Optimal Triangle Stripifications as Minimum Energy States in Hopfield Nets

Jiří Šíma *

Institute of Computer Science, Academy of Sciences of the Czech Republic,
P. O. Box 5, 18207 Prague 8, Czech Republic, sima@cs.cas.cz

Abstract. The important task of generating the minimum number of sequential triangle strips for a given triangulated surface model is motived by applications in computer graphics. This hard combinatorial optimization problem is reduced to the minimum energy problem in Hopfield nets by a linear-size construction. First practical experiments have confirmed that computing the semi-optimal stripifications by using Hopfield nets is a promising approach. In this work we provide a theoretical justification of this method by proving that the classes of equivalent optimal stripifications are mapped one to one to the minimum energy states.

1 Sequential Triangular Strips and Hopfield Nets

Piecewise-linear surfaces defined by sets of triangles (triangulation) are widely used representations for geometric models. Computing a succinct encoding of a triangulated surface model represents an important problem in graphics and visualization. Current 3D graphics rendering hardware often faces a memory bus bandwidth bottleneck in the processor-to-graphics pipeline. Apart from reducing the number of triangles that must be transmitted it is also important to encode the triangulated surface efficiently. A common encoding scheme is based on sequential triangle strips which avoid repeating the vertex coordinates of shared triangle edges. Triangle strips are supported by several graphics libraries (e.g. IGL, PHIGS, Inventor, OpenGL). A *sequential triangle strip* (*tristrip*) of length $m-2$ is an ordered sequence of $m \geq 3$ vertices $\sigma = (v_1, \ldots, v_m)$ which encodes the set of $n(\sigma) = m-2$ different triangles $T_\sigma = \{\{v_p, v_{p+1}, v_{p+2}\}\,;\, 1 \leq p \leq m-2\}$ so that their shared edges follow alternating left and right turns as indicated in Figure 1.a (*dashed line*). Thus a triangulated surface model T with n triangles that is decomposed into k tristrips $\Sigma = \{\sigma_1, \ldots, \sigma_k\}$ requires only $n + 2k$ (rather than $3n$) vertices to be transmitted. A crucial problem is to decompose a triangulation into the fewest tristrips which is NP-complete [2].

A new method of generating tristrips Σ for a given triangulation T with n triangles has recently been proposed [7] which is based on a linear-time reduction to the minimum energy problem in Hopfield network $\mathcal{H}_T$ having $O(n)$ units and connections. First practical experiments [5, 7] have confirmed that $\mathcal{H}_T$ powered

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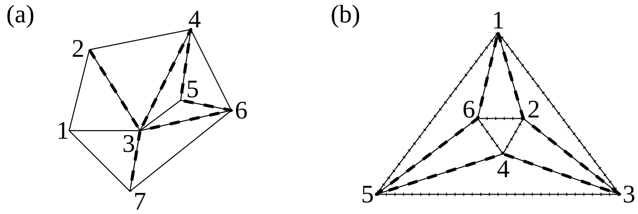

Fig. 1. (a) Tristrip (1,2,3,4,5,6,3,7,1) (b) Sequential cycle (1,2,3,4,5,6,1,2)

by simulated annealing (i.e. Boltzmann machine) can be used for computing the semi-optimal stripifications offline. For example, for "grid" models composed of 2312 triangles this method produced 297 tristrips on the average within 1–2 minutes as compared to 373 tristrips generated on-line by a leading stripification program FTSG [7]. In this paper the correctness of this reduction is formally shown providing a theoretical justification of the method. We prove a one-to-one correspondence between the classes of equivalent optimal stripifications of T and the minimum energy states reached by $\mathcal{H}_T$ during any sequential computation starting at the zero state (or $\mathcal{H}_T$ can be initialized arbitrarily if one asymmetric weight is introduced [7]). This provides another NP-completeness proof for the minimum energy problem in Hopfield nets [1, 8]. A possible direction of further research is a generalization of the method for tristrips with zero-area triangles [2].

