{1ANN} & \subseteq & \textbf{2ANN} & \subseteq & \textbf{3ANN} & \subseteq \ldots & = & \textbf{RE} & \equiv \ \textbf{TM} \\
 & & \uparrow & & & & & \times & & & & & & \uparrow & \\
 & & \textbf{Type 3} & & & & & \textbf{Chomsky hierarchy} & & & & & & \textbf{Type 0} & \\
 & & & & & & & \textbf{Type 1, 2} \ \ \textbf{?} & & & & & & &
\end{array}
$$

(the notation α**ANN** is also used for the class of languages accepted by αANNs)

The Analog Neuron Hierarchy as a Chomsky-Like NN Hierarchy



the **separation** of the first two levels **0ANN** $\overset{L_1}{\subsetneqq}$ **1ANN** $\overset{L_\#}{\subsetneqq}$ **2ANN**:

- LBA simulates 1ANN: **1ANN** $\subset$ **CSL** (Type 1)
- 1ANN accepts a non-CFL L_1: **1ANN** $\not\subset$ **CFL** (Type 2)

  $$L_1 = \left\{ x_1 \ldots x_n \in \{0,1\}^* \,\middle|\, \sum_{k=1}^{n} x_{n-k+1} \left(\tfrac{3}{2}\right)^{-k} < 1 \right\} \in \text{1ANN} \setminus \text{CFL}$$

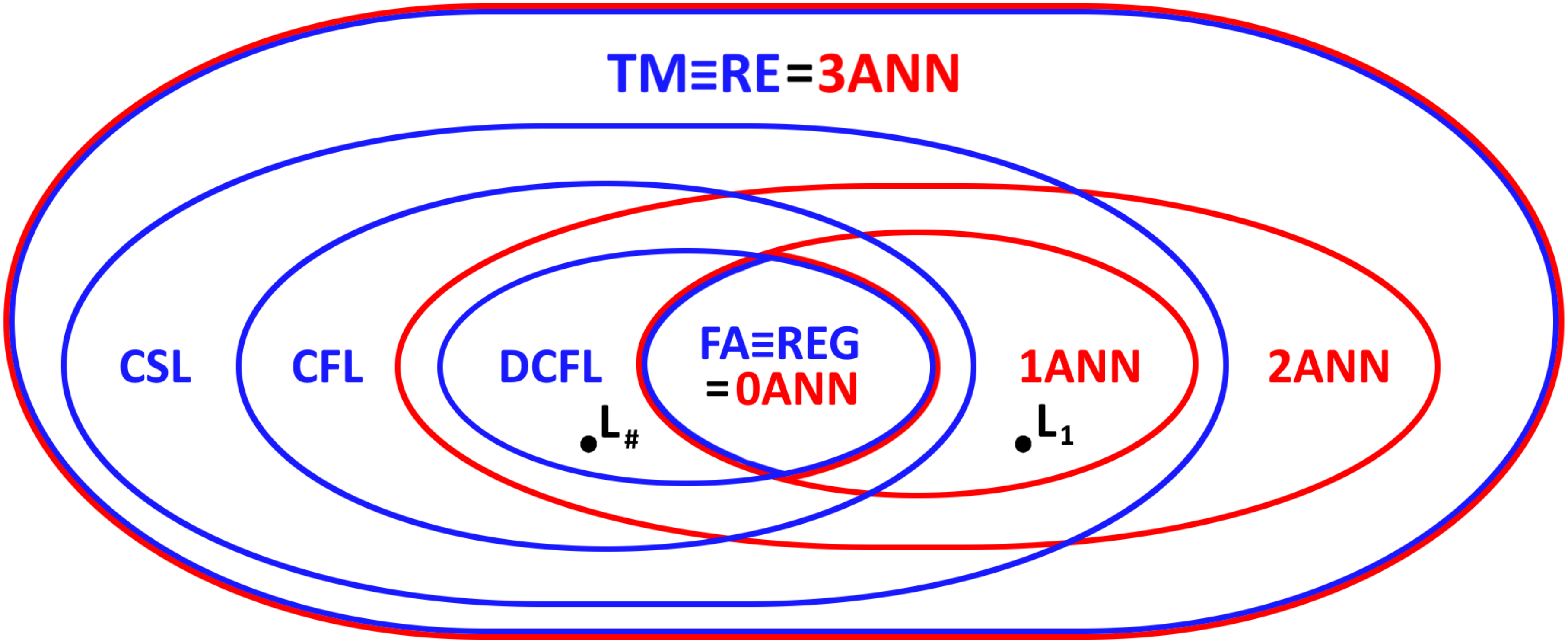
- 2ANN simulates deterministic PDA (DPDA $\equiv$ DCFL): **DCFL** $\subset$ **2ANN**
- 1ANN cannot count up to n (even with real weights): **DCFL** $\not\subset$ **1ANN**

  $$L_\# = \left\{ 0^n 1^n \,\middle|\, n \geq 1 \right\} \in \text{DCFL} \setminus \text{1ANN}$$

the **collapse** to the third level **3ANN** $=$ **4ANN** $= \ \ldots \ =$ **RE** $\equiv$ **TM** (Type 0):

- 3ANN simulates TM

The Chomsky Hierarchy vs. the Analog Neuron Hierarchy



the separation of some classes is still open, e.g. 2ANN $\overset{?}{\subsetneq}$ 3ANN, 1ANN $\cap$ CFL $\overset{?}{=}$ REG

the intermediate levels of the ANH and the Chomsky hierarchy seem incomparable

Positional Numeral Systems With Non-Integer Base

generalization of decimal expansions, which uses also non-integer numbers as the base and digits of a positional numeral system:

- $\beta \in \mathbb{R}$ is a real base (radix) such that $|\beta| > 1$
- $A \subset \mathbb{R}$ is a finite set of real digits such that $|A| \geq 2$

a finite β-expansion represents a number x in base β with digits a_i from A as

$$x = (0\,.\,a_1 \ldots a_n)_\beta = a_1\beta^{-1} + a_2\beta^{-2} + a_3\beta^{-3} + \cdots + a_n\beta^{-n} = \sum_{k=1}^{n} a_k\beta^{-k}$$

Examples:

1. $\beta = 10$, $A = \{0, 1, 2, \ldots, 9\}$
   decimal expansion of $\frac{3}{4} = (0\,.\,75)_{10} = 7 \cdot 10^{-1} + 5 \cdot 10^{-2}$
2. $\beta = 2$, $A = \{0, 1\}$
   binary expansion of $\frac{3}{4} = (0\,.\,11)_2 = 1 \cdot 2^{-1} + 1 \cdot 2^{-2}$
3. $\beta = \frac{5}{2}$, $A = \left\{\frac{5}{16}, \frac{7}{4}\right\}$
   $\frac{5}{2}$-expansion of $\frac{3}{4} \;=\; \left(0\;.\;\frac{7}{4}\;\frac{5}{16}\right)_{\frac{5}{2}} = \frac{7}{4} \cdot \left(\frac{5}{2}\right)^{-1} + \frac{5}{16} \cdot \left(\frac{5}{2}\right)^{-2}$

