Exponential Transients in Continuous-Time Symmetric Hopfield Nets

Jiří Šíma[1]* and Pekka Orponen[2]

[1] Institute of Computer Science, Academy of Sciences of the Czech Republic, P.O. Box 5, 182 07 Prague 8, Czech Republic, *sima@cs.cas.cz*
[2] Department of Mathematics, University of Jyväskylä, P.O. Box 35, FIN-40351 Jyväskylä, Finland, *orponen@math.jyu.fi*

Abstract. We establish a fundamental result in the theory of continuous-time neural computation, by showing that so called continuous-time symmetric Hopfield nets, whose asymptotic convergence is always guaranteed by the existence of a Liapunov function may, in the worst case, possess a transient period that is exponential in the network size. The result stands in contrast to e.g. the use of such network models in combinatorial optimization applications.

1 Introduction

Continuous-time recurrent neural networks are an attractive class of computational models with applications in, e.g., control, optimization, and signal processing (cf. [1, 5]). Recently there has also been increasing theoretical interest towards achieving a general understanding of the capabilities and limitations of these and other continuous-time computation models. (For overviews of this work, see e.g. [6, 7].)

Probably the best-known, and most widely-used continuous-time recurrent network model is that popularized by John Hopfield in 1984 [4], and known as the "continuous-time Hopfield model".[1] A fundamental property of this model is that if a given network has a symmetric coupling weight matrix, then its dynamics is governed by a *Liapunov*, or *energy function* [2, 4]. In particular, such a symmetric network always converges from any initial state towards some stable equilibrium state. This is a very useful property for obtaining guaranteed behavior in practical applications, but would at first sight seem to severely limit the networks' general dynamical capabilities. For instance, nondamping oscillations of the network state obviously cannot be created under this constraint, whereas such oscillations are easily obtained in networks with *asymmetric* coupling weights.

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[1] Although in fact the dynamics of this model were already analyzed earlier by Cohen and Grossberg in a more general setting [2].

Because of the apparent simplicity of symmetric Hopfield network dynamics, one might also assume that they always converge rapidly—an assumption that seems to often be implicitly made in e.g. discussing the potential of such networks as "fast analog solvers" for optimization problems. Contrary to this expectation, we shall in this paper construct for every n a Hopfield network $\mathcal{C}_n$ of $6n+1$ units with a symmetric coupling weight matrix and a saturated-linear "activation function" that simulates an $(n+1)$-bit binary counter and thus produces a sequence of $2^n - 1$ well-controlled oscillations before it converges. Besides suggesting some caution in applying neural networks to optimization problems, this result provides to our knowledge the first known example of a continuous-time, Liapunov-function controlled dynamical system with an exponential transient period. Such an exponential-transient oscillator can also be used to support a general Turing machine simulation by symmetric Hopfield networks [9].

In terms of bit representations, our convergence time lower bound can be compared to a general upper bound for discrete Hopfield networks [10]. It turns out that the continuous-time system $\mathcal{C}_n$ converges later than any discrete symmetric Hopfield network of the same description length, assuming that the time interval between two subsequent discrete updates corresponds to a continuous time unit. This suggests that continuous-time analog models of computation may be worth investigating more for their gains in representational efficiency than for their (theoretical) capability for arbitrary-precision real number computation [8].

2 A Simulated Binary Counter

A *(symmetric) Hopfield network*[2] consists of m computational *units* or "neurons" $p = 1, \ldots, m$, whose *states* are represented by real variables $y_1, \ldots, y_m \in [0,1]$. The dynamics of such a network is given by a system of m symmetrically coupled ordinary differential equations:

$$\frac{dy_p}{dt}(t) = -y_p(t) + \sigma(\xi_p(t)), \quad p = 1, \ldots, m, \tag{1}$$

where $\xi_p(t) = \sum_{q=0}^{m} v(p,q) y_q(t)$ is the real-valued *excitation* for unit $p = 1, \ldots, m$. Here, the real coupling coefficient $v(p,q) = v(q,p)$ corresponds to the *weight* on an edge connecting unit p to unit q whereas $v(0,p)$ is a local *bias* $v(0,p)$, associated with a formal constant variable $y_0(t) \equiv 1$. Further, σ is some nonlinear *activation function*, which we fix to be the *saturated linear* map: $\sigma(\xi) = 1$ for $\xi \geq 1$, $\sigma(\xi) = \xi$ for $0 < \xi < 1$, and $\sigma(\xi) = 0$ for $\xi \leq 0$. The *initial network state* $\mathbf{y}(0) \in [0,1]^m$ determines the boundary condition for the system (1).