Hopfield networks [3] represent a very influential associative memory model which is connected to the much-studied Ising spin glass model in statistical physics [1]. Part of the appeal of Hopfield nets also stems from their natural hardware implementations using electrical networks or optical computers, and their application in combinatorial optimization [4]. A Hopfield network is composed of s *units* (*neurons*), indexed as $1, \ldots, s$, that are connected into undirected graph, in which each connection between unit i and j is labeled with an integer *symmetric weight* $w(i,j) = w(i,j)$. The absence of a connection indicates a zero weight, and vice versa. Further assume $w(j,j) = 0$ for $j = 1, \ldots, s$. The network *state* $\mathbf{y}^{(t)} = (y_1^{(t)}, \ldots, y_s^{(t)}) \in \{0,1\}^s$ is updated at time instants $t = 0, 1, \ldots$. The initial state $\mathbf{y}^{(0)}$ may be chosen arbitrarily, e.g. $\mathbf{y}^{(0)} = (0, \ldots, 0)$. At discrete time $t \geq 0$, the *excitation* of any neuron j is defined as $\xi_j^{(t)} = \sum_{i=1}^{s} w(i,j) y_i^{(t)} - h(j)$ including an integer *threshold* $h(j)$ local to unit j. At the next instant $t+1$, one (e.g. randomly) selected neuron j (*sequential* mode) computes its new output $y_j^{(t+1)} = H(\xi_j^{(t)})$ by applying the Heaviside activation function H, that is, j is *active* when $H(\xi) = 1$ for $\xi \geq 0$ while j is *passive* when $H(\xi) = 0$ for $\xi < 0$. For the remaining units $y_i^{(t+1)} = y_i^{(t)}$, $i \neq j$. A Hopfield net reaches a *stable state* $\mathbf{y}^{(t^*)}$ at time $t^* \geq 0$ if $\mathbf{y}^{(t^*)} = \mathbf{y}^{(t^*+1)}$ provided that any unit in the network is updated. The fundamental property of a symmetric Hopfield net is that its dynamics is constrained by *energy* function $E(\mathbf{y}) = -\frac{1}{2} \sum_{j=1}^{s} \sum_{i=1}^{s} w(i,j) y_i y_j + \sum_{j=1}^{s} h(j) y_j$ which is a bounded function defined on its state space whose value decreases along any nonconstant computation path ($\xi_j^{(t)} \neq 0$ is assumed without loss of generality). It follows that starting from any state the network converges towards some stable state corresponding to a local minimum of E [3]. Thus the cost function of a hard combinatorial optimization problem can be encoded into the energy function of a Hopfield net which is then minimized in the course of computation. Hence, the *minimum energy problem* of finding a network state with minimum energy is of special interest which is NP-complete [1, 8].

2 The Reduction

Let T be a set of n triangles that represents a triangulated surface model homeomorphic to a sphere in which each edge is incident to at most two triangles. An edge is said to be *internal* if it is shared by exactly two triangles; otherwise it is a *boundary* edge. Denote by I and B the sets of internal and boundary edges, respectively. A *sequential cycle* is a "cycled tristrip" which is an ordered sequence of vertices $C = (v_1, \ldots, v_m)$ where $m \geq 4$ is even, encoding the set of $m - 2$ different triangles $T_C = \{\{v_p, v_{p+1}, v_{p+2}\}\,;\, 1 \leq p \leq m-2\}$ so that $v_{m-1} = v_1$ and $v_m = v_2$. Also denote by I_C and B_C the sets of internal and boundary edges of C, respectively, that is, $I_C = \{\{v_p, v_{p+1}\}\,;\, 1 \leq p \leq m-2\}$ and $B_C = \{\{v_p, v_{p+2}\}\,;\, 1 \leq p \leq m-2\}$. An example of the sequential cycle is depicted in Figure 1.b where its internal (*dashed line*) and boundary (*dotted line*) edges are indicated. Let $\mathcal{C}$ be the set of all sequential cycles in T. For each $C \in \mathcal{C}$ one unique *representative* internal edge $e_C \in I_C$ can be chosen as follows. Start with any $C \in \mathcal{C}$ and choose any edge from I_C to be its representative edge e_C. Observe that for a fixed orientation of triangulated surface any internal edge follows either left or right turn corresponding to at most two sequential cycles. Denote by C' the sequential cycle having no representative edge so far which shares its edge $e_C \in I_C \cap I_{C'}$ with C if such C' exists; otherwise let C' be any sequential cycle with no representative internal edge or stop if all the sequential cycles do have their representative edges. Further choose any edge from $I_{C'} \setminus \{e_C\}$ to be the representative edge $e_{C'}$ of C' and repeat the previous step with C replaced by C'. Clearly, each edge represents at most one cycle because set $I_{C'} \setminus \{e_C\} \neq \emptyset$ always contains only edges that do not represent any cycle so far. If it were not the case then another sequential cycle C'' different from C would obtain its representative edge $e_{C''}$ from $I_{C'} \cap I_{C''}$ and hence a representative edge would already be assigned to C' before C is considered.

