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DOCTORAL THESIS

Mgr. Jan Hladký, Ph.D.

Structural Graph Theory

Computer Science Institute of Charles University

Supervisor of the doctoral thesis:	doc. RNDr. Daniel Kráľ, Ph.D.
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This thesis deals with the Loebl-Komlós-Sós Conjecture, a problem in structural graph theory. I have enjoyed the last four years working on the project with Diana Piguet, Miki Simonovits, Maya Stein, and Endre Szemerédi. My collaboration with Diana has been particularly joyful. It has been a great experience, with each collaborator in quite a different way. I thank Maxim Sviridenko for discussions on the algorithmic aspects of some problems which appear in the thesis. My thanks also go to Dan Král who agreed to supervise this thesis. Even though Dan was not directly involved in the subject of the thesis he was immensely helpful in other aspects of my studies.

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Due to personal and professional circumstances it so happened that I was on leave from Prague for the entire four years of my studies. Anusch Taraz of Technical University Munich hosted me during my first year and I had a lot of fun talking maths with Julia Böttcher and Andreas Würfl, research students of his at that time. Then I was based for two years at the Department of Computer Science at the University of Warwick, where I also obtained my first PhD degree in 2011 under the supervision of Artur Czumaj. Afterwards, I moved from Computer Science to Maths and have stayed there since. My time at Warwick has been particularly enjoyable thanks to my colleagues Anna and Michal Adamaszek, Peter Allen, Amin Coja-Oghlan, Demetres Christofides, Charis Eftymiou, András Máthé, Oleg Pikhurko, and Juraj Stacho.

I spent extended time periods at the School of Mathematics of the University of Birmingham, and at the Center for Mathematical Modelling, Universidad de Chile while working on this thesis.

V Birminghamu, Spojené království, 10. září 2012
(Birmingham, United Kingdom, 10th September 2012)

Prohlášení (Declaration)

Prohlašuji, že jsem tuto disertační práci vypracoval samostatně a výhradně s použitím citovaných pramenů, literatury a dalších odborných zdrojů. Výsledky v druhé kapitole této práce jsou založeny na výzkumu s následujícími spolupracovníky: János Komlós, Diana Piguet, Miklós Simonovits, Maya J. Stein, Endre Szemerédi.

(I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources. Results in Chapter II of this thesis are based on research with János Komlós, Diana Piguet, Miklós Simonovits, Maya J. Stein, and Endre Szemerédi.)

Mgr. Jan Hladký, Ph.D.

Název práce: Strukturální teorie grafů

Autor: Mgr. Jan Hladký, Ph.D.

Pracoviště: Informatický ústav Univerzity Karlovy

Vedoucí disertační práce: doc. RNDr. Daniel Král', Ph.D.

Abstrakt: V práci se zabýváme domněnkou Loebla, Komlóse a Sósové, která je klasickým problémem extremální teorie grafů. Dokážeme následující slabou verzi domněnky: pro libovolné $\alpha > 0$ existuje číslo k_0 takové, že pro každé $k > k_0$ a každý n-vrcholový graf G obsahující alespoň $(\frac{1}{2}+\alpha)n$ vrcholů stupně alespoň $(1+\alpha)k$ platí, že G obsahuje každý strom T na k vrcholech jako podgraf.

Důkaz tohoto výsledku sleduje strategii běžnou v přístupech využívajících Szemerédiho regularity lemma: nejdřív je graf G rozložen a v tomto rozkladu je nalezena kombinatorická struktura s vhodnými vlastnostmi. V posledním kroku je strom T vnořen do G pomocí této struktury. Rozklad zaručený původním regularity lemmatem je ovšem triviálni pokud je G řídký. Abychom obešli toto omezení, vyvineme rozkladovou techniku která umožňuje postihnout i strukturu řídkých grafů: každý graf může být rozložen do vrcholů s velkým stupněm, regulárních párů (ve smyslu regularity lemmatu) a dvou dalších částí, které mají jisté expandující vlastnosti.

Výsledky v této práci byly dosaženy s následujícími spolupracovníky: János Komlós, Diana Piguet, Miklós Simonovits, Maya Jakobine Stein, Endre Szemerédi.

Klíčová slova: extremální teorie grafů; regularity lemma; domněnka Loebla, Komlóse a Sósové

Title: Structural Graph Theory

Author: Mgr. Jan Hladký, Ph.D.

Department: Computer Science Institute of Charles University

Supervisor: doc. RNDr. Daniel Král', Ph.D.

Abstract: In the thesis we make progress on the Loebl-Komlós-Sós Conjecture which is a classic problem in the field of Extremal Graph Theory. We prove the following weaker version of the Conjecture: For every $\alpha > 0$ there exists a number k_0 such that for every $k > k_0$ we have that every n-vertex graph G with at least $(\frac{1}{2} + \alpha)n$ vertices of degrees at least $(1 + \alpha)k$ contains each tree T of order k as a subgraph.

The proof of our result follows a strategy common to approaches which employ the Szemerédi Regularity Lemma: the graph G is decomposed, a suitable combinatorial structure inside the decomposition is found, and then the tree T is embedded into G using this structure. However the decomposition given by the Regularity Lemma is not of help when G sparse. To surmount this shortcoming we develop a decomposition technique that applies also to sparse graphs: each graph can be decomposed into vertices of huge degrees, regular pairs (in the sense of the Regularity Lemma), and two other components each exhibiting certain expander-like properties.

The results were achieved in a joint work with János Komlós, Diana Piguet, Miklós Simonovits, Maya Jakobine Stein and Endre Szemerédi.

Keywords: Extremal Graph Theory; Regularity Lemma; Loebl-Komlós-Sós Conjecture

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Chapter I

Graph theory: a brief introduction

Finite graphs are one of the simplest mathematical structures. For this reason there had been many graph-theoretic problems people had been puzzled with much before any systematic study of the graph theory itself. The notorious examples of such problems are Leonhard Euler's 1735 Königsberg Bridges Problem, and the Four-Colour Problem, originally posed by Francis Guthrie in 1852 as a problem of colouring the map of the counties of England. (While Euler himself found a simple but ingenious solution to the former, the latter needed more than 100 years and much developments in graph theory to be resolved in two steps in 1976 by Appel and Haken [AH89] and in 1997 by Robertson, Sanders, Seymour and Thomas [RSST97].) Other notable early works include studies of Thomas Kirkman and William Hamilton on cycles on polyhedra[i], Gustav Kirchhoff's circuit laws[ii], and Arthur Cayley's and James Sylvester's studies which had links to theoretical chemistry and to the structure of molecules in particular. It was Sylvester in 1878 [Syl78] who suggested the name of graph to the structure he was studying.

Many of these early problems were motivated by practical applications, and among those many called for an algorithm. The Shortest Path Problem[iii] considered by many researcher's independently in the 1950's is a primal such example. As opposed to these, in this thesis we deal mostly with structural (existential) results, with only a little care about the algorithmic counterpart.

[i] this led to the important graph theoretic notion of Hamilton cycles

[ii] even though Kirchhoff did not use explicitly graph theory his derivations are graph-theoretical in nature; see for example [Gri10, §1] for a modern approach

[iii] in which the task is to find efficiently the shortest path between two given vertices of a graph; this has numerous applications from automotive navigation systems to routing in computer networks to finding optimal turns in a Rubik's Cube

Graph theory mostly explores the structure of graphs. Understanding the structure is the key for many other problems, such as counting graphs with given restrictions, investigating the properties of random graphs, or devising efficient graph algorithms. Structural graph theory is naturally a very wide field itself, and the current state of art is more advanced in some parts than in others. For example, we have a fairly good understanding of the structure of matchings in graphs, in particular as there is a strong connection to the theory of linear programming. Another example is a monstrous project by Roberston and Seymour [RS83]–[RS12] which gives a precise description of graphs avoiding a fixed minor; a primal example of which are the class of planar graphs, i.e., graphs which can be embedded in the Euclidean plane without their edges crossing. A third example is that we have a most detailed description of the structure of the so-called dense graph; this description is given by the Szemerédi Regularity Lemma which we describe in Section I.1 (and then in greater detail in Section II.2.5). The structure given by the Szemerédi Regularity Lemma has been successfully used in hundreds of problems in graph theory, number theory, theoretical computer science, and elsewhere. However, it seems out of reach (if not impossible) to get a similar universal structural results for general graphs. One of the main contributions of this thesis is to work out a structural result in this direction. This structural result applies to all graphs. On the other hand, its applications seem to be restricted to only a relatively small class of problems. The method borrows heavily from a previous and ongoing work of Ajtai, Komlós, Simonovits, and Szemerédi.

There are many extensions and generalizations of finite graphs all of which bring additional challenges: directed graphs, graphs with weights on their vertices and edges, etc. The study of infinite graphs is intimately connected to set theory and point-set topology. Matroids are a certain abstraction of finite graphs. While a vivid area by itself they provide useful insights about some algorithmic aspects of graph theory, and provide the right framework for some graph optimization problems. Recently emerging theories of graph limits show a profound connection between discrete and continuous. We work with the simplest of these models, i.e., with finite graphs. Most of our results apply only to astronomically huge graphs though, thus making them inapplicable in practice. Such a limitation goes with some modern graph theoretical tools (such as the Graph Minors Project, or the Regularity Method mentioned above). On the other hand, from a mathematical prospective such results are fairly satisfactory as they typically describe the problem "up to finitely many possible exceptions".

The thesis deals with the so-called Loebl-Komlós-Sós Conjecture, a classical extremal graph theory problem. The basic question in extremal graph theory is what density conditions guarantee a containment of a certain subgraph. Such questions were first systematically studied by Hungarian mathematicians centred around Paul Erdős since Paul Turán's [Tur41] proof of what is now called the Turán Theorem in 1941. Indeed, the Turán Theorem is now considered the starting point of extremal graph theory itself. This initially fairly local group of researchers grew and internationalized with many of the members of the "Hungarian school" fleeing the country due to political and financial circumstances in the 1970's and 1980's and taking up positions mainly in the US. The scope of extremal graph theory has since widened, and now includes beside the Turán-type questions described above various counting questions, graph decomposition results, and parts of the Ramsey theory. The research in the field of extremal graph theory gave rise to or made a lasting impact on some other rich and beautiful theories; let us mention here the probabilistic method, or the recent theory of flag algebras. Some surprising breakthroughs came from algebra. For example, the known constructions of graphs with many edges without the four cycle C_4 are based on finite projective planes, basic objects of algebraic geometry. Algebraic methods, spectral techniques, and explicit constructions based on algebra form a field by itself, and from there it is not too far, for example, to expansion in groups (see [Lub94] for some beautiful but outdated highlights) which has been a central project in mathematics over the last four decades.

Extremal graph theory can be viewed as a subfield of extremal combinatorics. Extremal combinatorics asks the same kind of questions as extremal graph theory but in the context of other discrete structure. The simplest and most common questions concern finite sets and their systems: *What density conditions of a system of subsets of a given set guarantee a given pattern to exist?* The famous Erdős-Ko-Rado theorem illustrates this more concretely: Suppose that $\mathcal{A}$ is a family of k-sets of some n-vertex set, $2k \leqslant n$. If $|\mathcal{A}| > \binom{n-1}{k-1}$ then $\mathcal{A}$ contains two disjoint sets. Some of these questions are more natural to be phrased using the language of hypergraphs.[iv] Roughly speaking, when a problem concerns uniform hypergraphs of low uniformity we might expect tools similar to those available in extremal graph theory to be used for its solution. However, there are many techniques developed specifically for extremal combinatorics problems.

1 The Regularity Lemma

The Regularity Lemma of Szemerédi is nowadays a central tool in graph theory. In this section we survey the developments around the lemma as it relates to what we believe

[iv]A *hypergraph* is a family $\mathcal{A}$ of subsets of a given finite ground set. It is *uniform* if all the members of $\mathcal{A}$ have the same size k. The number k is called the *uniformity* of $\mathcal{A}$.

to be the most important contribution of this thesis — a certain general graph decomposition technique. Technical details including the statement of the lemma itself can be found in Section II.2.5. A much more detailed account is given in two slightly outdated surveys [KS96, KSSS02], and applications of the Regularity Lemma to problems very similar to the one considered in this thesis are surveyed in [KO09].

Seeds of the Regularity Lemma can be found in Szemerédi's proof [Sze75] of the Erdős-Turán Conjecture about arithmetic progressions in dense subsets of integers, now the Szemerédi Theorem. The lemma from [Sze75] was enough for the Szemerédi Theorem, and for some other problems among which the (6,3)-problem [RS78] is the most notable. Yet, that statement was still quite far the contemporary understanding of the Regularity Lemma: "*Each graph can be decomposed into a bounded collection of random-like bipartite graphs.*" It took several years until the lemma appeared in its current form [Sze78] in 1978. The lemma is indeed *the* structural result for dense graphs as it approximates a very wide range of graph parameters, local (such as the density of triangles) as well as global (such as the size of the MAX-CUT).