A Hopfield network $\mathcal{C} = \mathcal{C}_n$ with $m = 6n+1$ neurons will now be constructed which simulates an $(n+1)$-bit binary counter, and thus has a transient period that is exponential in the parameter m. The original idea for a corresponding discrete-time counter network stems from [3]. In our simulation, the binary states of the counter will be represented by excitations of the corresponding real-valued

[2] We shall henceforth discuss only symmetric networks.

units in $\mathcal{C}$ that are either above the upper saturation threshold of 1 or below the lower saturation threshold of 0 for the activation function σ. For brevity, we shall simply say that a unit p is *saturated* at 0 or 1 at time t if its excitation satisfies $\xi_p(t) \leq 0$ or $\xi_p(t) \geq 1$, respectively. We also say that p is *unsaturated* when $0 < \xi_p(t) < 1$. (Note that we use the *excitations*, not the actual *states* of the units to represent binary values.) The following theorem summarizes the result:

Theorem 1. *For every integer $n \geq 0$ there exists a continuous-time symmetric Hopfield net $\mathcal{C}$ with $m = 6n + 1$ neurons whose global state transition from saturation at 0 to saturation at 1 requires continuous time $\Omega(2^{m/6}/\varepsilon)$, for any $0 < \varepsilon < 0.05$ such that $2^{m/2} < \varepsilon 2^{1/\varepsilon}$. This convergence bound translates to $2^{\Omega(g(M))}$ time units, where M represents the number of bits that are sufficient for encoding the weights in $\mathcal{C}$ and $g(M)$ is an arbitrary continuous function such that $g(M) = o(M)$, $g(M) = \Omega(M^{2/3})$, and $M/g(M)$ is increasing.*

Proof. (Sketch.) The construction of symmetric Hopfield net $\mathcal{C} = \mathcal{C}_n$ with $m = 6n + 1$ units and zero initial state $\mathbf{y}(0) = 0^m$ simulating an $(n + 1)$-bit binary counter will be described by induction on n. The operation of the network will first be discussed intuitively, and its correctness will then be formally verified. The induction starts with a network $\mathcal{C}_0$ containing only a single unit c_0, with bias $v(0, c_0) = \varepsilon$ and feedback coupling $v(c_0, c_0) = 1 + \varepsilon$. This represents the first counter bit of "order 0". Because of its positive feedback the state of c_0 gradually grows from initial 0 towards 1. Eventually c_0 saturates at 1, at which point we say that the unit c_0 becomes *active* or *fires*. This trick of gradual transition from 0 to 1 (see Lemma 3 below) is used repeatedly throughout our construction of $\mathcal{C}$.