(Infinite) β-Expansions

introduced by Rényi (1957) and studied by Parry (1960); still an active research field with applications in coding theory, algorithmic complexity of arithmetic operations, models of quasicrystals, etc. (e.g. a research group at FNSPE CTU, Prague)

an infinite β-expansion of number x over digits a_i from A:

$$x = (0\,.\,a_1 a_2 a_3 \cdots)_\beta = a_1\beta^{-1} + a_2\beta^{-2} + a_3\beta^{-3} + \cdots = \sum_{k=1}^{\infty} a_k \beta^{-k}$$

which is a convergent power series due to $|\beta| > 1$

Example: $\beta = \frac{3}{2}$, $A = \{0, 1\}$

$\frac{3}{2}$-expansion of $\frac{16}{45}$: $(0\,.\,000\,10\,10\,10\,10\,10\,\ldots)_{\frac{3}{2}} = (0\,.\,000\,\overline{10})_{\frac{3}{2}}$

$$= \left(\frac{3}{2}\right)^{-4} + \left(\frac{3}{2}\right)^{-6} + \left(\frac{3}{2}\right)^{-8} + \cdots = \sum_{k=2}^{\infty} \left(\frac{3}{2}\right)^{-2k} = \sum_{k=2}^{\infty} \left(\frac{4}{9}\right)^{k} = \frac{16}{45}$$

(a geometric series)

Existence of β-Expansions

Let $\beta > 1$ and $A = \{\alpha_1, \ldots, \alpha_p\}$ where $\alpha_1 < \alpha_1 < \cdots < \alpha_p$.

Then every number in the interval $\left[\frac{\alpha_1}{\beta-1}, \frac{\alpha_p}{\beta-1}\right]$ has a β-expansion

iff $\max_{1<j\leq p}(\alpha_j - \alpha_{j-1}) \leq \frac{\alpha_p-\alpha_1}{\beta-1}$. (Pendicini, 2005)

Examples:

1. $\beta > 1$, $A = \{0, 1, \ldots, \lceil\beta\rceil - 1\}$ containing the standard integer digits

   every number in the interval $D_\beta = \left(0, \frac{\lceil\beta\rceil-1}{\beta-1}\right)$ (even $\overline{D_\beta}$) has a β-expansion,

   note that $(0,1) \subseteq D_\beta$, e.g. $D_\beta = (0,1)$ for integer base β

2. $\beta = 3$, $A = \{0, 2\}$ (i.e. $2 \not\leq \frac{2-0}{3-1} = 1$)

   any number from the complement of the Cantor ternary set

$$\bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^n+1}, \frac{3k+2}{3^n+1}\right) \subset (0,1)$$ has no 3-expansion

   (including iteratively the open middle third from a set of line segments, starting with (0,1))

Uniqueness of β-Expansions for Integer Base β

for an integer base $\beta > 1$ and the standard digits, $A = \{0, 1, \ldots, \beta - 1\}$,

almost any number from the interval $D_\beta = (0, 1)$ has a **unique** β-expansion,

e.g. the unique decimal expansion of $\frac{\sqrt{2}}{2} = (0\,.\,70710678118\ldots)_{10}$,

except for numbers with a finite β-expansion, which have **two distinct** (infinite) β-expansions,

e.g. two (infinite) decimal expansions of

$$\frac{3}{4} = (0\,.\,75)_{10} = (0\,.\,75000\ldots)_{10} = (0\,.\,74999\ldots)_{10}$$

Uniqueness of β-Expansions for Non-Integer Base β

for a non-integer base, almost every number has **infinitely (uncountably) many distinct** β-expansions (Sidorov, 2003)

Example: $1 < \beta < 2, \quad A = \{0, 1\}, \quad D_\beta = \left(0\,,\, \frac{1}{\beta-1}\right)$

- $1 < \beta < \varphi$ where $\varphi = (1+\sqrt{5})/2 \approx 1.618034$ is the golden ratio:

  every $x \in D_\beta$ has uncountably many distinct β-expansions (Erdös et al.,1990)

- $\varphi \le \beta < q$ where $q \approx 1.787232$ is the Komornik-Loreti constant
  (i.e. $\sum_{k=1}^{\infty} t_k q^{-k} = 1$ where $t_k = \text{parity}(\text{bin}(k))$ is the Thue-Morse sequence):

  countably many $x \in D_\beta$ have unique β-expansions (Glendinning,Sidorov,2001),

  e.g. the unique $\frac{5}{3}$-expansions of $\frac{9}{16}\left(\frac{3}{5}\right)^{k-1} = \left(0\,.\,(0)^k\,\overline{10}\right)_{\frac{5}{3}}$ for $k \ge 0$

  vs. countably many distinct φ-expansions of $1 = \left(0\,.\,(10)^k\,0\overline{1}\right)_\varphi$ for $k \ge 0$

- $q \le \beta < 2$: uncountably many $x \in D_\beta$ have unique β-expansions

partially generalizes to $\beta > 2$ and arbitrary A: two critical bases $1 < \varphi_A \le q_A$ such that the number of unique β-expansions is finite if $1 < \beta < \varphi_A$, countable if $\varphi_A < \beta < q_A$, and uncountable if $\beta > q_A$ (Komornik,Pedicini,2016)

Eventually Periodic β-Expansions

$$(0\,.\,a_1a_2\ldots a_{k_1}\,\overline{a_{k_1+1}a_{k_1+2}\ldots a_{k_2}})_\beta = (0\,.\,a_1a_2\ldots a_{k_1})_\beta + \beta^{-k_1}\varrho$$

where

- $a_1a_2\ldots a_{k_1} \in A^{k_1}$ is a preperiodic part of length $k_1 \geq 0$
  (purely periodic β-expansions for $k_1 = 0$)
- $a_{k_1+1}a_{k_1+2}\ldots a_{k_2} \in A^m$ is a repetend of $m = k_2 - k_1 > 0$ repeating digits
- $\varrho = (0\,.\,\overline{a_{k_1+1}a_{k_1+2}\ldots a_{k_2}})_\beta = \dfrac{\sum_{k=1}^{m} a_{k_1+k}\,\beta^{-k}}{1-\beta^{-m}}$ is a periodic point

Example: $\beta = \frac{3}{2}$, $A = \{0,1\}$

$$\frac{22}{15} = (0\,.\,1\,\overline{10})_{\frac{3}{2}} = (0\,.\,1)_{\frac{3}{2}} + \left(\frac{3}{2}\right)^{-1}\cdot\varrho = \left(\frac{3}{2}\right)^{-1} + \left(\frac{3}{2}\right)^{-1}\cdot(0\,.\,\overline{10})_{\frac{3}{2}}$$

where $\varrho = (0\,.\,\overline{10})_{\frac{3}{2}} = \sum_{k=0}^{\infty}\left(\frac{3}{2}\right)^{-2k-1} = \dfrac{1\cdot\left(\frac{3}{2}\right)^{-1} + 0\cdot\left(\frac{3}{2}\right)^{-2}}{1-\left(\frac{3}{2}\right)^{-2}} = \dfrac{6}{5}$