The Regularity Lemma did not find many applications in graph theory in the early years of its life. One of the first ones was a result of Chvátal, Rödl, Szemerédi and Trotter [CRST83] about the Ramsey numbers of bounded-degree graphs. The number of applications increased rapidly in the 1990's. This was perhaps given by the development of the Blow-up technique [KSS97] which made the work with the lemma cleaner and more efficient, and more importantly as some other prominent mathematicians including Noga Alon took interest around that time.

While a weak form of the Regularity Lemma was one of the keys for Szemerédi's proof of the Erdős-Turán Conjecture the other steps were by no means simple, and the original proof is still considered as one of the most intricate ones in mathematics. On the other hand, already Ruzsa and Szemerédi [RS78] observed that an instance of Szemerédi's Theorem for arithmetic progressions of length $k = 3$ — first proved by Roth as early as 1952 [Rot52] — is a simple consequence of the Regularity Lemma. Thus the hope was that there could be a simple proof of Szemerédi's Theorem provided that one finds a suitable extension of Szemerédi's Regularity Lemma. Such a programme was carried out by Rödl and his collaborators in the late 1990's and early 2000's. And indeed, Frankl and Rödl obtained a short proof of Szemerédi's Theorem for $k = 4$ based on their regularization of 3-uniform hypergraphs [FR02]. It actually turned out that the main difficulty was not a regularity lemma for hypergraphs itself but a "counting lemma", a tool accompanying the Regularity Lemma which is trivial in the graph case. This approach was then by generalized to arbitrary uniformity of the hypergraph by Nagle, Rödl, Schacht, and Skokan in [RS04, NRS06] and developed independently by

Gowers [Gow07] thereby giving an alternative proof of the full Szemerédi's Theorem. From a contemporary perspective it is the developments of the regularity method for hypergraphs which has brought the most fruits to other fields of mathematics. The link goes via the Gowers uniformity norms and has many implications in number theory.

The Regularity Lemma found some important applications in theoretical computer science. For example, it provides the ultimate answer for many problems in so-called property testing (see e.g. [AFNS09]).

Even though the Regularity Lemma is applicable to all graphs the statement carries a certain unavoidable error parameter which makes it void for sparse graphs, i.e., graphs which contain a negligible proportion of all possible edges. Kohayakawa [Koh97] and independently Rödl realized in the late 1990's that Szemerédi's proof can be transferred to give a useful regularity concept for a fairly wide class of sparse graphs, so-called "subgraphs of random graphs". This observation has been used successfully since with some exciting breakthroughs around the Kohayakawa-Rödl-Łuczak Conjecture [Sch10, CG10, ST12, BMS12] and around sparse counting lemmas [CFZ12] being achieved only very recently.

There has been quite some effort to avoid using the Regularity Lemma in some problems. That is, to find regularity-free alternatives to proofs of some existing results. The main motivation is that proofs employing the Regularity Lemma often lead to statements which have very poor quantitative bounds. There does not seem to exist a unifying solution to circumvent the lemma. However, there seem to be some general techniques. One of them seems to exist for problems concerning embedding large graphs (or hypergraphs) into a host structure. In that setting the Regularity Lemma is typically used to give a simplified macroscopic picture of the structure. Then, one can often use more down to earth graph theoretic tools te get an answer even without seeing this complete macroscopic picture. The so-called absorption method is often employed then. Examples of work in this area include [LSS10, HPS09]. The so-called dependant random choice has helped in several other important instances. There is an excellent and up-to-date survey on the technique by Fox and Sudakov [FS11]. The work in the area of finding alternatives to the Regularity Lemma will certainly remain most active in the near future.

The Regularity Lemma has been a key to a number of results in extremal graph theory but even more importantly has brought graph theoretical tools to areas like number theory, combinatorial group theory, or discrete geometry, and has stimulated research in ergodic theory. It turns out that the Regularity Lemma is a bridge between the worlds of discrete and continuous in a wide sense, a fact which is being evidenced by the developing theory of graph limits. In this thesis we contribute to the theory of the regularity method by a decomposition technique which applies to all graphs, dense and sparse alike, and which extends the original Szemerédi Regularity Lemma. The technique unfortunately seems to be rather narrow in applications (compared to the almost universal applicability of the Szemerédi Regularity Lemma), yet there are no outlooks for anything more powerful. A more detailed description of these results is given in Section II.1.2.

Chapter II

Loebl-Komlós-Sós Conjecture

The material presented in this chapter is based on a joint work with János Komlós, Diana Piguet, Miklós Simonovits, Maya J. Stein, and Endre Szemerédi. A text based on this chapter will be made available online in a form of a monograph coauthored by these collaborators soon.

1 Introduction

1.1 Statement of the problem

In this paper we provide an approximate solution of the Loebl-Komlós-Sós Conjecture. This is a problem in extremal graph theory which fits the classical form *Does a certain density condition imposed on the host graph guarantee a certain subgraph?* Results of this type include Dirac's Theorem which determines the minimum degree threshold for containment of a Hamilton cycle, or Mantel's Theorem which determines the average degree threshold for containment of a triangle. Indeed, most of these extremal problems are formulated in terms of the minimum or average degree of the host graph.

We investigate density conditions which guarantee that a host graph contains *each* tree of order k. The greedy tree-embedding strategy shows that minimum degree more of than $k-2$ is a sufficient condition. Further, this bound is best possible as any $(k-2)$-regular graph avoids the k-vertex star. However, Erdős and Sós conjectured that the minimum degree condition can be relaxed to an average degree one still giving the same conclusion.

Conjecture 1.1 (Erdős-Sós Conjecture 1963)**.** *Let G be a graph of average degree greater than $k-2$. Then G contains each tree of order k as a subgraph.*

A solution of the Erdős-Sós Conjecture for all k bigger than an absolute constant was announced by Ajtai, Komlós, Simonovits, and Szemerédi in the early 1990's. In a similar spirit, Loebl, Komlós, and Sós conjectured that a *median degree* of more than $k-2$ is sufficient for containment of any tree of order k. By median degree we mean the degree of a vertex in the middle of the ordered degree sequence.

Conjecture 1.2 (Loebl-Komlós-Sós Conjecture 1995 [EFLS95])**.** *Suppose that G is an n-vertex graph with at least $n/2$ vertices of degrees more than $k-2$. Then G contains each tree of order k.*

We discuss in detail Conjectures 1.1 and 1.2 in Section 1.3. Here, we just state the main result of the paper, an approximate solution of the Loebl-Komlós-Sós Conjecture.

Theorem 1.3 (Main result)**.** *For every $\alpha > 0$ there exists k_0 such that for any $k > k_0$ we have the following. Each n-vertex graph G with at least $(\frac{1}{2}+\alpha)n$ vertices of degrees at least $(1+\alpha)k$ contains each T tree of order k.*

1.2 Regularity lemma and dense graph theory

The Szemerédi Regularity Lemma has been a major tool in extremal graph theory for three decades. It provides an approximate representation of a graph with a so-called *cluster graph*. This cluster graph representation is the key for graph-containment problems. The usual strategy here is that instead of solving the original problem one focuses on a modified simpler problem on the cluster graph.

The applicability of the Szemerédi Regularity Lemma is, however, limited to *dense graphs*, i.e., graphs that contain a substantial proportion of all possible edges. Luckily enough many graphs arising in extremal graph theory are dense, as for example those coming from Dirac's and Mantel's Theorem above. Indeed, while the proofs of these two sample results are elementary many of their extensions rely on the Regularity Lemma.

So, the theory of dense graphs is well understood due to the Szemerédi Regularity Lemma, but no such tool is available for sparse graphs. A regularity type representation of general (possibly sparse) graphs is one of the most important goals of contemporary discrete mathematics. By such a representation we mean an approximation of the input graph by a structure of bounded complexity carrying all important information about the graph.

A central tool in the proof of Theorem 1.3 is a structural decomposition of the graph $G_{\triangleright T1.3}$. This decomposition — which we call *sparse decomposition* — applies to any graph whose average degree is bigger than an absolute constant. The sparse decomposition provides a partition of any graph into vertices of huge degrees and into a bounded degree part. The bounded degree part is further decomposed into dense regular pairs, an edge set with certain expander-like properties, and a vertex set which is expanding in a different way. In case of dense graphs this decomposition produces a Szemerédi regularity partition, and thus the sparse decomposition extends the Szemerédi Regularity Lemma. It should be said however that the sparse decomposition lacks many features of the Szemerédi Regularity Lemma which make it applicable in combinatorics and other areas. In this sense this decomposition seems substantially less universal than the Szemerédi regularity partition. Even within graph-containment problems our decomposition seems to be limited to problems concerning containment of trees.

This kind of decomposition was first used by Ajtai, Komlós, Simonovits, and Szemerédi in their work on the Erdős-Sós Conjecture.

1.3 Loebl-Komlós-Sós Conjecture and Erdős-Sós Conjecture

Let us introduce first some notation. We say that H *embeds* in a graph G and write $H \subseteq G$ if H is a (not necessarily induced) subgraph of G. The associated map $\phi : V(H) \to V(G)$ is called an *embedding of H in G*. More generally, for a graph class $\mathcal{H}$ we write $\mathcal{H} \subseteq G$ if $H \subseteq G$ for every $H \in \mathcal{H}$. Let $\mathbf{trees}(k)$ be the class of all trees of order k.

Conjecture 1.2 is dominated by two parameters: one quantifies the number of vertices of 'large' degree, and the other tells us how large this degree should actually be. Strengthening either of these bounds sufficiently, the conjecture becomes trivial. [i]

On the other hand, one may ask whether lower bounds would suffice. For the bound $k-2$, this is not the case, since stars of order k require a vertex of degree at least $k-1$ in the host graph. As for the bound $n/2$, the following example shows that this number cannot be decreased much.

First, assume that n is even, and that $n = k$. Let G^* be obtained from the complete graph on n vertices by deleting all edges inside a set of $\frac{n}{2}+1$ vertices. It is easy to check that G^* does not contain the path[ii] $P_k \in \mathbf{trees}(k)$. Now, taking the union of several disjoint copies of G^* we obtain examples for other values of n. (And adding a small complete component we can get to *any* value of n.) See Figure 1.1 for an illustration.

However, we do not know of any example attaining the exact bound $n/2$. Thus it might be possible to lower the bound $n/2$ from Conjecture 1.2 to the one attained in our example above:

Conjecture 1.4. *Let $k \in \mathbb{N}$ and let G be a graph on n vertices, with more than $\frac{n}{2} - \lfloor \frac{n}{k} \rfloor - (n \mod k)$ vertices of degree at least $k-1$. Then $\mathbf{trees}(k) \subseteq G$.*

[i] Indeed, if we replace $n/2$ with n, then any tree of order k can be embedded greedily. Also, if we replace $k-2$ with $4k-4$, then G, being a graph of average degree at least $2k-2$, has a subgraph G' of minimum degree at least $k-1$. Again we can greedily embed any tree of order k.

[ii] In general, G^* does not contain any tree $T \in \mathbf{trees}(k)$ which has an equitable two-coloring.

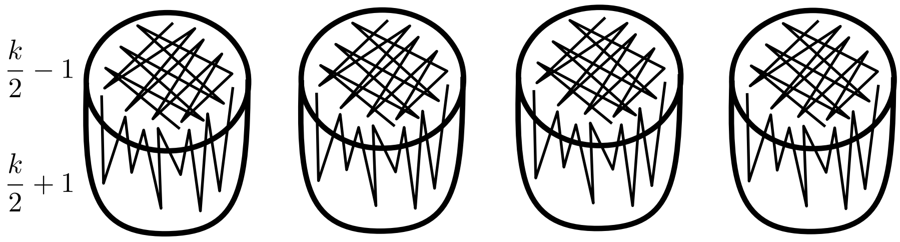

Figure 1.1: An extremal graph for the Loebl-Komlós-Sós Conjecture.

It might even be that if n/k is far from integrality, a slightly lower bound on the number of vertices of large degree still works (see [Hla, HP]).

Several partial results concerning Conjecture 1.2 have been obtained; we briefly summarize the most important ones. Two main directions can be distinguished among those results that prove the conjecture for special classes of graphs: either one places restrictions on the host graph, or on the class of trees to be embedded. Of the latter type is the result by Bazgan, Li, and Woźniak [BLW00], who proved the conjecture for paths. Also, Piguet and Stein [PS08] proved that Conjecture 1.2 is true for trees of diameter at most 5, which improved earlier results of Barr and Johansson [BJ] and Sun [Sun07].