For the induction step depicted in Figure 1 (the edges in this graph drawn without an originating unit correspond to the biases), assume that an "order $(k-1)$" counter network $\mathcal{C}_{k-1}$ $(1 \leq k \leq n)$ has been constructed, containing the first k counter units $c_0, \ldots, c_{k-1}$, together with auxiliary units $a_\ell, x_\ell, b_\ell, d_\ell, z_\ell$ $(\ell = 1, \ldots, k-1)$, for a total of $m_k = 6k-5$ units. Then the next counter unit c_k is connected to all the m_k units $p \in \mathcal{C}_{k-1}$ via unit weights which, together with c_k's bias, make c_k to fire shortly after all these units are active, i.e. when the simulated counting from 0 to $2^k - 1$ has been accomplished. In addition, unit c_k is connected to a sequence of five auxiliary units a_k, x_k, b_k, d_k, z_k, which are being, one by one, activated after c_k fires (Lemma 3). The purpose of the auxiliary units a_k, b_k, d_k is only to slow down the continuous-time state flow. The unit x_k is used to reset all the lower-order units in $\mathcal{C}_{k-1}$ back to values near 0 after c_k fires (Lemma 2.2b). To achieve this effect, x_k is linked with each $p \in \mathcal{C}_{k-1}$ via a large negative weight $v(x_k, p) = -\lceil v(c_k, p) + \sum_{q \in \mathcal{C}_{k-1};\, v(q,p)>0} v(q, p) \rceil$ that exceeds the mutual positive influence of units in $\mathcal{C}_{k-1} \cup \{c_k\}$. The value of parameter $V_k = 1 - \sum_{p \in \mathcal{C}_{k-1}} v(x_k, p)$ is determined so that the state of x_k is independent of the states of $p \in \mathcal{C}_{k-1}$. Finally, unit z_k balances the negative influence of x_k on $\mathcal{C}_{k-1}$ so that the first k counter bits can again count from 0 to $2^k - 1$ but now with c_k being active. This is achieved by exact weights $v(z_k, p) = -v(x_k, p) - 1$ for $p \in \mathcal{C}_{k-1}$ in which the -1 compensates for $v(c_k, p) = 1$. Clearly, units $p \in \mathcal{C}_{k-1}$ cannot reversely affect z_k since their maximal contribution $\sum_{p \in \mathcal{C}_{k-1}} v(p, z_k) =$

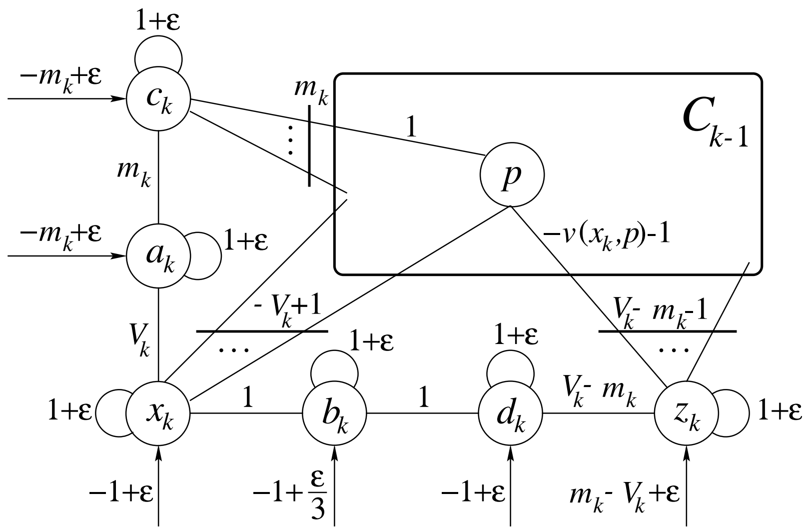

Fig. 1. Inductive construction of $\mathcal{C}_k$

$-m_k - \sum_{p \in \mathcal{C}_{k-1}} v(x_k, p) = V_k - m_k - 1$ to the excitation of z_k cannot overcome its bias. This completes the inductive step of the counter network construction.

Now the correct state evolution of the Hopfield network $\mathcal{C}$ described above needs to be verified. Thus, a sequence of lemmas analyzing the corresponding system (1) is presented. Due to lack of space, the proofs are only sketched here. Lemma 1 first upper bounds the maximum sum of absolute values of weights incident on any unit in $\mathcal{C}$. Lemma 2 then describes explicitly the continuous-time state evolution for saturated units. An analysis of how the decreasing *defects*, i.e. distances from limit values in the states of saturated units, affect the excitation of any other unit reveals that the units in $\mathcal{C}$ actually approximate the discrete update rule of corresponding threshold gates after a certain transient time. The proof of Lemma 2 follows from the dynamics equations (1) and Lemma 1. Furthermore, the transfer of the activity in $\mathcal{C}$ from a unit to a subsequent one, when all the incident units are saturated, will be analyzed explicitly and its duration time will be calculated in Lemma 3. (But note that the analysis for c_0 at $t = 0$ slightly differs.) The result is also generalized to the case when some of the incident units may become unsaturated.