Eventually Quasi-Periodic β-Expansions

$$(0\,.\,a_1 \ldots a_{k_1}\, a_{k_1+1} \ldots a_{k_2}\, a_{k_2+1} \ldots a_{k_3}\, a_{k_3+1} \ldots a_{k_4} \ldots)_\beta$$
$$= (0\,.\,a_1 a_2 \ldots a_{k_1})_\beta + \beta^{-k_1} \varrho$$

where

- $a_1 a_2 \ldots a_{k_1} \in A^{k_1}$ is a preperiodic part of length (purely quasi-periodic β-expansions for $k_1 = 0$)
- $a_{k_i+1} \ldots a_{k_{i+1}} \in A^{m_i}$ is a quasi-repetend of length $m_i = k_{i+1} - k_i > 0$
- $\varrho = (0\,.\,\overline{a_{k_i+1} \ldots a_{k_{i+1}}})_\beta = \dfrac{\sum_{k=1}^{m_i} a_{k_i+k}\, \beta^{-k}}{1 - \beta^{-m_i}}$ is the same periodic point for every $i \geq 1$

$\longrightarrow$ quasi-repetends can be interchanged with each other arbitrarily

- a generalization of eventually periodic β-expansions

$$a_{k_1+1} \ldots a_{k_2} = a_{k_2+1} \ldots a_{k_3} = a_{k_3+1} \ldots a_{k_4} = \cdots$$

Example: $\beta \approx 1.220744$ satisfying $\beta^4 - \beta - 1 = 0$ $(\star)$, $A = \{0, 1\}$

$$1 = (0\,.\,00\,010\,1000\,1000\,010 \ldots)_\beta = (0\,.\,00)_\beta + \beta^{-2} \varrho$$

where 00 is a preperiodic part and 010, 1000 are two quasi-repetends with same periodic point $\varrho = (0\,.\,\overline{010})_\beta = \frac{\beta^{-2}}{1-\beta^{-3}} \overset{\star}{=} \beta^2 \overset{\star}{=} \frac{\beta^{-1}}{1-\beta^{-4}} = (0\,.\,\overline{1000})_\beta$

An Example of Repetends With Unbounded Length

base $\beta = \frac{5}{2}$, digits $A = \{0\,,\ \frac{1}{2}\,,\ \frac{7}{4}\}$

for every $n \geq 0$, the quasi-repetends $\frac{7}{4}\ \underbrace{\frac{1}{2} \cdots \frac{1}{2}}_{n \text{ times}}\ 0 \in A^{n+2}$ have

the same periodic point $\varrho = \frac{3}{4}$:

$$\Big(0\,.\,\overline{\frac{7}{4}\ \underbrace{\frac{1}{2} \cdots \frac{1}{2}}_{n \text{ times}}\ 0}\Big)_{\frac{5}{2}} = \frac{\frac{7}{4} \cdot \left(\frac{5}{2}\right)^{-1} + \sum_{i=2}^{n+1} \frac{1}{2} \cdot \left(\frac{5}{2}\right)^{-i} + 0 \cdot \left(\frac{5}{2}\right)^{-n-2}}{1 - \left(\frac{5}{2}\right)^{-n-2}} = \frac{3}{4}$$

$\longrightarrow$ $\frac{3}{4}$ has uncountably many distinct quasi-periodic $\frac{5}{2}$-expansions:

$$\frac{3}{4} = \Big(0\,.\,\frac{7}{4}\ \underbrace{\frac{1}{2} \cdots \frac{1}{2}}_{n_1 \text{ times}}\ 0\ \frac{7}{4}\ \underbrace{\frac{1}{2} \cdots \frac{1}{2}}_{n_2 \text{ times}}\ 0\ \frac{7}{4}\ \underbrace{\frac{1}{2} \cdots \frac{1}{2}}_{n_3 \text{ times}}\ 0\ \frac{7}{4}\ \underbrace{\frac{1}{2} \cdots \frac{1}{2}}_{n_4 \text{ times}}\ 0\ \cdots\Big)_{\frac{5}{2}}$$

where $n_1, n_2, n_3, \ldots$ is any infinite sequence of nonnegative integers

(there are examples of exponentially many quasi-repetends in terms of their length)

Eventually Quasi-Periodic β-Expansions and Tail Sequences

$(r_n)_{n=0}^{\infty}$ is a tail sequence of β-expansion $\varepsilon = \big(0\,.\,a_1\,a_2\,a_3\,\ldots\big)_\beta$ if

$$r_n = (0\,.\,a_{n+1}a_{n+2}\ldots)_\beta = \sum_{k=1}^{\infty} a_{n+k}\beta^{-k} \quad \text{for every } n \geq 0$$

denote by $R_\varepsilon = \{r_n \mid n \geq 0\}$ its range

Lemma. *If R_ε is finite (i.e. the tail sequence contains a constant infinite subsequence), then the β-expansion ε is eventually quasi-periodic.*

Theorem. *Let β be a real algebraic number ($|\beta| > 1$) whose all conjugates β' (i.e. the other roots of minimal polynomial of β) meet $|\beta'| \neq 1$. Then a β-expansion ε is eventually quasi-periodic **iff** R_ε is finite.*

Theorem. *Let β be a real algebraic number ($|\beta| > 1$) whose conjugate β' meets $|\beta'| = 1$. Then there exists a finite set $A \subset \mathbb{Z}$ of integer digits and a quasi-periodic β-expansion ε over A of the number 0 that has infinite R_ε.*

(solves an important open problem in algebraic number theory)

Quasi-Periodic Numbers

a real number $x \in \mathbb{R}$ is β-quasi-periodic within A if **every** infinite β-expansion of x over A, is eventually quasi-periodic

Examples:

- x with no β-expansion at all, **is** formally quasi-periodic (e.g. any number from the complement of the Cantor ternary set is 3-quasi-periodic within $A = \{0, 2\}$)
- $x = \frac{3}{4}$ **is** $\frac{5}{2}$-quasi-periodic within $A = \left\{0\,,\,\frac{1}{2}\,,\,\frac{7}{4}\right\}$:
  all the $\frac{5}{2}$-expansions of $\frac{3}{4}$ using the digits from A, are eventually quasi-periodic
- $x = \frac{40}{57} = (0\,.\,0\,\overline{011})_{\frac{3}{2}}$ **is not** $\frac{3}{2}$-quasi-periodic within $A = \{0, 1\}$:
  the greedy (i.e. lexicographically maximal) $\frac{3}{2}$-expansion $(0\,.\,100000001\,\ldots)_{\frac{3}{2}}$ of $\frac{40}{57}$ is not eventually quasi-periodic

Theorem. *Let $\beta > 1$ be a Pisot number (i.e. a real algebraic integer whose all conjugates β' meet $|\beta'| < 1$) and $A \subset \mathbb{Q}(\beta)$. Then any $x \in \mathbb{Q}(\beta)$ is β-quasi-periodic within A.*

- $x = 1$ **is** β-quasi-periodic within $A = \{0, 1\}$ for the plastic constant $\beta \approx 1.324718$ (i.e. the minimal Pisot number satisfying $\beta^3 - \beta - 1 = 0$)