Restrictions on the host graph have led to the following results. Soffer [Sof00] showed that Conjecture 1.2 is true if the host graph has girth at least 7. Dobson [Dob02] proved the conjecture for host graphs whose complement does not contain a $K_{2,3}$. This has been extended by Matsumoto and Sakamoto [MS] who replace the $K_{2,3}$ with a slightly larger graph.

A different approach is to solve the conjecture for special values of k. One such case, known as the Loebl conjecture, or also as the $(n/2$–$n/2$–$n/2)$–Conjecture, is the case $k = n/2$. Ajtai, Komlós, and Szemerédi [AKS95] solved an approximate version of this conjecture, and later Zhao [Zha11] used a refinement of this approach to prove the sharp version of the conjecture for large graphs.

An approximate version of Conjecture 1.2 for dense graphs, that is, for k linear in n, was proved by Piguet and Stein [PS12]. Let us take this opportunity to introduce a useful notation. Write $\mathbf{LKS}(n,k,\alpha)$ for the class of all n-vertex graphs with at least $(\frac{1}{2}+\alpha)n$ vertices of degrees at least $(1+\alpha)k$. With this notation Conjecture 1.2 states that every graph in $\mathbf{LKS}(n,k,0)$ contains every tree from $\mathbf{trees}(k+1)$.

Theorem 1.5 (Piguet-Stein [PS12])**.** *For any $q > 0$ and $\alpha > 0$ there exists a number n_0 such that for any $n > n_0$ and $k > qn$ the following holds. If $G \in \mathbf{LKS}(n,k,\alpha)$ then $\mathbf{trees}(k+1) \subseteq G$.*

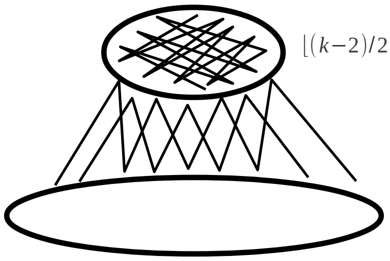

Figure 1.2: An almost extremal graph for the Erdős-Sós Conjecture.

This result was proved using the regularity method. Adding stability arguments, Hladký and Piguet [HP], and independently Cooley [Coo09] proved Conjecture 1.2 for large dense graphs.

Theorem 1.6 (Hladký-Piguet [HP], Cooley [Coo09])**.** *For any $q > 0$ there exists a number $n_0 = n_0(q)$ such that for any $n > n_0$ and $k > qn$ the following holds. If $G \in \mathbf{LKS}(n, k, 0)$ then $\mathbf{trees}(k + 1) \subseteq G$.*

Let us now turn our attention to the Erdős-Sós Conjecture. It is particularly important to compare the structure of the respective extremal graph with the extremal graphs for the Loebl-Kómlos-Sós Conjecture. The Erdős-Sós Conjecture 1.1 is best possible whenever $n(k-2)$ is even. Indeed, in that case it suffices to consider a $(k-2)$-regular graph. This is a graph with average degree exactly $k-2$ which does not contain the star of order k. Even when the star[iii] is excluded from the considerations, we can — at least when $k-1$ divides n — consider a disjoint union of $\frac{n}{k-1}$ cliques K_{k-1}. This graph contains *no* tree from $\mathbf{trees}(k)$.

There is another important graph with many edges which does not contain for example the path P_k, depicted in Figure 1.2. This graph has $\frac{1}{2}(k-2)n - O(k^2)$ edges when k is even and $\frac{1}{2}(k-3)n - O(k^2)$ edges otherwise, and therefore gets close to the conjectured bound when $k \ll n$. Apart from the already mentioned announced breakthrough by Ajtai, Komlós, Simonovits, and Szemerédi, work on this conjecture includes [BD96, Hax01, MS, SW97, Woź96].

Ramsey theory. The field is named in honour of Frank P. Ramsey who initiated the work with the following fundamental result.

Theorem 1.7 (Ramsey 1930, [Ram30])**.** *For every number $\ell \in \mathbb{N}$ there exists a number $n \in \mathbb{N}$ such that for each 2-edge-colouring of the complete graph K_n contains a monochromatic copy of the complete graph K_ℓ.*

[iii] which in a sense is a pathological tree

The smallest such number is called the *Ramsey number* $R(K_\ell, K_\ell)$. This implies that sufficiently large 2-edge-coloured (say, by red and blue) complete graphs contain a red copy of a fixed graph H_1 or a blue copy of a fixed graph H_2; the smallest order of the complete graph with this universality property is denoted by $R(H_1, H_2)$. Theorem 1.7 is a qualitative statement, i.e., even the fact that a finite n with this property exists is quite remarkable. In this direction there has been a great study to understand to what other structures does such "Ramsey property" extend.[iv] See [GGL95, Chapter 25] for a survey in this direction. But there is also an obvious qualitative question: How does $R(H_1, H_2)$ behave as a function of H_1 and H_2? For example, $R(K_\ell, K_\ell)$ grows roughly exponentially in ℓ, but the exact value of the exponent is not known, $\sqrt{2}^\ell \lesssim R(K_\ell, K_\ell) \lesssim 4^\ell$. Both Conjecture 1.2 and Conjecture 1.1 have an important application in this direction. They (each) imply that the Ramsey number of two trees $T_{k+1} \in \mathbf{trees}(k+1)$, $T_{\ell+1} \in \mathbf{trees}(\ell+1)$ is bounded by $R(T_{k+1}, T_{\ell+1}) \leqslant k+\ell+1$. Actually more is implied: Any 2-edge-colouring of $K_{k+\ell+1}$ contains either *all* trees in $\mathbf{trees}(k+1)$ in red, or *all* trees in $\mathbf{trees}(\ell+1)$ in blue.

The bound $R(T_{k+1}, T_{\ell+1}) \leqslant k+\ell+1$ is almost tight only for certain types of trees: Harary [Har72] shows $R(S_k, S_\ell) = k+\ell-2-\varepsilon$ for stars $S_k \in \mathbf{trees}(k)$, $S_\ell \in \mathbf{trees}(\ell)$, where $\varepsilon \in \{0,1\}$ depends on the parity of k and ℓ. On the other hand, Gerencsér and Gyárfás [GG67] showed $R(P_k, P_\ell) = \max\{k,\ell\} + \left\lfloor \frac{\min\{k,\ell\}}{2} \right\rfloor - 1$ for paths $P_k \in \mathbf{trees}(k)$, $P_\ell \in \mathbf{trees}(\ell)$. Haxell, Łuczak, and Tingley confirmed asymptotically [HLT02] that the discrepancy of the Ramsey bounds for trees depends on their balancedness, at least when the maximum degrees of the trees considered are moderately bounded.

Trees in random graphs. To complete the picture of research involving tree containment problems we mention two rich and vivid (and also closely connected, as we shall see) areas: trees in random graphs, and trees in expanding graphs. The former area is centered around the following question: *What is the probability threshold $p = p(n)$ for the Erdős-Rényi random graph $G_{n,p}$ to contain asymptotically almost surely (a.a.s.) each tree/all trees from a given class of trees $\mathcal{F}_n$?* Note that there is a difference between containing "each tree" and "all trees" as the error probabilities for missing individual trees might sum up.

Most research focused on containment of spanning trees, or almost spanning trees. The only well-understood case is when $\mathcal{F}_n = \{P_{k_n}\}$ is a path. The threshold $p = \frac{(1+o(1))\ln n}{n}$ for appearance of a spanning path (i.e., $k_n = n$) was determined by Komlós and Szemerédi [KS83], and independently by Bollobás [Bol84]. Note that this threshold

[iv] Of course these extensions are not always straightforward. For example it is not immediate what statement hides under the fact that "finite-dimensional vector spaces are Ramsey" proven in [GLR72].

is the same as the threshold for a weaker property for connectedness. We should also mention a previous result of Pósa [Pós76] which determined the order of magnitude of the threshold, $p = \Theta(\frac{\ln n}{n})$. The heart of Pósa's proof, the celebrated rotation-extension technique, is an argument about expanding graphs, and indeed many other results about trees in random graphs exploit the expansion properties of $G_{n,p}$ in the first place.

The appearance of almost spanning paths in $G_{n,p}$ was determined by Fernandez de la Vega [FdlV79] and independently by Ajtai, Komlós, and Szemerédi [AKS81]. Their results say that a path of length $(1-\varepsilon)n$ appears a.a.s. in $G_{n,\frac{C}{n}}$ for $C = C(\varepsilon)$ sufficiently large. This behavior extends to bounded degree trees. Indeed, Alon, Krivelevich, and Sudakov [AKS07] proved that $G_{n,\frac{C}{n}}$ (for a suitable $C = C(\varepsilon, \Delta)$) a.a.s. contains all trees of order $(1-\varepsilon)n$ with maximum degree at most Δ (the constant C was later improved in [BCPS10]).

Let us now turn to spanning trees in random graphs. The paper [AKS07] also gives that a single spanning tree T with bounded maximum degree and linearly many leaves is a.a.s. contained in $G_{n,\frac{C \ln n}{n}}$. This result can be reduced to the main result of [AKS07] regarding almost spanning trees quite easily. The constant C can be taken $C = 1+o(1)$, as was shown recently by Hefetz, Krivelevich, and Szabó [HKS]; obviously this is best possible. The same result also applies to trees which contain a path of linear length whose all vertices have degree two. A breakthrough in the area was achieved by Krivelevich [Kri10] who gave an upper bound on the threshold $p = p(n, \Delta)$ for embedding a single spanning tree of a given maximum degree Δ. This bound is essentially tight for $\Delta = n^c$, $c \in (0, 1)$. Even though the argument in [Kri10] is not difficult, it relies on a deep result of Johansson, Kahn and Vu [JKV08] about factors in random graphs.

Trees in expanders. By an expander graph we mean a graph with a large Cheeger constant, i.e., a graph which satisfies a certain isoperimetric property. There are other ways how to parametrize an expander, of which a spectral one is often the most useful. As indicated above, random graphs are very good expanders, and this is the main motivation for studying tree containment problems in expanders. Another motivation comes from studying the universality phenomenon. Here the goal is to construct sparse graphs which contain all trees from a given class, and expanders are natural candidates for this. The study of sparse tree-universal graphs is a remarkable area by itself which brings challenges both in probabilistic and explicit constructions. For example, Bhatt, Chung, Leighton, and Rosenberg [BCLR89] give an explicit construction of a graph with only $O_\Delta(n)$ edges which contains all n-vertex trees with maximum degree at most Δ. A more recent paper of Johannsen, Krivelevich, and Samotij [JKS12] contains a number of universality results for spanning trees of maximum degree $\Delta = \Delta(n)$ both for random graphs, and for expanders. For example, they show universality for this class of each graph with a large Cheeger constant which satisfies a certain connectivity condition.

Pósa's rotation-extension technique was extended from paths to trees by Friedman and Pippenger [FP87], and found many applications (e.g. [HK95, Hax01, BCPS10]). Sudakov and Vondák [SV10] use tree-indexed random walks to embed trees in $K_{s,t}$-free graphs[v]; a similar approach employed in a beautiful paper by Benjamini and Schramm [BS97] in the setting of infinite graphs.

In our proof of Theorem 1.3, embedding trees in expanders play a crucial role, too. However, our notion of expansion[vi] is very unlike to those studied previously.

Minimum degree conditions for spanning trees. Recall that the tight min-degree condition for containment of a general spanning tree T in an n-vertex graph G is the trivial one, $\mathrm{deg}^{\mathrm{min}}(G) \geqslant n-1$. However, the only tree which requires this bound is the star. This indicates that this threshold can be lowered substantially if we have a control of $\mathrm{deg}^{\mathrm{max}}(T)$. Szemerédi and his collaborators [KSS01, CLNGS10] showed that this is indeed the case, and obtained tight min-degree bounds for certain ranges of $\mathrm{deg}^{\mathrm{max}}(T)$. For example, when $\mathrm{deg}^{\mathrm{max}}(T) \leqslant n^{o(1)}$, then $\mathrm{deg}^{\mathrm{min}}(G) \geqslant (\frac{1}{2} + o(1))n$ is a sufficient condition. (Note that G may become disconnected close to this bound.)