Hopfield network $\mathcal{H}_T$ corresponding to T will now be constructed. With each internal edge $e = \{v_1, v_2\} \in I$ two neurons ℓ_e and r_e are associated whose states either $y_{\ell_e} = 1$ or $y_{r_e} = 1$ indicate that e follows the left or right turn, respectively, along a tristrip. Let $L_e = \{e, e_1, e_2, e_3, e_4\}$ with $e_1 = \{v_1, v_3\}$, $e_2 = \{v_2, v_3\}$, $e_3 = \{v_2, v_4\}$, and $e_4 = \{v_1, v_4\}$ be the set of edges of the two triangles $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_4\}$ that share edge e. Denote by $J_e = \{\ell_f, r_f\,;\, f \in L_e \cap I\}$ the set of corresponding neurons. Unit ℓ_e is connected with all neurons from J_e via weights $w(i, \ell_e) = -7$ for $i \in J_{\ell_e} = J_e \setminus \{r_{e_2}, \ell_e, r_{e_4}\}$ except for units r_{e_2} (if $e_2 \in I$), ℓ_e, and r_{e_4} (if $e_4 \in I$) whose states may encode a tristrip that traverses edge e by the left turn. Such a situation (for $L_e \subseteq I$) is depicted in

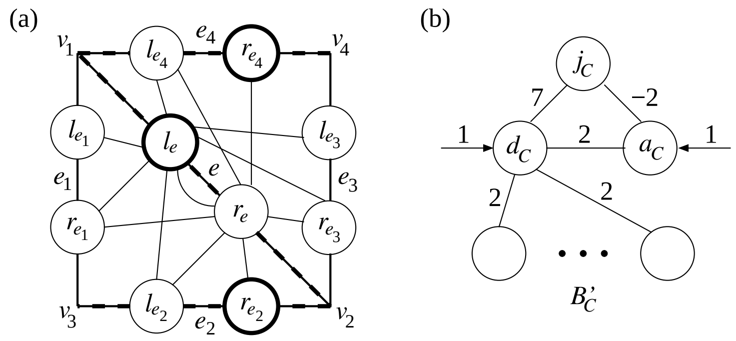

Fig. 2. The construction of Hopfield network $\mathcal{H}_T$

Figure 2.a where the tristrip together with associated active neurons r_{e_2}, ℓ_e, r_{e_4} are marked. Similarly, unit r_e is connected with $i \in J_{r_e} = J_e \setminus \{\ell_{e_1}, r_e, \ell_{e_3}\}$ via $w(i, r_e) = -7$ corresponding to the right turn. Hence, the states of $\mathcal{H}_T$ with these negative weights that enforce locally the alternation of left and right turns encode tristrips. For each e_C, $C \in \mathcal{C}$, define $j_C = \ell_{e_C}$ if e_C follows the left turn along C or $j_C = r_{e_C}$ if e_C follows the right turn along C. Denote $J = \{j_C\,;\, C \in \mathcal{C}\}$ and $J' = \{\ell_e, r_e \notin J\,;\, e \in I\}$. Define the thresholds of underlying neurons as $h(j) = -5 + 2b_{e(j)}$ for $j \in J'$ and $h(j) = 1 + 2b_{e(j)}$ for $j \in J$ where $e(j) = e$ for $j \in \{\ell_e, r_e\}$ and $b_e = |\{C \in \mathcal{C}\,;\, e \in B'_C\}| \leq 2$ for $B'_C = B_C \setminus L_{e_C}$.