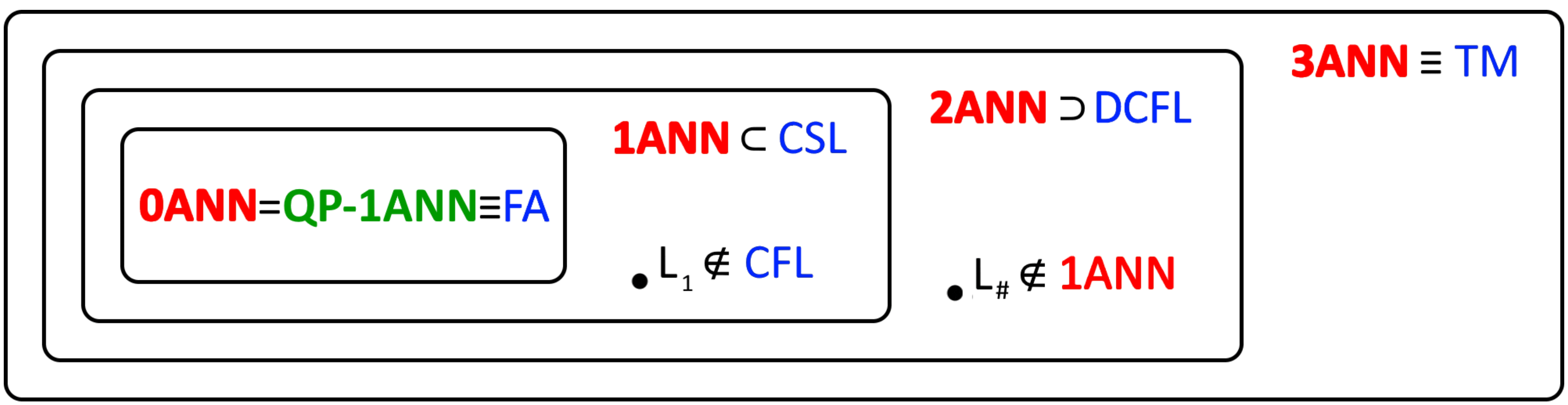
Quasi-Periodic 1ANN (QP-1ANN): for a 1ANN, denote:

- $\beta = 1/w_{11}$ is the base ($|\beta| > 1$) where w_{11} is the self-loop weight of the one analog-state neuron ($0 < |w_{11}| < 1$)
- $A = \left\{ \sum_{i=0\,;\,i\neq 1}^{s} \frac{w_{1i}}{w_{11}}\, y_i \;\middle|\; y_2,\ldots,y_s \in \{0,1\} \right\} \cup \{0,\beta\}$ are the digits
- $X = \left\{ \sum_{i=0\,;\,i\neq 1}^{s} \frac{w_{ji}}{w_{j1}}\, y_i \;\middle|\; j \neq 1\,,\; w_{j1} \neq 0\,,\; y_2,\ldots,y_s \in \{0,1\} \right\} \cup \{0,1\}$

we say that 1ANN (even with real weights) is quasi-periodic and denote **QP-1ANN** if every $x \in X$ is β-quasi-periodic within A

Example: 1ANN with rational weights $+$ the self-loop weight $w_{11} = 1/\beta$ where β is an integer or the plastic constant or the golden ratio

Theorem. QP-1ANN $=$ REG $=$ 0ANN $\equiv$ FA (Type 3)

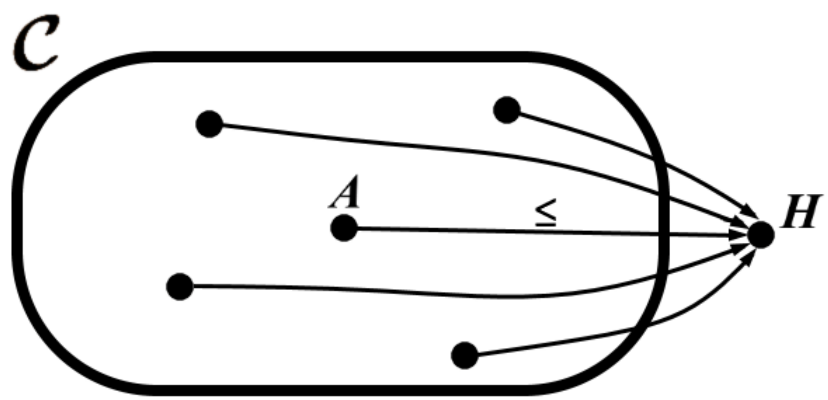


$\mathcal{C}$-Hard Problems

$\mathcal{C}$ is a complexity class of decision problems (i.e. formal languages)

$A \leq B$ is a reduction transforming a problem A to a problem B (a preorder), which is assumed not to have a higher computational complexity than $\mathcal{C}$

H is a $\mathcal{C}$-hard problem (under the reduction $\leq$) if for every $A \in \mathcal{C}$, $A \leq H$



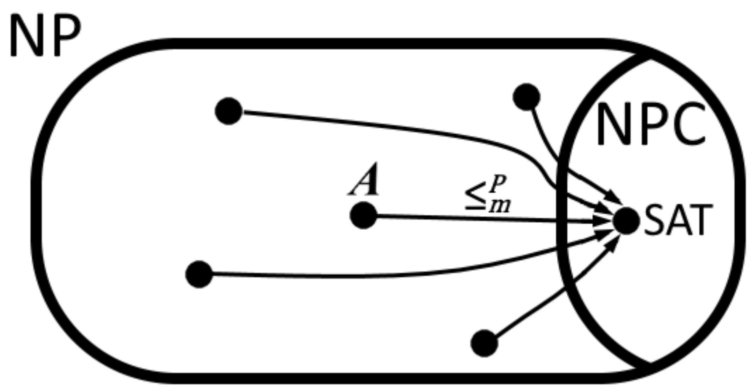
- If a $\mathcal{C}$-hard problem has a (computationally) "easy" solution, then each problem in $\mathcal{C}$ has an "easy" solution (via the reduction).
- If a $\mathcal{C}$-hard problem H is in $\mathcal{C}$ (a so-called $\mathcal{C}$-complete problem), then H belongs to the hardest problems in the class $\mathcal{C}$.

The Most Prominent Example: NP-Hard Problems

$\mathcal{C} =$ NP is the class of decision problems solvable in polynomial time by a nondeterministic Turing machine

$A \leq_m^P B$ is a polynomial-time many-one reduction (Karp reduction) from A to B

the satisfiability problem SAT is NP-hard: for every $A \in$ NP, $A \leq_m^P$ SAT

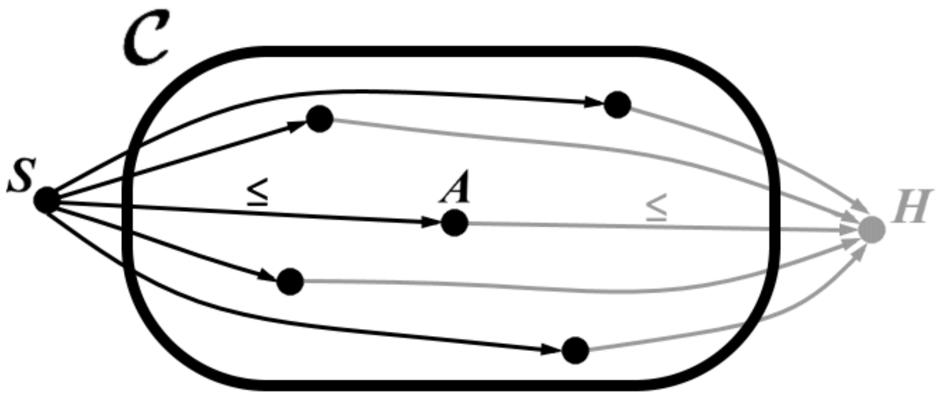


- If an NP-hard problem is polynomial-time solvable,
  then each NP problem would be solved in polynomial time (i.e. P = NP)
- The NP-hard problem SAT is in NP (i.e. SAT is NP-complete),
  that is, SAT belongs to the hardest problems (NPC) in the class NP.