Lemma 1. *For any unit $p \in \mathcal{C}$ in the Hopfield network constructed above, the sum of absolute values of its incident weights (excluding its local bias) is upper bounded by $\Xi_p = \sum_{q=1}^{m} |v(q,p)| < \varepsilon 2^{1/\varepsilon}$.*

Proof. (Sketch.) The maximum value of Ξ_p among $p \in \mathcal{C}$ is reached by unit x_n of the highest order n, that is $\Xi_{x_n} = 2V_n + 1 + \varepsilon$. Parameter $V_n = 2(11 \cdot 7^{n-1} - 5)/3$ is computed by induction on n in which recursive formula $v(x_k, p) = 2v(x_{k-1}, p)$ for $p \in \mathcal{C}_{k-2}$ $(k > 1)$, and Figure 1 are employed. Hence, $\Xi_p \leq 4(11 \cdot 7^{n-1} - 5)/3 + 1 + \varepsilon < \varepsilon 2^{1/\varepsilon}$ by assumptions on ε in Theorem 1. $\square$

Lemma 2.
1. *Let $p \in \mathcal{C}$ be a unit saturated at $b \in \{0,1\}$ with a* defect $\delta_p(t) = |y_p(t) - b|$, *for the duration of a continuous time interval $\tau = [t_0, t_f]$ for some $t_0 \geq 0$. Then the state dynamics of p converging towards value b can be explicitly solved as $y_p(t) = |b - \delta_p e^{-(t-t_0)}|$ for $t \in \tau$, where $\delta_p = \delta_p(t_0)$ is p's initial defect.*
2a. *Let $Q \subseteq \mathcal{C}$ be a subset of units saturated for the duration of time interval $\tau = [t_0, t_f]$. Then the dynamics of $\xi_p(t)$ for any unit $p \in \mathcal{C}$ can be described as*

$$\xi_p(t) = v(0,p) + \sum_{q \in Q;\, \xi_q(t) \geq 1} v(q,p) + \sum_{q \notin Q} v(q,p) y_q(t) + \Delta_{pQ} e^{-(t-t_0)} \qquad (2)$$

for $t \in \tau$, where $\Delta_{pQ} = \sum_{q \in Q;\, \xi_q(t_0) \leq 0} v(q,p)\delta_p - \sum_{q \in Q;\, \xi_q(t_0) \geq 1} v(q,p)\delta_p$ is the initial total weighted defect *of Q affecting $\xi_p(t_0)$.*
2b. *In addition, let $t_f > t_0 + t_1$ where $t_1 = (\ln 2)/\varepsilon$, and assume the respective weights in $\mathcal{C}$ satisfy either $v(0,p) + \sum_{q \in Q;\, \xi_q(t_0) \geq 1} v(q,p) + \sum_{q \notin Q;\, v(q,p) > 0} v(q,p) < -\varepsilon$ or $v(0,p) + \sum_{q \in Q;\, \xi_q(t_0) \geq 1} v(q,p) + \sum_{q \notin Q;\, v(q,p) < 0} v(q,p) > 1 + \varepsilon$. Then p is saturated at either 0 or 1, respectively, for the duration of time interval $[t_0 + t_1, t_f]$.*

Lemma 3.
1. *Consider a situation where a unit $p \in \mathcal{C}$ (e.g. $c_0, c_k, a_k, b_k, d_k, z_k$ for $1 \leq k \leq n$) with fractional part of bias $\varepsilon' \in \{\varepsilon, \varepsilon/3\}$ and feedback weight $v(p,p) = 1 + \varepsilon$ is supposed to activate and transfer a signal to the subsequent unit r (i.e. $c_k, a_k, x_k, d_k, z_k, c_0$, respectively) with bias fraction ε and $v(r,r) = 1 + \varepsilon$ via weight $v(p,r) \geq 1$. Let all the units incident on p, r excluding p, r be saturated for the duration of some sufficiently large time interval $\tau = [t_0, t_f]$ (e.g. $t_f > t_0 + t_2$ where t_2 is defined below), starting at a time $t_0 \geq 0$ when $\xi_p(t_0) = 0$. Assume that the initial defects $\delta_p + \Delta_{rQ} < \varepsilon$ for $Q = \mathcal{C} \setminus \{p\}$ are bounded. Further assume that the respective weights satisfy $v(0,p) + \sum_{q \in Q;\, \xi_q(t_0) \geq 1} v(q,p) = \varepsilon'$ and $v(0,r) + \sum_{q \in Q;\, \xi_q(t_0) \geq 1} v(q,r) = \varepsilon - v(p,r)$. Then p is unsaturated with the state dynamics*