Hopfield network $\mathcal{H}_T$ must avoid the states encoding cycled strips of triangles around sequential cycles [2] which would have less energy E than those encoding the optimal stripifications. Thus two auxiliary neurons d_C, a_C are introduced for each $C \in \mathcal{C}$. Unit d_C computes the disjunction of outputs from all neurons associated with boundary edges B'_C of C which, being active, enables the activation of unit j_C associated with e_C. Hence, any tristrip may pass through e_C along the direction of C only if a boundary edge of C is a part of another tristrip crossing C. This ensures that the states of $\mathcal{H}_T$ do not encode sequential cycles. Unit a_C balances the contribution of d_C to energy E when j_C is passive. Figure 2.b defines the weights and thresholds of d_C, a_C for $C \in \mathcal{C}$. This completes the construction of $\mathcal{H}_T$. Observe that the number of units $s = 2|I| + 2|\mathcal{C}|$ (similarly the number of connections) in $\mathcal{H}_T$ is linear in terms of $n = |T|$ because $|\mathcal{C}| \leq 2|I| = O(n)$ since each internal edge can belong to at most two cycles.

3 The Correctness

Let $\mathcal{S}_T$ be the set of optimal stripifications with the minimum number of tristrips for T. Define $\Sigma \in \mathcal{S}_T$ is *equivalent* with $\Sigma' \in \mathcal{S}_T$, i.e. $\Sigma \sim \Sigma'$ iff $\{T_\sigma\,;\, \sigma \in \Sigma\} = \{T_{\sigma'}\,;\, \sigma' \in \Sigma'\}$. Two equivalent optimal stripifications may differ in a tristrip σ encoding triangles $T_\sigma = T_C$ of sequential cycle C that is split at two different positions. Let $[\Sigma]_\sim = \{\Sigma' \in \mathcal{S}_T\,;\, \Sigma' \sim \Sigma\}$ and denote by $\mathcal{S}_T/_\sim = \{[\Sigma]_\sim\,;\, \Sigma \in \mathcal{S}_T\}$ the partition of $\mathcal{S}_T$ into equivalence classes.

Theorem 1. *Let $\mathcal{H}_T$ be a Hopfield network corresponding to triangulation T with n triangles and denote by $Y^* \subseteq \{0,1\}^s$ the set of stable states that can be reached during any sequential computation by $\mathcal{H}_T$ starting at the zero state. Then each state $\mathbf{y} \in Y^*$ encodes a correct stripification $\Sigma_{\mathbf{y}}$ of T into k tristrips and has energy*

$$E(\mathbf{y}) = 5(k-n)\,. \tag{1}$$

In addition, there is a one-to-one correspondence between the classes of equivalent optimal stripifications $[\Sigma]_\sim \in \mathcal{S}_T/_\sim$ having the minimum number of tristrips for T and the states in Y^* with minimum energy $\min_{\mathbf{y}\in Y^*} E(\mathbf{y})$.