$\mathcal{C}$-Simple Problems

a conceptual counterpart to $\mathcal{C}$-hard problems:

S is a $\mathcal{C}$-simple problem (under the reduction $\leq$) if for every $A \in \mathcal{C}$, $S \leq A$



- If a $\mathcal{C}$-simple problem S proves to be not "easy",
  e.g. S is not solvable by a machine M that can compute the reduction $\leq$,
  then all problems in $\mathcal{C}$ are not "easy", i.e. $\mathcal{C}$ cannot be solved by M.

  $\longrightarrow$ **New Proof Technique:** a lower bound known for one $\mathcal{C}$-simple problem S extends to the whole class of problems $\mathcal{C}$

- If a $\mathcal{C}$-simple problem S is in $\mathcal{C}$, then S is the simplest problem in the class $\mathcal{C}$.

A Trivial Example: SAT is simple for the class of NP-hard problems under $\leq_m^P$

A Nontrivial Example of a $\mathcal{C}$-Simple Problem

$\mathcal{C}$ = DCFL' = DCFL \ REG

is the class of non-regular deterministic context-free languages

$L_1 \leq_{tt}^{A} L_2$ is a truth-table reduction (a stronger Turing reduction) from L_1 to L_2

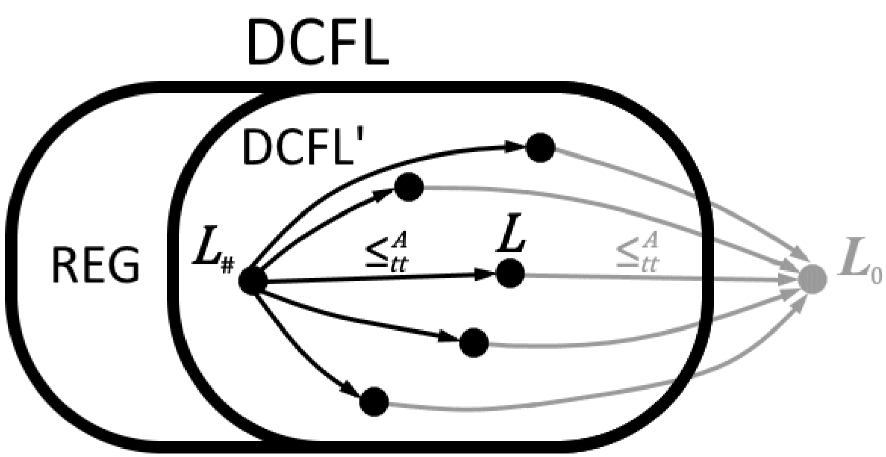
implemented by a Mealy machine with the oracle L_2

The Technical Result:

the language $L_\# = \{0^n 1^n \mid n \geq 1\}$ over the binary alphabet $\{0, 1\}$ is DCFL'-simple under the reduction $\leq_{tt}^{A}$: for every $L \in$ DCFL', $L_\# \leq_{tt}^{A} L$

$\longrightarrow$ $L_\# \in$ DCFL' is the *simplest* non-regular deterministic context-free languages

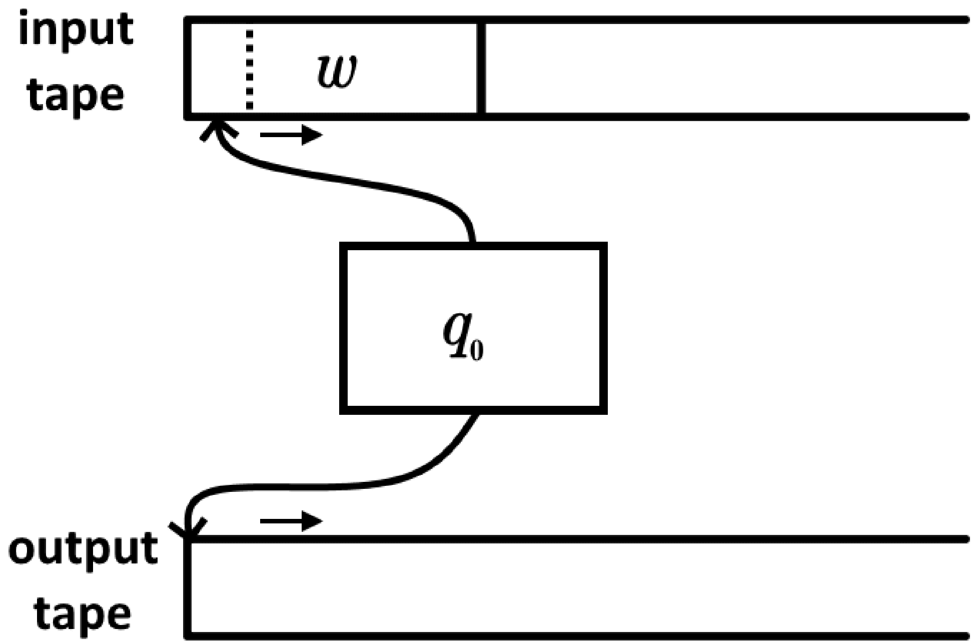
cf. the *hardest* context-free language L_0 due to `S. Greibach (1973)` is CFL-hard



Mealy Machines

$\mathcal{A}$ is a Mealy Machine with an input/output alphabet Σ/Δ

i.e. a deterministic finite automaton with an output tape:



input word $w \in \Sigma^*$

initial state q_0

Mealy Machines

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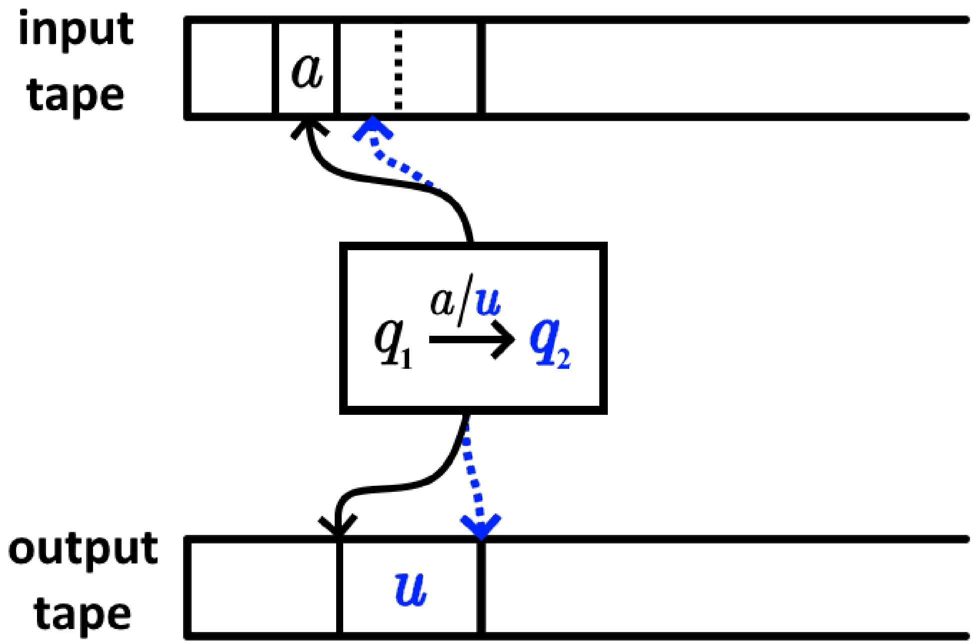