1.4 Overview of the proof of Theorem 1.3

We introduce various tools for the proof of Theorem 1.3 in Sections 2–8. Section 2 contains some general preliminaries, Section 3 deals with processing the tree $T_{\triangleright \mathrm{T}1.3}$, Sections 4–7 deal with processing the graph $G_{\triangleright \mathrm{T}1.3}$. In Section 8 we introduce techniques for embedding trees in a graph. These tools are then put together in a relatively short proof of Theorem 1.3 in Section 9. Section 10 contains some concluding remarks.

The scheme of the proof is given in Figure 1.3.

The proof structure resembles those of proofs of tree embedding problems in dense graph theory. We use the sparse decomposition to get an approximate representation of the graph $G_{\triangleright \mathrm{T}1.3}$, we find a suitable combinatorial structure inside the sparse decomposition, and then we embed the tree $T_{\triangleright \mathrm{T}1.3}$ — which is preprocessed by cutting it into tiny subtrees — into $G_{\triangleright \mathrm{T}1.3}$ using this structure. Dealing with a sparse decomposition is much more complex than dealing with the Szemerédi regularity partition.

[v]this property implies expansion

[vi]actually, two, very different notions, introduced in Definitions 4.2 and 4.6

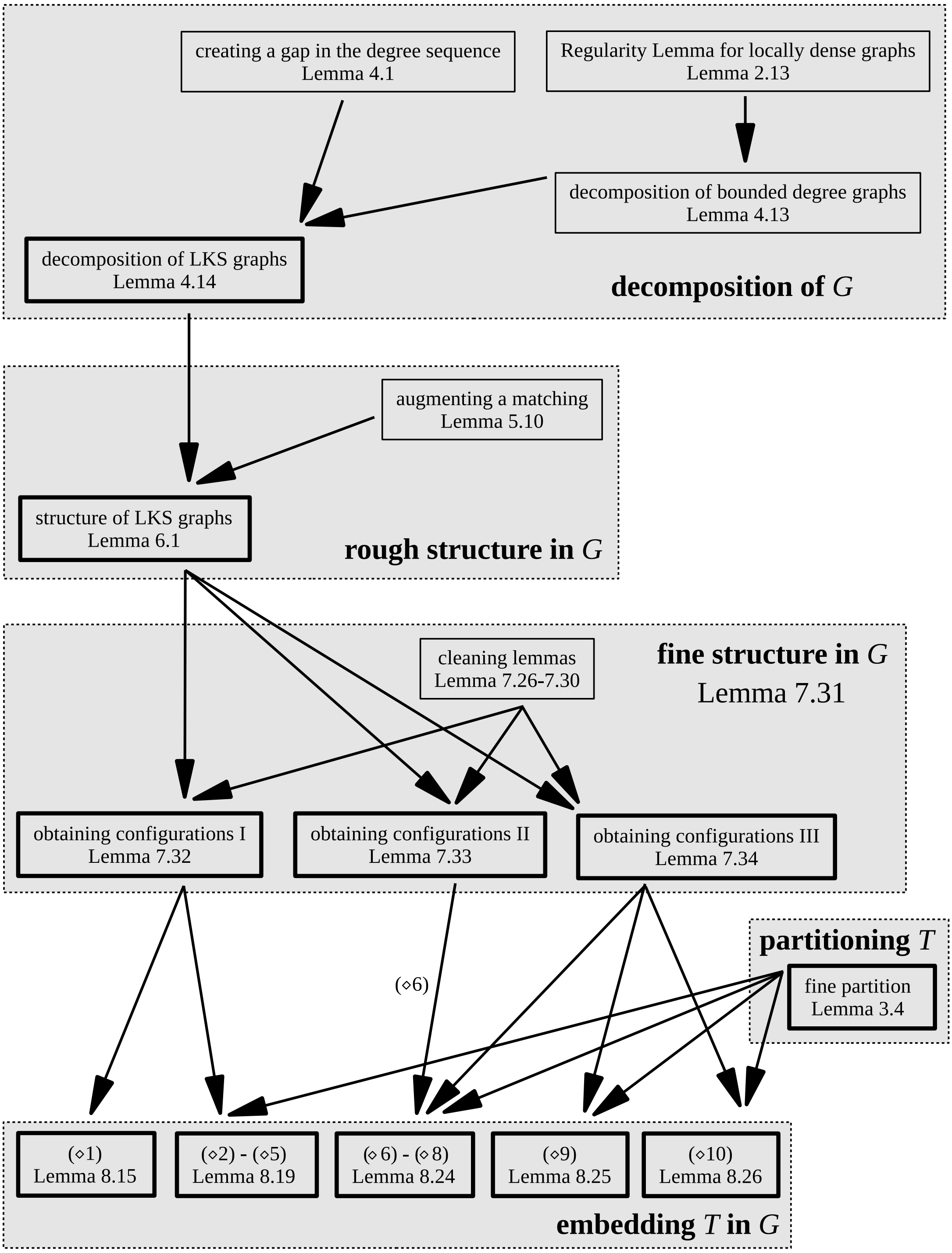

Figure 1.3: Structure of the proof of Theorem 1.3.

We need several techniques for embedding the tiny subtrees in addition to the standard filling-up-a-regular-pair technique used in conjunction with the regularity method. These techniques will be utilized for embedding to the various components of the sparse decomposition.

Let us now describe the proof structure in more detail. The starting point is a sparse decomposition of the input graph $G_{\triangleright T1.3} \in \mathbf{LKS}(n, k, \alpha)$. This is done in Lemma 4.14.

Lemma 4.14 is a combination of two lemmas: Lemma 4.1, which partitions $V(G_{\triangleright \mathrm{T1.3}})$ into vertices of huge degrees and vertices of bounded degree, and Lemma 4.13, which provides a sparse decomposition of any bounded degree graph; we expect this lemma to have applications to other tree containment problems. One of the ingredients of Lemma 4.13 is Lemma 2.13, a certain version of the Regularity Lemma which applies to locally dense graphs.

Having obtained the sparse decomposition — a counterpart to the Szemerédi regularity partition in the dense setting — the next step is to find some structure suitable for embedding $T_{\triangleright \mathrm{T1.3}}$. We do so in two stages: first we obtain a "rough structure" in Lemma 6.1 which we then refine to one of ten possible configurations, denoted (**◇1**)–(**◇10**); this second step is done in Lemma 7.31.

Obtaining the rough structure for Lemma 6.1 involves Lemma 5.10, a step which we call "augmenting a matching". Very roughly speaking, this means that the sparse decomposition might not exhibit structure strong enough for our purposes. Therefore we need to find an additional object — a so-called semiregular matching — on top of the sparse decomposition. This is discussed in detail in Section 6.2.

The step of obtaining a configuration from the rough structure, which is the main objective of Section 7, is based on a pigeonhole-type argument such as: if there are many edges between two sets, and few 'kinds' of edges, then many of the edges are of the same kind. The different kinds of edges come from the sparse decomposition (and allow for different kinds of embedding techniques, as will become clear in Section 8). Just "homogenizing" the situation by restricting to one particular kind is not enough. In addition, we need to employ certain "cleaning lemmas" — Lemmas 7.26–7.30. A simplest such lemma would be that a graph with many edges contains a subgraph with a large minimum degree; the latter property apparently being more directly applicable for a sequential embedding of a tree. The actual cleaning lemmas we use are rather complicated extensions of this simple idea. Lemma 7.31 distinguishes between three situations which are then treated separately in Lemmas 7.32–7.34.

Recall that we preprocess $T_{\triangleright \mathrm{T1.3}}$ in Lemma 3.4. More precisely, we consider a so-called *ℓ-fine partition* of $T_{\triangleright \mathrm{T1.3}}$ which is a decomposition into small subtrees (called *shrubs*) and cut-vertices. In Section 8 we show how each of the configurations (**◇1**)–(**◇10**) given by Lemmas 7.32–7.34 helps for embedding $T_{\triangleright \mathrm{T1.3}}$. We first work out techniques of embedding shrubs into various components of the sparse decomposition, and into other building blocks of the configurations. Combining these we then get an embedding of $T_{\triangleright \mathrm{T1.3}}$ in $G_{\triangleright \mathrm{T1.3}}$ in each configuration, thus finishing the proof of Theorem 1.3.

2 Notation and preliminaries

In this section we recall some standard terminology and introduce some further specific notation. We also state some basic results from graph theory.

2.1 Notation

The set $\{1, 2, \ldots, n\}$ of the first n positive integers is denoted by $[n]$. Suppose that we have a nonempty set A, and $\mathcal{X}$ and $\mathcal{Y}$ each partition A. Then $\boxplus$ denotes the coarsest common refinement of $\mathcal{X}$ and $\mathcal{Y}$, i.e.,

$$\mathcal{X} \boxplus \mathcal{Y} := \{X \cap Y \ : \ X \in \mathcal{X}, Y \in \mathcal{Y}\} \setminus \{\emptyset\} \ .$$

We frequently employ indexing by many indices. We write superscript indices in parentheses (such as $a^{(3)}$), as opposed to notation of powers (such as a^3). We use sometimes subscript to refer to parameters appearing in a fact/lemma/theorem. For example $\alpha_{\triangleright \mathrm{T1.3}}$ refers to the parameter α from Theorem 1.3. We omit rounding symbols when this does not affect the correctness of the arguments.

We use lower case greek letters to denote small positive constants. The exception is the letter ϕ which is reserved for embedding of a tree T in a graph G, $\phi : V(T) \to V(G)$. The capital greek letters are used for large constants.

2.2 Basic graph theory notation

All graphs considered in this paper are finite, undirected, without multiple edges, and without self-loops. We write $V(G)$ and $E(G)$ for the vertex set and edge set of a graph G, respectively. Further, $v(G) = |V(G)|$ is the order of G, and $e(G) = |E(G)|$ is its number of edges. If $X, Y \subseteq V(G)$ are two, not necessarily disjoint, sets of vertices we write $e(X)$ for the number of edges induced by X, and $e(X, Y)$ for the number of ordered pairs $(x, y) \in X \times Y$ such that $xy \in E(G)$. In particular, note that $2e(X) = e(X, X)$.

For a graph G, a vertex $v \in V(G)$ and a set $U \subseteq V(G)$, we write $\deg(v)$ and $\deg(v, U)$ for the degree of v, and for the number of neighbours of v in U, respectively. We write $\deg^{\min}(G)$ for the minimum degree of G, $\deg^{\min}(U) := \min\{\deg(u) \ : \ u \in U\}$, and $\deg^{\min}(V_1, V_2) = \min\{\deg(u, V_2) \ : \ u \in V_1\}$ for two sets $V_1, V_2 \subseteq V(G)$. Similar notation is used for the maximum degree, denoted by $\deg^{\max}(G)$. The neighbourhood of a vertex v is denoted by $\mathrm{N}(v)$. We set $\mathrm{N}(U) := \bigcup_{u \in U} \mathrm{N}(u)$. The symbol $-$ is used for two graph operations: if $U \subseteq V(G)$ is a vertex set then $G - U$ is the subgraph of G induced by the set $V(G) \setminus U$. If $H \subseteq G$ is a subgraph of G then the graph $G - H$ is defined on the vertex set $V(G)$ and corresponds to deletion of edges of H from G.

A subgraph $H \subseteq G$ of a graph G is called *spanning* if $V(H) = V(G)$.

The *null graph* is the unique graph on zero vertices, while any graph with zero edges is called *empty*.

A family $\mathcal{A}$ of pairwise disjoint subsets of $V(G)$ is an *ℓ-ensemble in G* if $|A| \geqslant \ell$ for each $A \in \mathcal{A}$. We say that $\mathcal{A}$ is *inside* X (or *outside* Y) if $A \subseteq X$ (or $A \cap Y = \emptyset$) for each $A \in \mathcal{A}$.

If T is a tree and $r \in V(T)$, then the pair (T, r) is a *rooted tree* with root r. We then write $V_{\text{odd}}(T, r) \subseteq V(T)$ for the set of vertices of T of odd distance from r. Analogously we define $V_{\text{even}}(T, r)$. Note that $r \in V_{\text{even}}(T, r) \subseteq V(T)$. The distance between two vertices v_1 and v_2 in a tree is denoted by $\text{dist}(v_1, v_2)$.

We next give two simple facts about the number of leaves in a tree. These have already appeared in [Zha11] and in [HP] (and most likely in some more classic texts as well). Nevertheless, for completeness we shall include their proofs here.