$$y_p(t) = \frac{\varepsilon' \left(e^{\varepsilon(t-t_0)} - 1 \right)}{\varepsilon(1+\varepsilon)} - \frac{\varepsilon' + \Delta_{pQ} e^{-(t-t_0)}}{1+\varepsilon} \qquad (3)$$

exactly for the duration of time interval $(t_0, t_0 + t_1')$, where $t_1' = (\ln(1 + \varepsilon/\varepsilon'))/\varepsilon$ (note $t_1' = t_1$ for $\varepsilon' = \varepsilon$ and $t_1' = 2t_1$ for $\varepsilon' = \varepsilon/3$), while r is saturated at 0. In addition, p is saturated at 1 for the duration of $[t_0 + t_1', t_f]$, while r unsaturates from 0 at time $t_0 + t_2$ where $t_2 = \ln((v(p,r)\delta_p(t_0 + t_1')(1 + \varepsilon/\varepsilon')^{1/\varepsilon} - \Delta_{rQ})/\varepsilon) \geq t_1'$.
2. *Consider a situation in $\mathcal{C}$ where unit x_k ($1 \leq k \leq n$) is supposed to receive a signal from preceding unit a_k, activate itself, and further transfer the signal to subsequent unit b_k while units in $\mathcal{C}_{k-1}$ incident on x_k may unsaturate from 1 after x_k unsaturates from 0. Let all the other units incident on x_k, b_k excluding x_k, b_k and $\mathcal{C}_{k-1}$ be saturated for the duration of a sufficiently large time interval $\tau = [t_0, t_f]$ (e.g. at least until b_k unsaturates from 0) starting at a time $t_0 \geq 0$ when $\xi_{x_k}(t_0) = 0$. Assume that the initial defects meet $\delta_{b_k}, \Delta_{b_k Q'} < \varepsilon 2^{-1/\varepsilon}$ for $Q' = \mathcal{C} \setminus (\mathcal{C}_{k-1} \cup \{x_k\})$, and also $(1+\varepsilon)\delta_{x_k} - \sum_{p \in \mathcal{C}_{k-1}} v(p, x_k)\delta_p \leq \varepsilon 2^{-1/\varepsilon}$ outside*

Q', are bounded. Further, assume that the respective weights satisfy $v(0,x_k) + \sum_{q\in Q';\,\xi_q(t_0)\geq 1} v(q,x_k) + \sum_{p\in\mathcal{C}_{k-1}} v(p,x_k) = \varepsilon$ and $\sum_{q\in Q';\,\xi_q(t_0)\geq 1} v(q,b_k) = 0$. Then x_k saturates at 1 in time at most $t_0 + 2t_1$, remaining then saturated until time at least t_f, and b_k unsaturates from 0 only after x_k is saturated at 1.