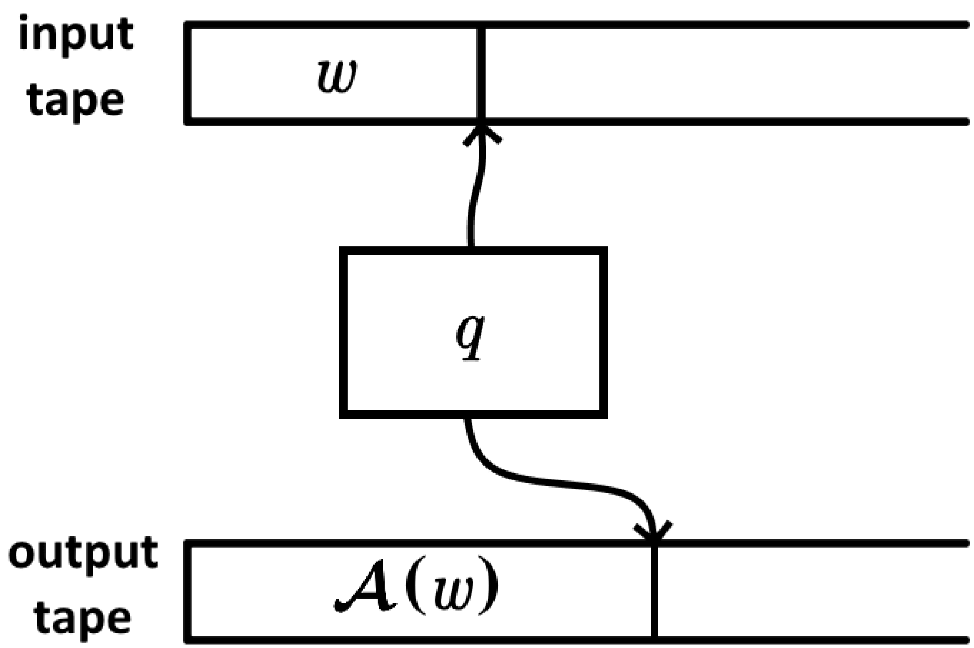
current input symbol $a \in \Sigma$

state transition
from q_1 to q_2

output string $u \in \Delta^*$

Mealy Machines

$\mathcal{A}$ is a Mealy Machine with an input/output alphabet Σ/Δ
i.e. a deterministic finite automaton with an output tape:



input $w \in \Sigma^*$

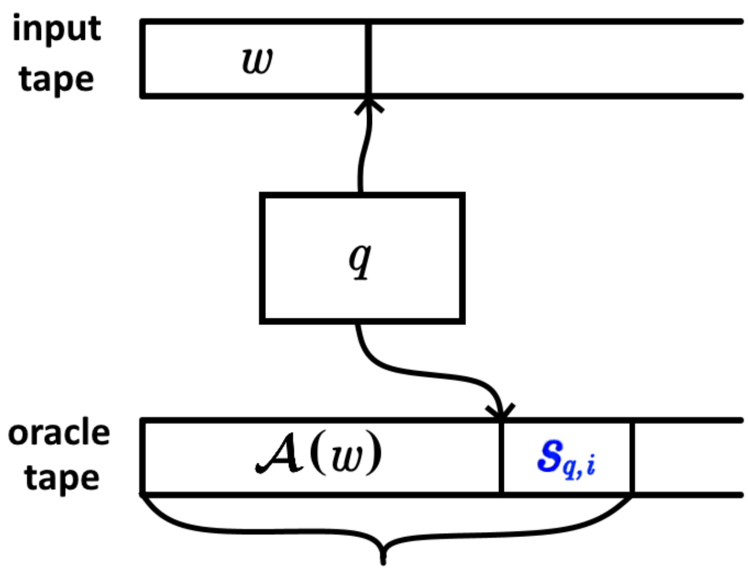
"final" state q

output $\mathcal{A}(w) \in \Delta^*$

$\longrightarrow$ a deterministic finite-state transducer: $w \in \Sigma^* \longmapsto \mathcal{A}(w) \in \Delta^*$

The Truth-Table Reduction by Oracle Mealy Machines

$\mathcal{A}^{L_2}$ is a Mealy Machine $\mathcal{A}$ with an oracle $L_2 \subseteq \Delta^*$:



for each state q of $\mathcal{A}$:

- r_q suffixes $s_{q,1}, \ldots, s_{q,r_q} \in \Delta^*$
- truth table $T_q : \{0,1\}^{r_q} \to \{0,1\}$ with r_q variables

r_q queries: $\overset{?}{\in} L_2$ for every $i = 1, \ldots, r_q$

$w \in \Sigma^*$ is accepted by $\mathcal{A}^{L_2}$ **iff** w brings $\mathcal{A}$ to the state q such that

$$T_q\left(\mathcal{A}(w) \cdot s_{q,1} \overset{?}{\in} L_2\,,\ \mathcal{A}(w) \cdot s_{q,2} \overset{?}{\in} L_2\,, \ldots,\ \mathcal{A}(w) \cdot s_{q,r_q} \overset{?}{\in} L_2\right) = 1$$

$L_1 \le_{tt}^{A} L_2$: $L_1 \subseteq \Sigma^*$ is truth-table reducible to $L_2 \subseteq \Delta^*$ **iff**
$L_1 = \mathcal{L}(\mathcal{A}^{L_2})$ is accepted by some Mealy machine $\mathcal{A}^{L_2}$ with oracle L_2

Proposition: *The relation* $\le_{tt}^{A}$ *is a preorder.*

Why $L_{\#} = \{0^n 1^n \mid n \geq 1\}$ Is the Simplest DCFL' language?

any reduced context-free grammar G generating a non-regular language $L \subseteq \Delta^*$

is self-embedding: there is a self-embedding nonterminal A admitting the derivation

$A \Rightarrow^* xAy$ for some non-empty strings $x, y \in \Delta^+$ (Chomsky, 1959)

G is reduced $\longrightarrow$ $S \Rightarrow^* vAz$ and $A \Rightarrow^* w$ for some $v, w, z \in \Delta^*$

$$\longrightarrow \qquad S \Rightarrow^* vx^m wy^m z \in L \text{ for every } m \geq 0 \qquad (1)$$

??? a conceivable (one-one) reduction from $L_{\#}$ to L: for every $m, n \geq 1$,

$$0^m 1^n \in \{0,1\}^* \longmapsto vx^m wy^n z \in \Delta^*$$

(the inputs outside 0^+1^+ are mapped onto some fixed string outside L)

since $0^m 1^n \in L_{\#}$ **implies** $vx^m wy^n z \in L$ by (1)

!!! however, the opposite implication may not be true:

Why $L_\#$ Is the Simplest DCFL' language? (cont.)