Fact 2.1. *Let T be a tree with color-classes X and Y, and $v(T) \geqslant 2$. Then the set X contains at least $|X| - |Y| + 1$ leaves of T.*

Proof. Root T at an arbitrary vertex $r \in Y$. Let I be the set of internal vertices of T that belong to X. Each $v \in I$ has at least one immediate successor in the tree order induced by r. These successors are distinct for distinct $v \in I$ and all lie in $Y \setminus \{r\}$. Thus $|I| \leqslant |Y| - 1$. The claim follows. □

Fact 2.2. *Let T be a tree with ℓ vertices of degree at least three. Then T has at least $\ell + 2$ leaves.*

Proof. Let D_1 be the set of leaves, D_2 the set of vertices of degree two and D_3 be the set of vertices of degree of at least three. Then

$$2(|D_1| + |D_2| + |D_3|) - 2 = 2v(T) - 2 = 2e(T) = \sum_{v \in V(T)} \deg(v) \geqslant |D_1| + 2|D_2| + 3|D_3| ,$$

and the statement follows. □

For the next lemma, note that for us, the minimum degree of the null graph is ∞.

Lemma 2.3. *For all $\ell, n \in \mathbb{N}$, every n-vertex graph G contains a (possibly empty) subgraph G' such that $\deg^{\min}(G') \geqslant \ell$ and $e(G') \geqslant e(G) - (\ell - 1)n$.*

Proof. We construct the graph G' by sequentially removing vertices of degree less than ℓ from the graph G. In each step we remove at most $\ell - 1$ edges. Thus the statement follows. □

We finish this section with stating the Gallai-Edmonds matching theorem. A graph H is called *factor-critical* if $H - v$ has a perfect matching for each $v \in V(H)$. The following statement is a fundamental result in matching theory. See [LP86], for example.

Theorem 2.4 (Gallai-Edmonds matching theorem)**.** *Let H be a graph. Then there exist a set $Q \subseteq V(H)$ and a matching M of size $|Q|$ in H such that*

1) every component of $H - Q$ is factor-critical, and

2) M matches every vertex in Q to a different component of $H - Q$.

The set Q in Theorem 2.4 is often referred to as a *separator*.

2.3 LKS-minimal graphs

Given a graph G, denote by $\mathbb{S}_{\eta,k}(G)$ the set of those vertices of G that have degree less than $(1+\eta)k$ and by $\mathbb{L}_{\eta,k}(G)$ the set of those vertices of G that have degree at least $(1+\eta)k$.[vii] Thus the sizes of the sets $\mathbb{S}_{\eta,k}(G)$ and $\mathbb{L}_{\eta,k}(G)$ are what specifies the membership to $\mathbf{LKS}(n,k,\eta)$ (which we had defined as the class of all n-vertex graphs with at least $(\frac{1}{2}+\eta)n$ vertices of degrees at least $(1+\eta)k$).

Define $\mathbf{LKSmin}(n,k,\eta)$ as the set of all graphs $G \in \mathbf{LKS}(n,k,\eta)$ that are edge-minimal with respect to the membership in $\mathbf{LKS}(n,k,\eta)$. In order to prove Theorem 1.3 it suffices to restrict our attention to graphs from $\mathbf{LKSmin}(n,k,\eta)$, and this is why we introduce the class. Let us collect some properties of graphs in $\mathbf{LKSmin}(n,k,\eta)$ which follow directly from the definition.

Fact 2.5. *For any graph $G \in \mathbf{LKSmin}(n,k,\eta)$ the following is true.*

1. *$\mathbb{S}_{\eta,k}(G)$ is an independent set.*
2. *All the neighbours of every vertex $v \in V(G)$ with $\deg(v) > \lceil (1+\eta)k \rceil$ have degree exactly $\lceil (1+\eta)k \rceil$.*
3. *$|\mathbb{L}_{\eta,k}(G)| \leqslant \lceil (1/2+\eta)n \rceil + 1$.*

Observe that every edge in a graph $G \in \mathbf{LKSmin}(n,k,\eta)$ is incident to at least one vertex of degree exactly $\lceil (1+\eta)k \rceil$. This gives the following inequality.

$$e(G) \leqslant \lceil (1+\eta)k \rceil \, |\mathbb{L}_{\eta,k}(G)| \overset{\text{F2.5(3.)}}{\leqslant} \lceil (1+\eta)k \rceil \left(\left\lceil \left(\frac{1}{2} + \eta \right) n \right\rceil + 1 \right) < kn \; . \tag{2.1}$$

(The last inequality is valid under the additional mild assumption that, say, $\eta < \frac{1}{20}$ and $n > k > 20$. This can be assumed throughout the paper.)

Definition 2.6. *Let $\mathbf{LKSsmall}(n,k,\eta)$ be the class of those graphs $G \in \mathbf{LKS}(n,k,\eta)$ for which we have the following three properties:*

1. *All the neighbours of every vertex $v \in V(G)$ with $\deg(v) > \lceil (1+2\eta)k \rceil$ have degrees at most $\lceil (1+2\eta)k \rceil$.*

[vii] "$\mathbb{S}$" stands for "small", and "$\mathbb{L}$" for "large".

2. *All the neighbours of every vertex of* $\mathbb{S}_{\eta,k}(G)$ *have degree exactly* $\lceil (1+\eta)k \rceil$.

3. *We have* $e(G) \leqslant kn$.

Observe that the graphs from $\mathbf{LKSsmall}(n,k,\eta)$ also satisfy 1., and a quantitatively somewhat weaker version of 2. of Fact 2.5. This suggests that in some sense $\mathbf{LKSsmall}(n,k,\eta)$ is a good approximation of $\mathbf{LKSmin}(n,k,\eta)$.

As said, we will prove Theorem 1.3 only for graphs from $\mathbf{LKSmin}(n,k,\eta)$. However, it turns out that the structure of $\mathbf{LKSmin}(n,k,\eta)$ is too rigid. In particular, $\mathbf{LKSmin}(n,k,\eta)$ is not closed under discarding a small amount of edges during our cleaning procedures. This is why the class $\mathbf{LKSsmall}(n,k,\eta)$ comes into play: starting with a graph in $\mathbf{LKSmin}(n,k,\eta)$ we perform some initial cleaning and obtain a graph that lies in $\mathbf{LKSsmall}(n,k,\eta/2)$. We then heavily use its structural properties from Definition 2.6 throughout the proof.

2.4 Regular pairs

In this section we introduce the notion of regular pairs which is central for Szemerédi's Regularity Lemma and its extension which we discuss in Section 2.5. We also list some simple properties of regular pairs.

Given a graph H and a pair (U,W) of disjoint sets $U,W \subseteq V(H)$ the *density of the pair* (U,W) is defined as

$$\mathrm{d}(U,W) := \frac{e(U,W)}{|U||W|} \ .$$

Similarly, for a bipartite graph G with colour classes U, W we talk about its *bipartite density* $\mathrm{d}(G) = \frac{e(G)}{|U||W|}$. For a given $\varepsilon > 0$, a pair (U,W) of disjoint sets $U,W \subseteq V(H)$ is called an *ε-regular pair* if $|\mathrm{d}(U,W) - \mathrm{d}(U',W')| < \varepsilon$ for every $U' \subseteq U$, $W' \subseteq W$ with $|U'| \geqslant \varepsilon|U|$, $|W'| \geqslant \varepsilon|W|$. If the pair (U,W) is not ε-regular, then we call it *ε-irregular*.

We list two useful and well-known properties of regular pairs.

Fact 2.7. *Suppose that* (U,W) *is an* ε*-regular pair of density* d. *Let* $U' \subseteq W, W' \subseteq W$ *be sets of vertices with* $|U'| \geqslant \alpha|U|$, $|W'| \geqslant \alpha|W|$, *where* $\alpha > \varepsilon$. *Then the pair* (U',W') *is a* $2\varepsilon/\alpha$*-regular pair of density at least* $d - \varepsilon$.

Fact 2.8. *Suppose that* (U,W) *is an* ε*-regular pair of density* d. *Then all but at most* $\varepsilon|U|$ *vertices* $v \in U$ *satisfy* $\deg(v,W) \geqslant (d-\varepsilon)|W|$.

The following fact states a simple relation between the density of a (not necessarily regular) pair and the densities of its subpairs.

Fact 2.9. *Let* $H = (U,W;E)$ *be a bipartite graph of* $\mathrm{d}(U,W) \geqslant \alpha$. *Suppose that the sets* U *and* W *are partitioned into sets* $\{U_i\}_{i\in I}$ *and* $\{W_j\}_{j\in J}$, *respectively. Then at most* $\beta e(H)/\alpha$ *edges of* H *belong to a pair* (U_i,W_j) *with* $\mathrm{d}(U_i,W_j) \leqslant \beta$.

Proof. Trivially, we have

$$\sum_{i\in I, j\in J} \frac{|U_i||W_j|}{|U||W|} = 1 \ . \tag{2.2}$$

Consider a pair (U_i, W_j) of $\mathrm{d}(U_i, W_j) \leqslant \beta$. Then

$$e(U_i, W_j) \leqslant \beta |U_i||W_j| = \frac{\beta}{\alpha} \frac{|U_i||W_j|}{|U||W|} \alpha |U||W| \leqslant \frac{\beta}{\alpha} \frac{|U_i||W_j|}{|U||W|} e(U, W) \ .$$

Summing over all such pairs (U_i, W_j) and using (2.2) yields the statement. □

The next lemma asserts that if we have many ε-regular pairs (R, Q_i), then most vertices in R have approximately the total degree into the set $\bigcup_i Q_i$ that we would expect.

Lemma 2.10. *Let $Q_1, \ldots, Q_\ell$ and R be disjoint vertex sets. Suppose further that for each $i \in [\ell]$, the pair (R, Q_i) is ε-regular. Then we have*

(a) $\deg(v, \bigcup_i Q_i) \geqslant \frac{e(R, \bigcup_i Q_i)}{|R|} - \varepsilon \, |\bigcup_i Q_i|$ *for all but at most $\varepsilon|R|$ vertices $v \in R$, and*

(b) $\deg(v, \bigcup_i Q_i) \leqslant \frac{e(R, \bigcup_i Q_i)}{|R|} + \varepsilon \, |\bigcup_i Q_i|$ *for all but at most $\varepsilon|R|$ vertices $v \in R$.*

Proof. We prove (a), the other item is analogous. Suppose for contradiction that (a) does not hold. Without loss of generality, assume that there is a set $X \subseteq R$, $|X| > \varepsilon|R|$ such that $\frac{e(R, \bigcup Q_i)}{|R|} - \varepsilon|\bigcup Q_i| > \deg(v, \bigcup Q_i)$ for each $v \in X$. By averaging, there is an index $i \in [\ell]$ such that $\frac{|X|}{|R|} e(R, Q_i) - \varepsilon|X||Q_i| > e(X, Q_i)$, or equivalently,

$$\mathrm{d}(R, Q_i) - \varepsilon > \mathrm{d}(X, Q_i) \ .$$

This is a contradiction to the ε-regularity of the pair (R, Q_i). □

We use Lemma 2.10 to obtain the following.

Corollary 2.11. *Let $Q_1, \ldots, Q_\ell$ and R be disjoint vertex sets, each of size at most q, such that for each $i \in [\ell]$, the pair (R, Q_i) is ε-regular. Assume that more than $\varepsilon|R|$ vertices of R have degree at least x into $\bigcup Q_i$, but each $v \in R$ has neighbours in at most z of the sets Q_i. Then $\deg(v, \bigcup_i Q_i) \geqslant x - 2\varepsilon z q$ for all but at most $\varepsilon|R|$ vertices of R.*

Proof. For each $w \in R$, let $I_w \subseteq [\ell]$ be the set of those indices i for which there is at least one edge from w to Q_i. Now, by Lemma 2.10(b) there is a vertex $v \in R$ whose degree into $\bigcup_{i\in[\ell]} Q_i$ is at least x and whose degree into $\bigcup_{i\in I_v} Q_i$ is at most $\frac{e(R, \bigcup_{i\in I_v} Q_i)}{|R|} + \varepsilon \left|\bigcup_{i\in I_v} Q_i\right|$. So,

$$x \leqslant \deg(v, \bigcup_{i\in[\ell]} Q_i) = \deg(v, \bigcup_{i\in I_v} Q_i) \leqslant \frac{e\left(R, \bigcup_{i\in I_v} Q_i\right)}{|R|} + \varepsilon|\bigcup_{i\in I_v} Q_i| \leqslant \frac{e\left(R, \bigcup_{i\in I_v} Q_i\right)}{|R|} + \varepsilon z q.$$

Thus by Lemma 2.10(a) all but at most $\varepsilon|R|$ vertices of R have degree at least $x - 2\varepsilon z q$ into $\bigcup_i Q_i$. □

A stronger notion than regularity is that of super-regularity which we recall now. A pair (A, B) is (ε, γ)-*super-regular* if it is ε-regular, and we have $\deg^{\min}(A, B) \geqslant \gamma|B|$, and $\deg^{\min}(B, A) \geqslant \gamma|A|$. Note that then (A, B) has bipartite density at least γ.