Proof. (Sketch.)
1. Excitation $\xi_p(t) = \varepsilon' + (1+\varepsilon)y_p(t) + \Delta_{pQ}e^{-(t-t_0)}$ of p for $t \in [t_0, t_0+t_2]$ is obtained from (2) which determines p's state dynamics (1) by differential equation $(dy_p/dt)(t) = -y_p(t) + \varepsilon' + (1+\varepsilon)y_p(t) + \Delta_{pQ}e^{-(t-t_0)}$ when p is unsaturated. The corresponding initial condition $y_p(t_0) = (-\varepsilon' - \Delta_{pQ})/(1+\varepsilon) = \delta_p$ comes from $\xi_p(t_0) = 0$ which also bounds the initial defect as $-1-\varepsilon-\varepsilon' \leq \Delta_{pQ} \leq -\varepsilon' < 0$, due to $1 \geq \delta_p \geq 0$. Hence solution (3) follows, which provides dynamics $\xi_p(t) = \varepsilon'(e^{\varepsilon(t-t_0)} - 1)/\varepsilon > 0$, ensuring that p is unsaturated exactly for the duration of (t_0, t_0+t_1'), even though its state $y_p(t)$ is initially decreasing for $t \in (t_0, t_0+t_g)$ where $t_g = (\ln(-\Delta_{pQ}/\varepsilon'))/(1+\varepsilon) < t_1'$. Excitation $\xi_r(t) = \varepsilon - v(p,r) + v(p,r)y_p(t) + \Delta_{rQ}e^{-(t-t_0)}$ should prove to be nonpositive for all $t \in (t_0, t_0+t_1')$. By using $v(p,r) \geq 1$, $\delta_p + \Delta_{rQ} < \varepsilon$, and dynamics (3) in which $-\Delta_{pQ} = \varepsilon' + (1+\varepsilon)\delta_p$, this reduces to $\varepsilon(\varepsilon' + \varepsilon(1+\varepsilon))e^{-(t-t_0)} + \varepsilon'(e^{\varepsilon(t-t_0)} - 1) - \varepsilon \leq \varepsilon(\varepsilon' - \varepsilon^2)$. For $t \in [t_0, t_0+t_\varepsilon]$ where $t_\varepsilon = \ln((\varepsilon' + \varepsilon(1+\varepsilon))/(\varepsilon' - \varepsilon^2))$, term $e^{\varepsilon(t-t_0)}$ reaches its maximum at $t_0 + t_\varepsilon$ which implies the underlying inequality. For $t \in [t_0+t_\varepsilon, t_0+t_1']$, term $\varepsilon(\varepsilon' + \varepsilon(1+\varepsilon))e^{-(t-t_0)}$ achieves its maximum $\varepsilon(\varepsilon' - \varepsilon^2)$ at $t_0 + t_\varepsilon$ while $\varepsilon'(e^{\varepsilon(t-t_0)} - 1) - \varepsilon \leq 0$. Hence, r is saturated for the duration of (t_0, t_0+t_1'). Furthermore, $\xi_p(t) = 1 + (\varepsilon + \varepsilon')(1 - e^{-(t-t_0-t_1')}) \geq 1$ of saturated p derived from Lemma 2.1 ensures that p stays saturated at 1 at least for the duration of $[t_0+t_1', t_0+t_2]$, where t_2 comes from $\xi_r(t_0+t_2) = 0$. It must also be checked that $\xi_p(t) = \varepsilon' + 1 + \varepsilon + v(p,r)y_r(t) - (1+\varepsilon)\delta_p(t_0+t_2)e^{-(t-t_0-t_2)} + (\Delta_{pQ} - v(p,r)\delta_r)\,e^{-(t-t_0)} \geq 1$ for all $t \in [t_0+t_2, t_f]$. Here, $v(p,r)y_r(t) > 0$ whereas the respective defect terms having the least value at t_0+t_2 can be lower bounded by $-\varepsilon' - \varepsilon$ when the explicit formulas are substituted for $\delta_p(t_0+t_2)$, t_2, Δ_{pQ}, and inequalities $\delta_p + \Delta_{rQ} < \varepsilon$, $\delta_r \leq 1$, $v(p,r) \geq 1$, $\varepsilon' \geq \varepsilon/3$ are applied.
2. Notice that unit a_k saturates at 1 before x_k is unsaturated from 0 according to Lemma 3.1. Excitation $\xi_{x_k}(t) \geq \varepsilon + (1+\varepsilon)y_{x_k}(t) + \Delta_{x_kQ'}\,e^{-(t-t_0)}$ of x_k for $t \in \tau$ is lower bounded from formula (2) and $v(p,x_k) < 0$ for all $p \in \mathcal{C}_{k-1}$, which gives $(dy_{x_k}/dt)(t) \geq \varepsilon y_{x_k}(t) + \varepsilon + \Delta_{x_kQ'}\,e^{-(t-t_0)}$ for x_k unsaturated, according to (1). In the beginning of interval τ, state $y_{x_k}(t)$ is determined by (3) before the first $p \in \mathcal{C}_{k-1}$ unsaturates, since the assumption of Lemma 3.1 concerning the weights incident on x_k coincides with that of Lemma 3.2 due to $\varepsilon' = \varepsilon$ and $\xi_p(t_0) \geq 1$ for all $p \in \mathcal{C}_{k-1}$. Hence, $\Delta_{x_kQ'}$ can appropriately be expressed in terms of $\Delta_{x_kQ} = -\varepsilon - (1+\varepsilon)\delta_{x_k}$ for $Q = \mathcal{C} \setminus \{x_k\}$ from Lemma 3.1 so that the bound assumed on the initial defect outside Q' can be used to lower bound $\Delta_{x_kQ'} \geq -\varepsilon(1 + 2^{-1/\varepsilon})$ which gives $(dy_{x_k}/dt)(t) \geq \varepsilon y_{x_k}(t) + \varepsilon - \varepsilon(1+2^{-1/\varepsilon})e^{-(t-t_0)}$. It follows that $(dy_{x_k}/dt)(t) \geq \varepsilon - \varepsilon^2 > 0$ for $t \geq t_0 + t_d$ where $t_d = \ln((1+2^{-1/\varepsilon})/\varepsilon)$, provided that x_k is still unsaturated. This implies that $y_{x_k}(t)$ grows at least as fast as the straight line with equation $(\varepsilon - \varepsilon^2)(t - t_0 - t_d) - y = 0$ until x_k saturates at 1. Thus, x_k saturates at 1 certainly before $t_0 + t_d + t_s < t_0 + 2t_1$ where $t_s = 1/(\varepsilon - \varepsilon^2)$ because $\xi_{x_k}(t) > y_{x_k}(t)$ from (1) due to its state derivative is positive for $t \geq t_0 + t_d$.