Proof. Stripification $\Sigma_{\mathbf{y}}$ is decoded from $\mathbf{y} \in Y^*$ as follows. Denote $I_0 = \{e \in I\,;\, y_{\ell_e} = y_{r_e} = 0\}$ and let $I_1 = I \setminus I_0$. Set $\Sigma_{\mathbf{y}}$ contains each ordered sequence $\sigma = (v_1, \ldots, v_m)$ of $m \geq 3$ vertices that encodes $n(\sigma) = m-2$ different triangles $\{v_p, v_{p+1}, v_{p+2}\} \in T$ for $1 \leq p \leq m-2$ such that their edges $e_0 = \{v_1, v_3\}$, $e_m = \{v_{m-2}, v_m\}$, and $e_p = \{v_p, v_{p+1}\}$ for $1 \leq p \leq m-1$ satisfy $e_0, e_1, e_{m-1}, e_m \in I_0 \cup B$ and $e_2, \ldots, e_{m-2} \in I_1$. We will prove that $\Sigma_{\mathbf{y}}$ is a correct stripification of T. Observe first that every $j \in J \cup J'$ is passive if there is active $i \in J_j$. This ensures that each $\sigma \in \Sigma_{\mathbf{y}}$ encodes a set T_σ of different triangles whose shared edges follow alternating left and right turns and that sets T_σ, $\sigma \in \Sigma_{\mathbf{y}}$, are pairwise disjoint. In particular, for each $j \in J \cup J'$ the number of weights 2 from some d_C contributing to ξ_j is at most $b_{e(j)} \leq 2$ which are subtracted within threshold $h(j)$. Hence, if all $i \in J_j$ are passive then $\xi_j \leq 5$ for $j \in J'$, and $\xi_j \leq 6$ for $j \in J$ which may include weight 7 from d_C. Thus, active $i \in J_j$ contributing to ξ_j via weight -7 makes j passive due to $\mathbf{y}$ is a stable state. Further, we check $\bigcup_{\sigma\in\Sigma_{\mathbf{y}}} T_\sigma = T$. It suffice to prove that there is no $C = (v_1, \ldots, v_m) \in \mathcal{C}$ such that $e_p = \{v_p, v_{p+1}\} \in I_1$ for all $p = 1, \ldots, m-2$. Suppose that such C exists which implies $B_C \cap I \subseteq I_0$. Then $j_C \in J$ associated with e_C could not be activated during sequential computation of $\mathcal{H}_T$ starting at the zero state since $h(j_C)$ can be reached only by weight 7 from d_C which computes the disjunction of outputs from units i for $e(i) \in B'_C \cap I \subseteq I_0$. Hence, $e_C \in I_0$ which is a contradiction. This completes the argument for $\Sigma_{\mathbf{y}}$ to be a correct stripification of T.

Let $\Sigma_{\mathbf{y}}$ contain k tristrips. Each $\sigma \in \Sigma_{\mathbf{y}}$ is encoded using $n(\sigma)-1$ edges from I_1. The number of active units in $J' \cup J$ equals $|I_1| = \sum_{\sigma\in\Sigma_{\mathbf{y}}} (n(\sigma)-1) = n-k$. We will prove that each active $j \in J' \cup J$ is accompanied with a contribution of -5 to energy E which gives (1). Active $j \in J' \cup J$ makes all $i \in J_j$ passive. Neuron j is also connected to $b_{e(j)}$ active units d_C for $e(j) \in B'_C$ computing the disjunctions that include active j. Thus active $j \in J'$ produces contribution $-\frac{1}{2} b_{e(j)} w(d_C, j) - \frac{1}{2} b_{e(j)} w(j, d_C) + h(j) = -b_{e(j)} w(d_C, j) + h(j) = -5$ to E. Similarly, active $j_C \in J$ assumes active d_C and makes a_C passive, which contributes $-b_{e(j_C)} w(j_C, d_C) - w(d_C, j_C) + h(j_C) + h(d_C) = -5$ to E. Unit a_C balances the contribution of active d_C to E when j_C is passive, that is, $-w(a_C, d_C) + h(d_C) + h(a_C) = 0$.

Optimal stripification $\Sigma \in \mathcal{S}_T$ is encoded by state $\mathbf{y}$ of $\mathcal{H}_T$ so that $\Sigma \in [\Sigma_{\mathbf{y}}]_\sim$. Equivalent $\Sigma' \sim \Sigma$ is used to determine $\mathbf{y}$ such that $\Sigma_{\mathbf{y}} = \Sigma'$. Each $\sigma \in \Sigma$ encoding triangles $T_\sigma = T_C$ of some $C \in \mathcal{C}$ is replaced with $\sigma' = (v_1, \ldots, v_m) \in \Sigma'$ having $T_{\sigma'} = T_\sigma$ so that σ' starts with representative $e_C = \{v_1, v_2\}$. Let ℓ_e