!!! however, the opposite implication may not be true:

for the DCFL' language $L_1 = \{a^m b^n \mid 1 \leq m \leq n\}$ over $\Delta = \{a, b\}$

there are **no** words $v, x, w, y, z \in \Delta^*$ such that for every $m, n \geq 1$,

$$vx^m wy^n z \in L_1 \text{ would ensure } m = n$$

nevertheless, already **two** inputs $a^m b^{n-1} \overset{?}{\in} L_1$ and $a^m b^n \overset{?}{\in} L_1$ decides $m \overset{?}{=} n$

$\longrightarrow$ the truth-table reduction from $L_\#$ to L_1 with two queries to the oracle L_1 :

$$0^m 1^n \in \{0,1\}^* \quad \longmapsto \quad vx^m wy^{n-1} z \in \Delta^*, \quad vx^m wy^n z \in \Delta^*$$

where $x = a$, $y = b$, $v = w = z = \varepsilon$ is the empty string

satisfying $0^m 1^n \in L_\#$ **iff** $(vx^m wy^{n-1} z \notin L_1$ **and** $vx^m wy^n z \in L_1)$

this can be generalized to any DCFL' language L :

The Main Technical Result

Theorem: *Let $L \subseteq \Delta^*$ be a non-regular deterministic context-free language over an alphabet Δ. There exist non-empty words $v, x, w, y, z \in \Delta^+$ and a language $L' \in \{L, \overline{L}\}$ (where $\overline{L} = \Delta^* \setminus L$ is the complement of L) such that*

1. **either** *for all* $m, n \geq 0$, $vx^m wy^n z \in L'$ **iff** $m = n$,
2. **or** *for all* $m, n \geq 0$, $vx^m wy^n z \in L'$ **iff** $m \leq n$.

1.

m \ n	0	1	2	3	…
0	$\in L'$	$\notin L'$	$\notin L'$	$\notin L'$	
1	$\notin L'$	$\in L'$	$\notin L'$	$\notin L'$	
2	$\notin L'$	$\notin L'$	$\in L'$	$\notin L'$	
3	$\notin L'$	$\notin L'$	$\notin L'$	$\in L'$	
⋮					⋱

2.

m \ n	0	1	2	3	…
0	$\in L'$	$\in L'$	$\in L'$	$\in L'$	
1	$\notin L'$	$\in L'$	$\in L'$	$\in L'$	
2	$\notin L'$	$\notin L'$	$\in L'$	$\in L'$	
3	$\notin L'$	$\notin L'$	$\notin L'$	$\in L'$	
⋮					⋱

In particular, for all $m \geq 0$ *and* $n > 0$,

$(vx^m wy^{n-1} z \notin L'$ **and** $vx^m wy^n z \in L')$ **iff** $m = n$.

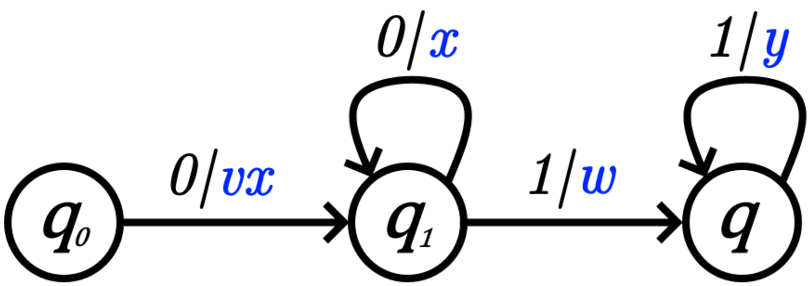
The Truth-Table Reduction From $L_\#$ to Any DCFL' L

implemented by a **Mealy machine** $\mathcal{A}^L$ with two queries to the oracle L:

For any DCFL' language $L \subseteq \Delta^*$, Theorem provides $v, x, w, y, z \in \Delta^+$ and $L' \in \{L, \overline{L}\}$, say $L' = L$ (analogously for $L' = \overline{L}$), such that

$$(\, vx^m wy^{n-1}z \notin L \;\; \textbf{and} \;\; vx^m wy^n z \in L \,) \;\; \textbf{iff} \;\; m = n\,. \qquad (2)$$

$\mathcal{A}^L$ transforms the input $0^m 1^n$ to the output $\mathcal{A}(0^m 1^n) = vx^m wy^{n-1} \in \Delta^+$ (the inputs outside 0^+1^+ are rejected), while moving to the state q with $r_q = 2$ suffixes $s_{q,1}$, $s_{q,2}$ and the truth table $T_q : \{0,1\}^2 \longrightarrow \{0,1\}$



$s_{q,1} = z$

$s_{q,2} = yz$

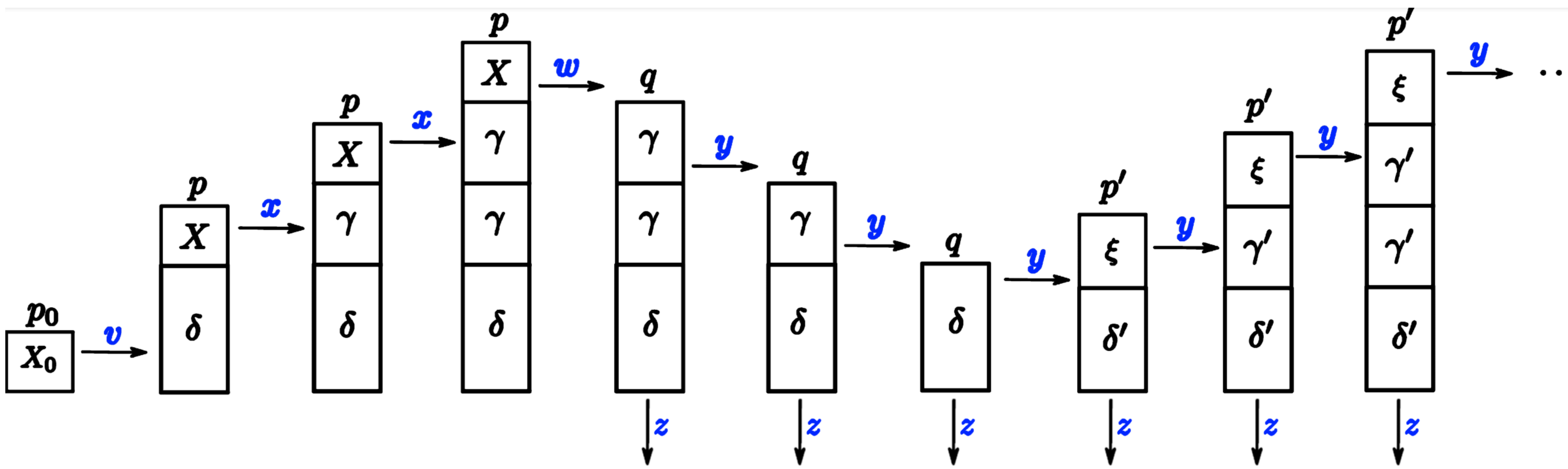
$\mathcal{A}(0^m1^n) \cdot z = vx^m wy^{n-1}z \overset{?}{\in} L$	$\mathcal{A}(0^m1^n) \cdot yz = vx^m wy^n z \overset{?}{\in} L$	T_q
0	1	1
0	0	0
1	1	0
1	0	0

It follows from (2) that $\mathcal{L}(\mathcal{A}^L) = L_\#$, i.e. $L_\# \leq_{tt}^{A} L$.