Similarly, $\xi_{b_k}(t) = -1 + \varepsilon/3 + y_{x_k}(t) + \Delta_{b_k Q'}\, e^{-(t-t_0)}$ for b_k saturated at 0. Let $t_y > 0$ be the least local time instant at which $y_{x_k}(t_0+t_y) = 1-\varepsilon/3-\Delta_{b_k Q'}\, e^{-t_y}$ when b_k is still saturated at 0 since $\xi_{b_k}(t_0+t_y) = 0$. Excitation $\xi_{x_k}(t_0+t_y) \geq \varepsilon + (1+\varepsilon)(1-\varepsilon/3-\Delta_{b_k Q'}\, e^{-t_y}) + \Delta_{x_k Q'}\, e^{-t_y}$ of x_k at t_0+t_y can be lower bounded by 1 from $\Delta_{x_k Q'} \geq -\varepsilon(1+2^{-1/\varepsilon})$ and $\Delta_{b_k Q'} < \varepsilon 2^{-1/\varepsilon}$, ensuring x_k is already saturated at 1 at t_0+t_y. Finally, it must be checked that $\xi_{x_k}(t) \geq \varepsilon+(1+\varepsilon)(1-(\varepsilon/3+\Delta_{b_k Q'}\, e^{-t_y})e^{-(t-t_0-t_y)})+y_{b_k}(t)+(\Delta_{x_k Q'}-\delta_{b_k})e^{-(t-t_0)} \geq 1$ for all $t \in [t_0+t_y, t_f]$ when b_k may unsaturate, which follows from $y_{b_k}(t) \geq 0$ and the respective defect bounds for δ_{b_k}, $\Delta_{b_k Q'}$, and $\Delta_{x_k Q'}$. $\square$

The correct timing of the counter simulation must ensure a sufficiently fast decrease of the defects as assumed in Lemma 3. According to Lemma 2.2b, the absolute value of the total weighted defect affecting any unit in $\mathcal{C}$ is bounded by ε after time t_1, decreasing further to $\varepsilon 2^{-1/\varepsilon}$ by time $2t_1$. On the other hand, t_1 lower bounds the time necessary for activating unit p in Lemma 3.1. Hence, the subsequent unit r has always time at least t_1 for decreasing the defect induced by its incident saturated units below ε even before unit p starts its activation. Similarly, the stronger defect bounds in Lemma 3.2 are met since time $2t_1$ is guaranteed before unit x_k unsaturates. The lower bound $\Omega(2^n/\varepsilon) = \Omega(2^{m/6}/\varepsilon)$ on the total simulation time follows immediately from the previous time analysis.