2.5 Regularizing locally dense graphs

The Regularity Lemma [Sze78] has proved to be a powerful tool for attacking graph embedding problems; see [KO09] for a survey. We first state the lemma in its original form.

Lemma 2.12 (Regularity lemma)**.** *For all $\varepsilon > 0$ and $\ell \in \mathbb{N}$ there exist $n_0, M \in \mathbb{N}$ such that for every $n \geqslant n_0$ the following holds. Let G be an n-vertex graph whose vertex set is pre-partitioned into sets $V_1, \ldots, V_{\ell'}$, $\ell' \leqslant \ell$. Then there exists a partition $U_0, U_1, \ldots, U_p$ of $V(G)$, $\ell < p < M$, with the following properties.*

1) For every $i, j \in [p]$ we have $|U_i| = |U_j|$, and $|U_0| < \varepsilon n$.

2) For every $i \in [p]$ and every $j \in [\ell']$ either $U_i \cap V_j = \emptyset$ or $U_i \subseteq V_j$.

3) All but at most εp^2 pairs (U_i, U_j), $i, j \in [p]$, $i \neq j$, are ε-regular.

We shall use Lemma 2.12 for auxiliary purposes only as it is helpful only in the setting of dense graphs (i.e., graphs which have n vertices and $\Omega(n^2)$ edges). This is not necessarily the case in Theorem 1.3. For this reason, we give a version of the Regularity Lemma — Lemma 2.13 below — which allows us to regularize even sparse graphs.

More precisely, suppose that we have an n-vertex graph H whose edges lie in bipartite graphs $H[W_i, W_j]$, where $\{W_1, \ldots, W_\ell\}$ is an ensemble of sets of size $\Theta(k)$. Although ℓ may be unbounded, for a fixed $i \in [\ell]$ there are only a bounded number, say m, of indices $j \in [\ell]$ such that $H[W_i, W_j]$ is non-empty. See Figure 2.1 for an example. Lemma 2.13 then allows us to regularize (in the sense of the Regularity Lemma 2.12) all the bipartite graphs $G[W_i, W_j]$ using the same partition $\{W_i^{(0)} \dot\cup W_i^{(1)} \dot\cup \ldots \dot\cup W_i^{(p_i)} = W_i\}_{i=1}^{\ell}$. Note that when $|W_i| = \Theta(k)$ for all $i \in [\ell]$ then H has at most

$$\Theta(k^2) \cdot m \cdot \ell \leqslant \Theta(k^2) \cdot m \cdot \frac{n}{\Theta(k)} = \Theta(kn)$$

edges. Thus, when $k \ll n$, this is a regularization of a sparse graph. This "sparse Regularity Lemma" is very different to that of Kohayakawa [Koh97]). Indeed, Kohayakawa's Regularity Lemma deals with graphs which have no local condensation of edges, such as subgraphs of random graphs. Consequently, the resulting regular pairs are of density $o(1)$. In contrast, Lemma 2.13 provides us with regular pairs of density $\Theta(1)$, but, on the other hand, is useful only for graphs which are locally dense.

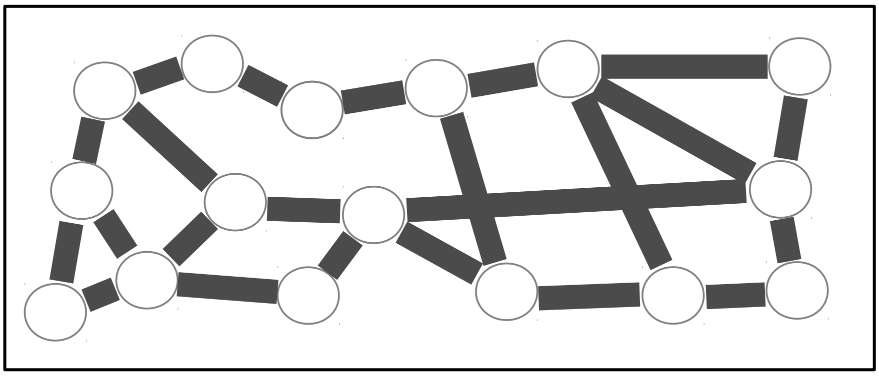

Figure 2.1: A locally dense graph as in Lemma 2.13. The sets $W_1, \ldots, W_\ell$ are depicted with grey circles. Even though there is a large number of them, each W_i is linked to only boundedly many other W_j's (at most four, in this example). Lemma 2.13 allows us to regularize all the bipartite graphs using the same system of partitions of the sets W_i.

Lemma 2.13 (Regularity Lemma for locally dense graphs)**.** *For all $m, z \in \mathbb{N}$ and $\varepsilon > 0$ there exists $q_{\mathrm{MAXCL}} \in \mathbb{N}$ such that the following is true. Suppose H and F are two graphs, $V(F) = [\ell]$ for some $\ell \in \mathbb{N}$, and $\deg^{\max}(F) \leqslant m$. Suppose that $\mathcal{Z} = \{Z_1, \ldots, Z_z\}$ is a partition of $V(H)$. Let $\{W_1, \ldots, W_\ell\}$ be a q_{MAXCL}-ensemble in H, such that for all $i, j \in [\ell]$ we have*

$$2|W_i| \geqslant |W_j| \; . \tag{2.3}$$

Then for each $i \in [\ell]$ there exists a partition $W_i^{(0)}, W_i^{(1)}, \ldots, W_i^{(p_i)}$ of the set W_i such that for all $i, j \in [\ell]$ we have

(a) $1/\varepsilon \leqslant p_i \leqslant q_{\mathrm{MAXCL}}$,

(b) $|W_i^{(i')}| = |W_j^{(j')}|$ *for each* $i' \in [p_i]$, $j' \in [p_j]$,

(c) for each $i' \in [p_i]$ *there exists* $x \in [z]$ *such that* $W_i^{(i')} \subseteq Z_x$,

(d) $\sum_i |W_i^{(0)}| < \varepsilon \sum_i |W_i|$, *and*

(e) at most $\varepsilon\,|\mathcal{Y}|$ *pairs* $\left(W_i^{(i')}, W_j^{(j')}\right) \in \mathcal{Y}$ *form an* ε*-irregular pair in* H*, where*

$$\mathcal{Y} := \left\{ \left(W_i^{(i')}, W_j^{(j')}\right) \; : \; ij \in E(F), i' \in [p_i], j' \in [p_j] \right\} \; .$$

We use Lemma 2.13 in Lemma 4.13. Lemma 4.13 is in turn the main tool in the proof of our main structural decomposition of the graph $G_{\triangleright \mathrm{T1.3}}$, Lemma 4.14. In the proof of Lemma 4.13 we decompose $G_{\triangleright \mathrm{T1.3}}$ into several parts with very different properties, and one of these parts is a locally dense graph which can be then regularized by Lemma 4.13. A similar Regularity Lemma is used in [AKSS].

The proof of Lemma 2.13 is similar to the proof of the standard Regularity Lemma 2.12, as given for example in [Sze78]. We assume the reader's familiarity with the notion of the index (a.k.a. the mean square density), and of the Index-pumping Lemma from there. We sketch the proof of Lemma 2.13 below.

Sketch of a proof of Lemma 2.13. For the sake of brevity, we omit respecting the prepartition $\mathcal{Z}$ in this sketch; this step is standard.

Before sketching a proof of the lemma, let us describe how a more naive approach fails. For each edge $ij \in E(F)$ consider a regularization of the bipartite graph $H[W_i, W_j]$, let $\{U_{i,j}^{(i')}\}_{i' \in [q_{i,j}]}$ be the partition of W_i into clusters, and let $\{U_{j,i}^{(j')}\}_{j' \in [q_{j,i}]}$ be the partition of W_j into clusters such that almost all pairs $(U_{i,j}^{(i')}, U_{j,i}^{(j')}) \subseteq (W_i, W_j)$ form an ε'-regular pair (for some ε' of our taste). We would now be done if the partition $\{U_{i,j}^{(i')}\}_{i' \in [q_{i,j}]}$ of W_i was independent of the choice of the edge ij. This however need not be the case. The natural next step would therefore be to consider the common refinement

$$\boxplus_{j:ij \in E(F)} \{U^{(i')_{i,j}}\}_{i' \in [q_{ij}]}$$

of all the obtained partitions of W_i. The pairs obtained in this way lack however any regularity properties as they are too small. Indeed, it is a notorious drawback of the Regularity Lemma that the number of clusters in the partition is enormous as a function of the regularity parameter. In our setting, this means that $q_{i,j} \gg \frac{1}{\varepsilon'}$. Thus a typical cluster $U_{i,j_1}^{(i'_1)}$ occupies on average only a $\frac{1}{q_{i,j_1}}$-fraction of the cluster $U_{i,j_2}^{(i'_2)}$, and thus already the set $U_{i,j_1}^{(i'_1)} \cap U_{i,j_2}^{(i'_2)} \subseteq U_{i,j_2}^{(i'_2)}$ is not substantial (in the sense of the regularity). The same issue arises when regularizing multicolored graphs (cf. [KS96, Theorem 1.18]). The solution is to impel the regularizations to happen in a synchronized way.

We first recall the proof of the original Regularity Lemma 2.12 which we then modify. Actually, it better suits our situation to illustrate this on a procedure which regularizes a given bipartite graph $G = (A, B; E)$. We start with arbitrary bounded partitions $\mathcal{W}_A$ and $\mathcal{W}_B$ of A and B. Sequentially, we look whether there is a witness of irregularity of $\mathcal{W}_A$ and $\mathcal{W}_B$. If there is, then the partition $\mathcal{W}_A$ and $\mathcal{W}_B$ can be refined so that the index increases. The facts that one can control the increase of the complexity of the partitions, and that the index increases substantially are the keys for guaranteeing that the iteration terminates in a bounded number of steps.

By Vizing's Theorem we can cover the edges of F by disjoint matchings $M_1, \ldots, M_{m+1}$. For each $i \in [m+1]$ we shall introduce a variable ind_i. The variable ind_i is the average index of the bipartite graphs which correspond to the edges of M_i and the current partitions of the sets W_x. In each step $i \in [m+1]$, we refine simultanously partitions in all bipartite graphs $G[W_x, W_y]$ ($xy \in M_i$) which possess witnesses of irregularity. More precisely, assume that in a certain step each set W_z is partitioned into sets $\mathcal{W}_z$. We then define

$$\mathrm{ind}_i = \frac{1}{|M_i|}\sum_{xy\in M_i} \mathrm{ind}(\mathcal{W}_x,\mathcal{W}_y)\ , \qquad \text{if } M_i\neq\emptyset, \text{ and}$$

$$\mathrm{ind}_i = 1\ , \qquad \text{otherwise.}$$

where ind is the usual index. The Index-pumping Lemma asserts that when refining the partition of $G[W_x, W_y]$ the value $\mathrm{ind}(\mathcal{W}_x,\mathcal{W}_y)$ increases substantially. The fact that M_i is a matching allows us to perform these simultaneous refinements without interference. It is well-known that none of ind_j $(j < i)$ did decrease during pumping ind_i up. Thus after a bounded number of steps there are no witnesses of irregularity in the graphs $G[W_x, W_y]$ $(xy \in E(H))$ with respect to the partitions $\mathcal{W}_x,\mathcal{W}_y$. This suffices to give the statement. □

Usually after applying the Regularity Lemma to some graph G, one bounds the number of edges which correspond to irregular pairs, to regular, but sparse pairs, or are incident with the exceptional sets U_0. We shall do the same for the setting of Lemma 2.13.

Lemma 2.14. *In the situation of Lemma 2.13, suppose that* $\deg^{\max}(H) \leqslant \Omega k$ *and* $e(H) \leqslant kn$, *and that each edge* $xy \in E(H)$ *is captured by some edge* $ij \in E(F)$, *i.e.,* $x \in W_i$, $y \in W_j$. *Moreover suppose that*

$$\mathrm{d}(W_i, W_j) \geqslant \gamma \text{ if } ij \in E(F). \tag{2.4}$$

Then all but at most $(\frac{4\varepsilon}{\gamma} + \varepsilon\Omega + \gamma)nk$ *edges of* H *belong to regular pairs* $(W^{(i)}_{i'}, W^{(j)}_{j'})$, $i, j \neq 0$, *of density at least* γ^2.