Ideas of the Proof of the Theorem

(inspired by some ideas on regularity of pushdown processes due to `Janar`, 2020)

- any non-regular DCFL language $L \subseteq \Delta^*$ is accepted
  by a deterministic pushdown automaton $\mathcal{M}$ by the empty stack
- since $L \notin$ REG, there is a computation by $\mathcal{M}$, reaching configurations with an arbitrary large stack which is being erased afterwards,
  corresponding to $v, x, w, y, z \in \Delta^+$ such that $vx^m wy^m z \in L$ for all $m \geq 1$
- in addition, we aim to ensure that for all $m \geq 0$ and $n > 0$,
  $(\, vx^m wy^{n-1} z \notin L'$ **and** $vx^m wy^n z \in L' \,)$ **iff** $m = n$



$(L' = L)$

case "$m = n$":	REJ	REJ	ACC	REJ	REJ	REJ
case "$m \leq n$":	REJ	REJ	ACC	ACC	ACC	ACC

Ideas of the Proof of the Theorem

(inspired by some ideas on regularity of pushdown processes due to Janar, 2020)

- any non-regular DCFL language $L \subseteq \Delta^*$ is accepted
  by a deterministic pushdown automaton $\mathcal{M}$ by the empty stack
- since $L \notin$ REG, there is a computation by $\mathcal{M}$, reaching configurations with an arbitrary large stack which is being erased afterwards, corresponding to $v, x, w, y, z \in \Delta^+$ such that $vx^m wy^m z \in L$ for all $m \geq 1$
- in addition, we aim to ensure that for all $m \geq 0$ and $n > 0$,
  $(vx^m wy^{n-1} z \notin L'$ **and** $vx^m wy^n z \in L')$ **iff** $m = n$
- we study the computation of $\mathcal{M}$ on an infinite word that traverses infinitely many pairwise non-equivalent configurations
- we use a natural congruence property of language equivalence on the set of configurations (determinism of $\mathcal{M}$ is essential)
- we apply Ramsey's theorem for extracting the required $v, x, w, y, z \in \Delta^+$ from the infinite computation

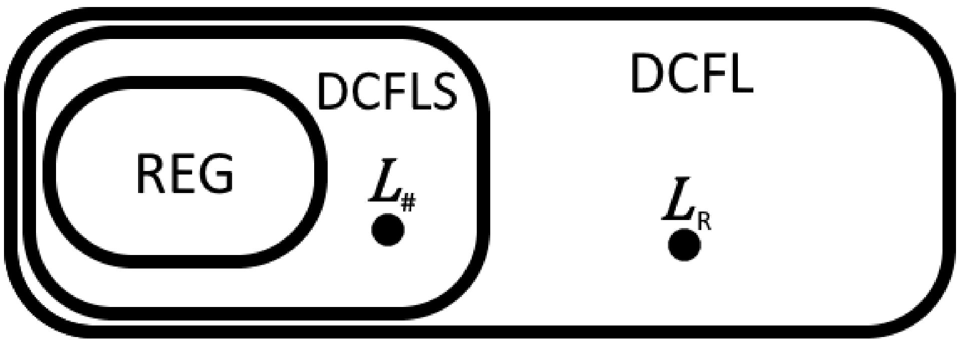
Basic Properties of DCFL'-Simple Problems

DCFLS is the class of DCFL'-simple problems

Proposition:

- REG $\subsetneq$ DCFLS $\subsetneq$ DCFL,

  e.g. $L_{\#} \in$ DCFLS, $L_R = \{wcw^R \mid w \in \{a,b\}^*\} \notin$ DCFLS



- The class DCFLS is closed under complement and intersection with regular languages.
- The class DCFLS is not closed under concatenation, intersection, and union.

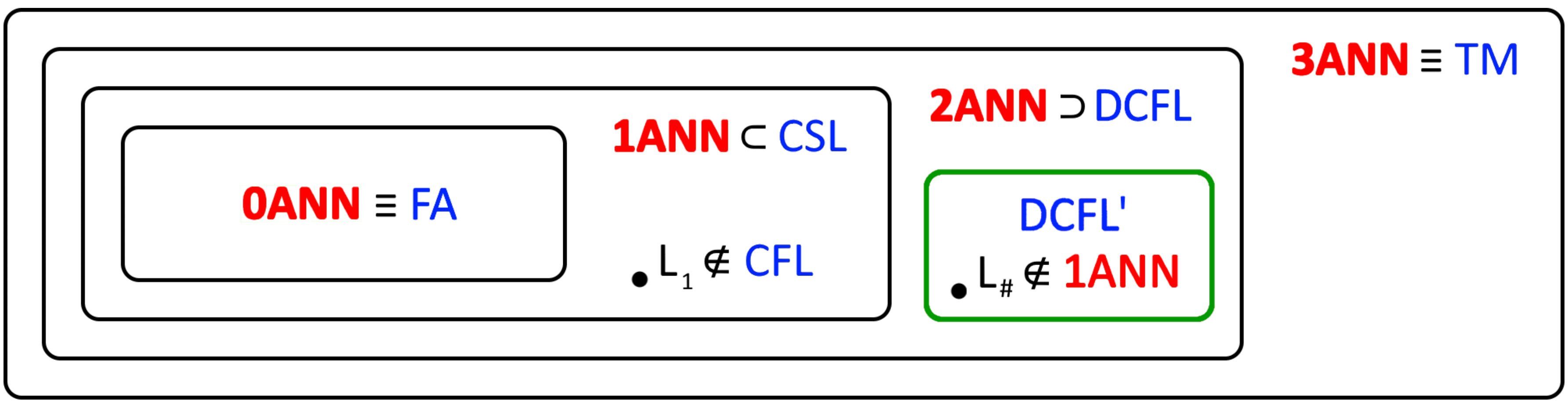
Application to the Analog Neuron Hierarchy

- $L_\# \notin$ **1ANN** by a nontrivial proof (based on the Bolzano–Weierstrass theorem) which can hardly be generalized to another DCFL' language

- $L_\#$ is DCFL'-simple under $\leq_{tt}^{A}$

- the reduction $\leq_{tt}^{A}$ to any $L \in$ 1ANN can be implemented by 1ANN

$\longrightarrow$ the known lower bound $L_\# \notin$ 1ANN for a single DCFL'-simple problem $L_\#$ is expanded to the whole class: **DCFL'** $\cap$ **1ANN** $= \emptyset$



$\longrightarrow$ **DCFL $\cap$ 1ANN = 0ANN**