From the proof of Lemma 1, the maximum integer weight parameter in $\mathcal{C}$ is of order $2^{O(m)}$. This corresponds to $O(m)$ bits per weight that is repeated $O(m^2)$ times, and thus yields at most $O(m^3)$ bits in the representation. In addition, the biases and feedbacks of the m units include fraction ε (or $\varepsilon/3$), and taking this into account requires $\Theta(m \log(1/\varepsilon))$ additional bits, say at least $\kappa m \log(1/\varepsilon)$ bits for some constant $\kappa > 0$. By choosing $\varepsilon = 2^{-f(m)/(\kappa m)}$ in which f is a continuous increasing function whose inverse is defined as $f^{-1}(\mu) = \mu/g(\mu)$, where g satisfies $g(\mu) = \Omega(\mu^{2/3})$ (implying $f(m) = \Omega(m^3)$) and $g(\mu) = o(\mu)$, it follows that $M = \Theta(f(m))$, especially $M \geq f(m)$ from $M \geq \kappa m \log(1/\varepsilon)$. The convergence time $\Omega(2^{m/6}/\varepsilon)$ can be translated to $\Omega(2^{f(m)/(\kappa m)+m/6}) = 2^{\Omega(f(m)/m)}$ which can be rewritten as $2^{\Omega(M/f^{-1}(M))} = 2^{\Omega(g(M))}$ since $f(m) = \Omega(M)$ from $M = \Theta(f(m))$ and $f^{-1}(M) \geq m$ from $M \geq f(m)$. This completes the proof of the theorem. $\square$

3 A Simulation Example

A computer program HCOUNT has been created to automate the construction from Theorem 1. For input $n \geq 0$, the program generates system (1) describing the Hopfield net dynamics in the form of a FORTRAN subroutine corresponding to the $(n+1)$-bit binary counter to be simulated. This FORTRAN procedure is then presented to a solver from the NAG library that provides a numerical solution for the system. For example, implementing a 4-bit counter on the HCOUNT generator results in a continuous-time symmetric Hopfield net $\mathcal{C}_3$ with 19 variables. Figure 2 shows the state evolution of counter units c_0, c_1, c_2, c_3 for a period of $2^3 - 1 = 7$ simulated discrete steps confirming the correctness of the construction. A parameter value of $\varepsilon = 0.1$ was used in this numerical simulation, showing that the theoretical estimate of ε in Theorem 1 is actually quite conservative.

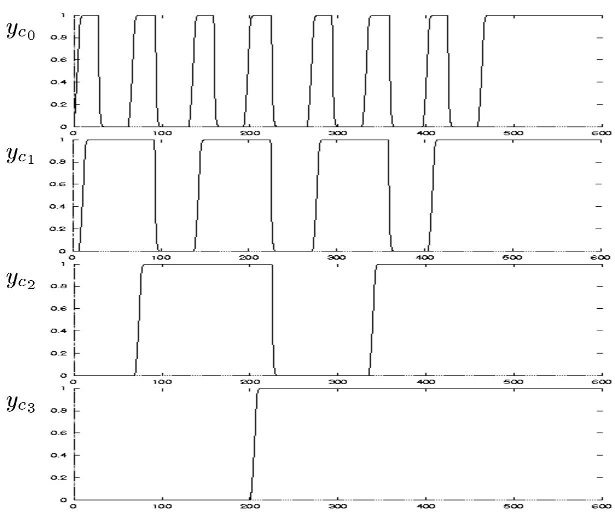

Fig. 2. Continuous-time simulation of 4-bit binary counter for $\varepsilon = 0.1$

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