Proof. Set $w := \min\{|W_i| : i \in V(F)\}$. By (2.4), each edge of F represents at least γw^2 edges of H. Since $e(H) \leqslant kn$ it follows that $e(F) \leqslant kn/(\gamma w^2)$. Thus, by the assumption (2.3), $\sum_{AB\in E(F)} |A||B| \leqslant e(F)(2w)^2 \leqslant \frac{4kn}{\gamma}$. Using (e) of Lemma 2.13 we get that the number of edges of H contained in ε-irregular pairs from $\mathcal{Y}$ is at most

$$\frac{4\varepsilon nk}{\gamma}\ . \tag{2.5}$$

Write E_1 for the set of edges of H which are incident with a vertex in $\bigcup_{i\in[\ell]} W^{(0)}_i$. Then by (d) of Lemma 2.13, and since $\deg^{\max}(H) \leqslant \Omega k$,

$$|E_1| \leqslant \varepsilon\Omega nk\ . \tag{2.6}$$

Let E_2 be the set of those edges of H which belong to ε-regular pairs $(W^{(i')}_i, W^{(j')}_j)$ with $ij \in E(F), i' \in [p_i], j' \in [p_j]$ of density at most γ^2. We claim that

$$|E_2| \leqslant \gamma kn\ . \tag{2.7}$$

Indeed, because of (2.4) and by Fact 2.9 (with $\alpha_{\triangleright\text{F2.9}} := \gamma$ and $\beta_{\triangleright\text{F2.9}} := \gamma^2$), for each $ij \in E(F)$ there are at most $\gamma e_H(W_i, W_j)$ edges contained in the bipartite graphs $H[W_i^{(i')}, W_j^{(j')}]$, $i' \in [p_i], j' \in [p_j]$, with $\mathrm{d}_H(W_i^{(i')}, W_j^{(j')}) \leqslant \gamma^2$. Since $\sum_{ij\in E(F)} e_H(W_i, W_j) \leqslant kn$, the validity of (2.7) follows. Combining (2.5), (2.6), and (2.7) we finish the proof. □

3 Cutting trees: ℓ-fine partitions

The purpose of this section is to introduce some notation related to trees. The notion of an ℓ-fine partition of a tree shall be of particular interest. Roughly speaking, an ℓ-fine partition of a tree $T \in \mathbf{trees}(k)$ is a partition of the T into a small number of cut-vertices and subtrees of order at most ℓ with some additional properties. This notion is essential for our proof of Theorem 1.3 as we use a certain sequential procedure to embed $T_{\triangleright\text{T1.3}}$ into the host graph $G_{\triangleright\text{T1.3}}$, embedding a subtree after subtree.

Let T be a tree rooted at r, inducing the partial order $\preceq$ on $V(T)$ (with r as the minimal element). If $a \preceq b$ and $ab \in E(T)$ then we say b is a *child of* a and a is the *parent of* b. $\mathrm{Ch}(a)$ denotes the set of children of a, and the parent of a vertex $b \neq r$ is denoted $\mathrm{Par}(b)$. For a set $U \subseteq V(T)$ write $\mathrm{Par}(U) := \bigcup_{u\in U\setminus\{r\}} \mathrm{Par}(u) \setminus U$ and $\mathrm{Ch}(U) := \bigcup_{u\in U} \mathrm{Ch}(u) \setminus U$.

We say that a tree $T' \subseteq T$ is *induced* by a vertex $x \in V(T)$ if $V(T')$ is the up-closure of x in $V(T)$, i.e., $V(T') = \{v \in V(T) : x \preceq v\}$. We then write $T' = T(r, \uparrow x)$, or $T' = T(\uparrow x)$, if the root is obvious from the context and call T' an *end subtree*. Subtrees of T that are not end subtrees are called *internal subtrees*.

Let T be a tree rooted at r and let $T' \subseteq T$ be a subtree with $r \notin V(T')$. The *seed of* T' is the $\preceq$-maximal vertex $x \in V(T) \setminus V(T')$ such that $x \preceq v$ for all $v \in V(T')$. We write $\mathrm{Seed}(T') = x$. A *fruit* in a rooted tree (T, r) is any vertex $u \in V(T)$ whose distance from r is even and at least four.

We can now state the most important definition of this section.

Definition 3.1 (**ℓ-fine partition**). *Let $T \in \mathbf{trees}(k)$ be a tree rooted at r. An* ℓ-fine partition of *T is a quadruple $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$, where $W_A, W_B \subseteq V(T)$ and $\mathcal{S}_A$, $\mathcal{S}_B$ are families of subtrees of T such that*

(a) the three sets W_A, W_B and $\{V(T^*)\}_{T^*\in\mathcal{S}_A\cup\mathcal{S}_B}$ partition $V(T)$,

(b) $r \in W_A \cup W_B$,

(c) $\max\{|W_A|, |W_B|\} \leqslant 336k/\ell$,

(d) *for* $w_1, w_2 \in W_A \cup W_B$ *the distance* $\mathrm{dist}(w_1, w_2)$ *is odd if and only if one of them lies in* W_A *and the other one in* W_B*,*

(e) $v(T^*) \leqslant \ell$ *for every tree* $T^* \in \mathcal{S}_A \cup \mathcal{S}_B$*,*

(f) $V(T^*) \cap \mathrm{N}(W_B) = \emptyset$ *for every* $T^* \in \mathcal{S}_A$ *and* $V(T^*) \cap \mathrm{N}(W_A) = \emptyset$ *for every* $T^* \in \mathcal{S}_B$*,*

(g) *each tree of* $\mathcal{S}_A \cup \mathcal{S}_B$ *has its seed in* $W_A \cup W_B$*,*

(h) $|V(T^*) \cap \mathrm{N}(W_A \cup W_B)| \leqslant 2$ *for each* $T^* \in \mathcal{S}_A \cup \mathcal{S}_B$*,*

(i) *if* $V(T^*) \cap \mathrm{N}(W_A \cup W_B)$ *contains two distinct vertices* y_1*,* y_2 *for some* $T^* \in \mathcal{S}_A \cup \mathcal{S}_B$*, then* $\mathrm{dist}(y_1, y_2) \geqslant 4$*,*

(j) *if* $T_1, T_2 \in \mathcal{S}_A \cup \mathcal{S}_B$ *are two internal subtrees of* T *such that* $v_1 \in T_1$ *precedes* $v_2 \in T_2$ *then* $\mathrm{dist}_T(v_1, v_2) > 2$*,*

(k) $\mathcal{S}_B$ *does not contain any internal tree of* T*, and*

(l) $\sum_{\substack{T^* \in \mathcal{S}_A \\ T^* \text{ end tree of } T}} v(T^*) \geqslant \sum_{T^* \in \mathcal{S}_B} v(T^*)$ *.*

Remark 3.2. *It is easy to see that any* ℓ*-fine partition* $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ *of a tree* (T, r) *is determined once we know the set* $W = W_A \cup W_B$*, except possibly for being able to swap* W_A *with* W_B *and* $\mathcal{S}_A$ *with* $\mathcal{S}_B$*. Indeed, the division of* W *into two sets* W' *and* W'' *follows the bipartition of* T*, and conditions* (k) *and* (l) *determine which of* W'*,* W'' *is* W_A *unless* $T - W$ *contains no internal trees and* (l) *would hold either way. During the proof of Lemma 3.4 below we shall therefore sometimes just say one of the conditions* (a)–(l) *holds for the set* W*, and not explicitly mention the tuple* $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$*.*

Remark 3.3. *Suppose that* $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ *is an* ℓ*-fine partition of a tree* (T, r)*, and suppose that* $T^* \in \mathcal{S}_A \cup \mathcal{S}_B$ *is such that* $|V(T^*) \cap \mathrm{N}(W_A \cup W_B)| = 2$*. Let us root* T^* *at the neighbour* r_1 *of its seed, and let* r_2 *be the other vertex of* $V(T^*) \cap \mathrm{N}(W_A \cup W_B)$*. Then* (d)*,* (f)*, and* (i) *imply that* r_2 *is a fruit in* (T^*, r_1)*.*

The following is the main lemma of this section. It asserts that each tree of order k has ℓ-fine partitions for all values of $\ell \leqslant k$.

Lemma 3.4. *Let* $T \in \mathbf{trees}(k)$ *be a tree rooted at* r *and let* $\ell \in \mathbb{N}$ *with* $\ell \leqslant k$*. Then* T *has an* ℓ*-fine partition.*

Similar but simpler tree-cutting procedures were used in other literature concerning the Loebl-Komlós-Sós Conjecture in the dense setting, cf. [AKS95, HP, PS12, Zha11]. There, using the notation of Conjecture 1.2, the trees in $\mathcal{S}_A \cup \mathcal{S}_B$ of an ℓ-fine partition of a tree $T \in \mathbf{trees}(k)$ are embedded in regular pairs of a Regularity Lemma decomposition of the host graph G. In the current paper however, a more complex decomposition result (Lemma 4.14) than the Regularity Lemma is used to capture the structure of G. To this end we had to further strengthen the features of the ℓ-fine partition. In particular, features (h), (i), (j) of Definition 3.1 were introduced to handle the more complex embedding procedures in our setting.

Remark 3.5. *(i) In our proof of Theorem 1.3, we shall apply Lemma 3.4 to a tree $T_{\triangleright\mathrm{T1.3}} \in \mathbf{trees}(k)$. The number $\ell_{\triangleright\mathrm{L3.4}}$ will be linear in k, and thus* (c) *of Definition 3.1 tells us that the size of the sets W_A and W_B is bounded by an absolute constant.*

(ii) Each internal tree in $\mathcal{S}_A$ of an ℓ-fine partition has a unique vertex from W_A above it. Thus with $\ell_{\triangleright\mathrm{L3.4}}$ as above also the number of internal trees in $\mathcal{S}_A$ is bounded by an absolute constant. This need not not be the case for the number of end trees. For instance, if $(T_{\triangleright\mathrm{T1.3}}, r)$ is a star with $k-1$ leaves and rooted at its centre r then $W_A = \{r\}$ while the $k-1$ leaves of $T_{\triangleright\mathrm{T1.3}}$ form the end shrubs in $\mathcal{S}_A$.

Proof of Lemma 3.4. First we shall use an inductive construction to get candidates for W_A, W_B, $\mathcal{S}_A$ and $\mathcal{S}_B$, which we shall modify later on, so that they satisfy all the conditions required by Definition 3.1.

Set $T_0 := T$. Now, inductively for $i \geqslant 1$ choose a $\preceq$-maximal vertex $x_i \in V(T_{i-1})$ with the property that $v(T_{i-1}(\uparrow x_i)) > \ell$. We set $T_i := T_{i-1} - (V(T_{i-1}(\uparrow x_i)) \setminus \{x_i\})$. If, say at step $i = i_{\mathrm{end}}$, no such x_i exists, then $v(T_{i-1}) \leqslant \ell$. In that case, set $x_i := r$, set $W_1 := \{x_i\}_{i=1}^{i_{\mathrm{end}}}$ and terminate. The fact that $v(T_{i-1} - V(T_i)) \geqslant \ell$ for each $i < i_{\mathrm{end}}$ implies that

$$|W_1| - 1 = i_{\mathrm{end}} - 1 \leqslant k/\ell\ . \tag{3.1}$$

Let $\mathcal{C}$ be the set of all components of the forest $T - W_1$. Observe that by the choice of the x_i each $T^* \in \mathcal{C}$ has order at most ℓ.

Let A and B be the colour classes of T such that $r \in A$. Now, choosing W_A as $W_1 \cap A$ and W_B as $W_1 \cap B$ and dividing $\mathcal{C}$ adequately into sets $\mathcal{S}_A$ and $\mathcal{S}_B$ would yield a quadruple that satisfies conditions (a), (b), (c), (d), (e) and (g). In order to find also the remaining properties satisfied, we shall refine our tree partition by adding more vertices to W_1, thus making the trees in $\mathcal{S}_A \cup \mathcal{S}_B$ smaller. In doing so, we have to be careful not to end up violating (c). We shall enlarge the set of cut vertices in several steps, accomplishing sequentially, in this order, also properties (h), (j), (f), (i), and in the last step at the same time (k) and (l). It will be easy to check that in each of the steps none of the previously established properties is lost, so we will not explicitly check them, except for (c).

For condition (h), first define T' as the subtree of T that contains all vertices of W_1 and all vertices that lie on paths in T which have both endvertices in W_1. Now, if a subtree $T^* \in \mathcal{C}$ does not already satisfy (h) for W_1, then $V(T^*) \cap V(T')$ must contain some vertices of degree at least three. We will add the set $Y(T^*)$ of all these vertices to W_1. Formally, let Y be the union of the sets $Y(T^*)$ over all $T^* \in \mathcal{C}$, and set $W_2 := W_1 \cup Y$. Then the components of $T - W_2$ satisfy (h).