or r_e from $J' \cup J$ be active iff there is $\sigma = (v_1, \ldots, v_m) \in \Sigma'$ such that its edge $e = \{v_p, v_{p+1}\}$ for some $2 \leq p \leq m-2$ follows the left or right turn, respectively. Further, unit d_C for $C \in \mathcal{C}$ is active iff there is active $i \in J' \cup J$ for $e(i) \in B'_C$ while a_C is active iff d_C is active and j_C is passive. It follows that $\mathbf{y}$ is a stable state of $\mathcal{H}_T$. It must still be proven that $\mathbf{y} \in Y^*$.

Define a directed graph $\mathcal{G} = (\mathcal{C}, \mathcal{A})$ where $(C_1, C_2) \in \mathcal{A}$ is an edge of $\mathcal{G}$ iff $e_{C_1} \in B'_{C_2}$. Let $\mathcal{C}'$ be a subset of all vertices $C \in \mathcal{C}$ with $y_{j_C} = 1$ that create directed cycles in $\mathcal{G}$. Suppose that units i are passive for all $e(i) \in \bigcup_{C \in \mathcal{C}'} B'_C \setminus E_{\mathcal{C}'}$ where $E_{\mathcal{C}'} = \{e_C\,;\, C \in \mathcal{C}'\}$. For each $C \in \mathcal{C}'$ units i for $e(i) \in B_C \cap L_{e_C}$ are also passive due to active j_C. It seems that such a stable state $\mathbf{y}$ could not be reached during any sequential computation by $\mathcal{H}_T$ starting at the zero state since j_C, $C \in \mathcal{C}'$, can be activated only by d_C, $C \in \mathcal{C}'$, which can be activated only by active j_C, $C \in \mathcal{C}'$. Since $\Sigma_{\mathbf{y}} \in \mathcal{S}_T$ the underlying tristrips follow internal edges of $C \in \mathcal{C}'$ as much as possible being interrupted only by edges from $\bigcup_{C \in \mathcal{C}'} B_C \setminus E_{\mathcal{C}'}$. Any $\sigma \in \Sigma_{\mathbf{y}}$ crossing some $C_1 \in \mathcal{C}'$ (i.e., $\emptyset \neq T_\sigma \cap T_{C_1} \neq T_{C_1}$) has one its end within C_1 because σ may enter C_1 only through $e_{C_2} \in B_{C_1}$ (i.e., $y_{j_{C_2}} = 1$) which is the only representative edge of $C_2 \in \mathcal{C}'$ that σ follows. It can be proven [6] that there exists $C \in \mathcal{C}'$ containing two $\sigma_1, \sigma_2 \in \Sigma_{\mathbf{y}}$ such that $T_{\sigma_1} \subseteq T_C$ and $T_{\sigma_2} \subseteq T_C$. Hence, $\Sigma'_{\mathbf{y}}$ with fewer tristrips can be constructed from $\Sigma_{\mathbf{y}}$ by introducing only one $\sigma^* \in \Sigma'_{\mathbf{y}}$ such that $T_{\sigma^*} = T_C$ (e.g. $y_{j_C} = 0$) instead of the two $\sigma_1, \sigma_2 \in \Sigma_{\mathbf{y}}$ while any $\sigma \in \Sigma_{\mathbf{y}}$ crossing and thus ending within C is shortened to $\sigma' \in \Sigma'_{\mathbf{y}}$ so that $T_{\sigma'} \cap T_C = \emptyset$ which does not increase the number tristrips. This is a contradiction with $\Sigma_{\mathbf{y}} \in \mathcal{S}_T$, and hence $\mathbf{y} \in Y^*$. Obviously, $[\Sigma_{\mathbf{y}}]_\sim$ with the minimum number of tristrips corresponds uniquely to $\mathbf{y} \in Y^*$ having minimum energy $\min_{\mathbf{y} \in Y^*} E(\mathbf{y})$ according to equation (1). $\square$

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