Let us upper-bound the size of the set W_2. For each $T^* \in \mathcal{C}$, note that by Fact 2.2 for $T^* \cap T'$, we know that $|Y(T^*)|$ is at most the number of leaves of $T^* \cap T'$ (minus two). On the other hand, each leaf of $T^* \cap T'$ has a child in W_1 (in T). As these children are distinct for different trees $T^* \in \mathcal{C}$, we find that $|Y| \leqslant |W_1|$ and thus

$$|W_2| \leqslant 2|W_1| \,. \tag{3.2}$$

Next, for condition (j), observe that by setting $W_3 := W_2 \cup \mathrm{Par}_T(W_2)$ the components of $T - W_3$ fulfill (j). We have

$$|W_3| \leqslant 2|W_2| \overset{(3.2)}{\leqslant} 4|W_1| \,. \tag{3.3}$$

In order to ensure condition (f), let R^* be the set of the roots ($\preceq$-minimal vertices) of those components T^* of $T - W_3$ which contain neighbours of both colour classes of T. Setting $W_4 := W_3 \cup R^*$ we see that (f) is satisfied for W_4. Furthermore, as for each vertex in R^* there is a distinct member of W_3 above it in the order on T, we obtain

$$|W_4| \leqslant 2|W_3| \overset{(3.3)}{\leqslant} 8|W_1|. \tag{3.4}$$

Next, we shall aim for a stronger version of property (i), namely,

(i') if $V(T^*) \cap \mathrm{N}_T(W_A \cup W_B) = \{y_1, y_2\}$ with $y_1 \neq y_2$ for some $T^* \in \mathcal{S}_A \cup \mathcal{S}_B$, then $\mathrm{dist}(y_1, y_2) \geqslant 6$.

The reason for requiring this strengthening is that later we might introduce additional cut vertices which would "shorten T^* by two".

Consider a component T^* of $T - W_4$ which is an internal tree of T. If T^* contains two distinct neighbours y_1, y_2 of W_4 such that $\mathrm{dist}_{T^*}(y_1, y_2) < 6$, then we call T^* *short*. Observe that there are at most $|W_4|$ short trees, because each of these trees has a unique vertex from W_4 above it. Let $Z(T^*) \subseteq V(T^*)$ be the vertices on the path from y_1 to y_2. Then $|Z(T^*)| \leqslant 6$. Letting Z be the union of the sets $Z(T^*)$ over all short trees in $T - W_4$, and set $W_5 := W_4 \cup Z$, we obtain

$$|W_5| \leqslant |W_4| + 6|W_4| \overset{(3.4)}{\leqslant} 56|W_1| \overset{(3.1)}{\leqslant} 112k/\ell. \tag{3.5}$$

We still need to ensure (k) and (l). To this end, consider the set $\mathcal{C}'$ of all components of $T - W_5$. Set $\mathcal{C}'_A := \{T^* \in \mathcal{C}' : \mathrm{Seed}(T^*) \in A\}$ and set $\mathcal{C}'_B := \mathcal{C}' \setminus \mathcal{C}'_A$. We assume that

$$\sum_{T^* \in \mathcal{C}'_A \,:\, T^* \text{ end tree of } T} v(T^*) \geqslant \sum_{T^* \in \mathcal{C}'_B \,:\, T^* \text{ end tree of } T} v(T^*)\ , \tag{3.6}$$

as otherwise we can simply swap A and B.

Now, for each $T^* \in \mathcal{C}'_B$ that is not a end subtree of T, set $X(T^*) := V(T^*) \cap \mathrm{N}_T(W_5)$. Let X be the union of all such sets $X(T^*)$. Observe that

$$|X| \leqslant 2|W_5 \cap B| \leqslant 2|W_5|. \tag{3.7}$$

For $W := W_5 \cup X$, all internal trees of $T - W$ have their seeds in A. This will guarantee (k), and, together with (3.6), also (l).

Finally, set $W_A := W \cap A$ and $W_B := W \cap B$, and let $\mathcal{S}_A$ and $\mathcal{S}_B$ be the sets of those components of $T - W$ that have their seeds in W_A and W_B, respectively. By construction, $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ has all the properties of an ℓ-fine partition. In particular, for (c), we find with (3.5) and (3.7) that $|W| \leqslant |W_5| + 2|W_5 \cap B| \leqslant 336k/\ell$. □

For an ℓ-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of a rooted tree (T, r), the trees $T^* \in \mathcal{S}_A \cup \mathcal{S}_B$ are called *shrubs*. An *end shrub* is a shrub which is an end subtree. An *internal shrub* is a shrub which is an internal subtree. A *knag* is a component of the forest $T[W_A \cup W_B]$. Suppose that $T^* \in \mathcal{S}_A$ is an internal shrub, and r^* its $\preceq_r$-minimal vertex. Then $T^* - r^*$ contains a unique component with a vertex from $\mathrm{N}_T(W_A)$. We call this component *principal subshrub*, and the other components *peripheral subshrubs*.

Definition 3.6 (ordered skeleton). *Let $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ be an ℓ-fine partition of a rooted tree (T, r). We then say that the sequence $\left(P_0^*, T_1^*, P_1^*, \ldots, T_m^*, P_m^*\right)$ is an* ordered skeleton *of the ℓ-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ if the following conditions are fulfilled:*

- *$(P_i^*)_{i=0}^m$ is an ordering of the knags of T, and P_0^* contains r,*
- *$(T_i^*)_{i=1}^m$ is an ordering of all internal shrubs, and*
- *for each $i = 1, \ldots, m$ we have that the subgraph formed by $P_0^*, T_1^*, P_1^*, \ldots, T_i^*$ and P_i^* is connected in T.*

The next lemma which follows directly from Definition 3.1 states that an ordered skeleton exists for any fine partition.

Lemma 3.7. *There exists an ordered skeleton of any ℓ-fine partition of any rooted tree.*

Figure 3.1 shows an (τk)-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of a binary tree $T \in$ $\mathbf{trees}(k)$, for a fixed $\tau > 0$ and k large. The vertices whose distance is $O(\log(\tau^{-1}))$ from the root comprise a sole knag of T (with respect to $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$). This example will be important in Section 4.5.

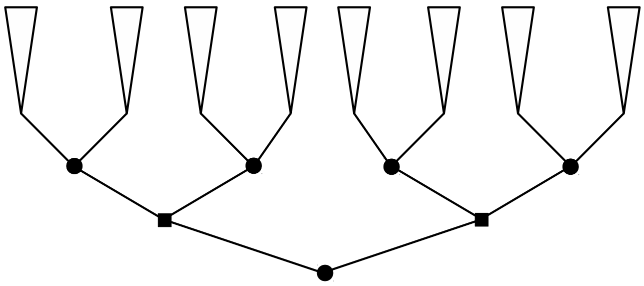

Figure 3.1: An (τk)-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of a binary tree $T \in \mathbf{trees}(k)$. The set W_A is denoted by circles and W_B by squares. The sole knag is of depth $O(\log(\tau^{-1}))$, two in this picture. Each schematic triangle represents one end shrub of $\mathcal{S}_A \cup \mathcal{S}_B$.

4 Decomposing sparse graphs

In this section, we work out a structural decomposition of a possibly sparse graph which is suitable for embedding trees. Our motivation comes from the success of the Regularity Method in the setting of dense graphs (see [KO09]). The main technical result of this section, the "decomposition lemma", Lemma 4.13, provides such a decomposition. Roughly speaking, each graph of a moderate maximum degree can be decomposed into regular pairs, and two different expanding parts.

We then combine Lemma 4.13 with a lemma on creating a gap in the degree sequence (Lemma 4.1) to get a decomposition lemma for graphs from $\mathbf{LKS}(n, k, \eta)$, Lemma 4.14. Lemma 4.14 asserts that each graph from $\mathbf{LKS}(n, k, \eta)$ can be decomposed into vertices of degree much larger than k, regular pairs, and expanding parts. As a careful reader can check from the proof of Lemma 4.14 below, such a decomposition is possible for *any* graph; in Lemma 4.14 however we use properties specific to the class $\mathbf{LKS}(n, k, \eta)$ to get some additional features of the decomposition. Indeed, we expect that our technique will find applications in other tree embedding problems, and possibly elsewhere.

4.1 Creating a gap in the degree sequence

The goal of this section is to show that any graph $G \in \mathbf{LKSmin}(n, k, \eta)$ has a subgraph $G' \in \mathbf{LKSsmall}(n, k, \eta/2)$ which has a gap in its degree sequence. Note that G' then contains almost all the edges of G. This is formulated in the next lemma.

Lemma 4.1. *Let $G \in \mathbf{LKSmin}(n,k,\eta)$ and let $(\Omega_i)_{i\in\mathbb{N}}$ be a sequence of positive numbers with $\Omega_j/\Omega_{j+1} \leqslant \eta^2/100$ for all $j \in \mathbb{N}$. Then there is an index $i^* \leqslant 100\eta^{-2}$ and a subgraph $G' \subseteq G$ such that*

(i) $G' \in \mathbf{LKSsmall}(n,k,\eta/2)$, and

(ii) no vertex $v \in V(G')$ has degree $\deg_{G'}(v) \in [\Omega_{i^*}k, \Omega_{i^*+1}k)$.

Proof. Set $R := \lfloor 100\eta^{-2} \rfloor$. For $i \in [R]$ and any graph $H \subseteq G$ define the sets $X_i(H) := \{v \in V(H) : \deg_H(v) \in [\Omega_i k, \Omega_{i+1}k)\}$ and for $i = R+1$ set $X_i(H) := \{v \in V(H) : \deg_H(v) \in [\Omega_i k, \infty)\}$. As

$$\sum_{i\in[R]} \quad \sum_{v\in X_i(G)\cup X_{i+1}(G)} \deg(v) \leqslant 4e(G) \ ,$$

by averaging we find an index $i^* \in [R]$ such that

$$\sum_{v\in X_{i^*}(G)\cup X_{i^*+1}(G)} \deg(v) \leqslant \frac{4e(G)}{R}. \tag{4.1}$$

Let E_0 be the set of all the edges incident with $X_{i^*}(G) \cup X_{i^*+1}(G)$. Now, starting with $G_0 := G - E_0$, successively define graphs $G_j \subsetneq G_{j-1}$ for $j \geqslant 1$ using any of the following two types of edge deletions:

(T1) If there is a vertex $v_j \in X_{i^*}(G_{j-1})$ then we choose an edge e_j that is incident with v_j, and set $G_j := G_{j-1} - e_j$.

(T2) If there is an edge $e_j = u_jv_j$ of G_{j-1} with $u_j \in \mathbb{S}_{\eta/2,k}(G_{j-1})$ and $v_j \in \bigcup_{i=i^*+1}^{R+1} X_i(G_{j-1})$ then we set $G_j := G_{j-1} - e_j$.

Since we keep deleting edges, the procedure stops at some point, say at step j^*, when neither of (T1), (T2) is applicable. Note that the resulting graph G_{j^*} already has Property (ii).

Let $E_1 \subseteq E(G)$ be the set of those edges deleted by applying (T1). We shall estimate the size of E_1. First, observe that

$$\left| \bigcup_{i=i^*+2}^{R+1} X_i(G) \right| \leqslant \frac{2e(G)}{\Omega_{i^*+2}k} \ .$$

Moreover, each vertex of $\bigcup_{i=i^*+2}^{R+1} X_i(G)$ appears at most $(\Omega_{i^*+1} - \Omega_{i^*})k < \Omega_{i^*+1}k$ times as the vertex v_j in the deletions of type (T1). Consequently,

$$|E_1| \leqslant \Omega_{i^*+1} \left| \bigcup_{i=i^*+2}^{R+1} X_i(G) \right| k \leqslant \frac{2\Omega_{i^*+1}e(G)}{\Omega_{i^*+2}} \ . \tag{4.2}$$

Now, observe that the vertices in $\mathbb{L}_{\eta,k}(G) \cap \mathbb{S}_{\eta/2,k}(G_{j^*})$ have dropped their degree from $(1+\eta)k$ to $(1+\eta/2)k$ by operations other than (T2). So each of these vertices is incident with at least $\eta k/2$ edges from the set $E_0 \cup E_1$. Therefore, by the definition of E_0, by (4.1), and by (4.2),

$$\left|\mathbb{L}_{\eta,k}(G) \cap \mathbb{S}_{\eta/2,k}(G_{j^*})\right| \leqslant \frac{2 \cdot |E_0 \cup E_1|}{\eta k/2} \leqslant \left(\frac{4}{R} + \frac{2\Omega_{i^*+1}}{\Omega_{i^*+2}}\right) \cdot \frac{4e(G)}{\eta k} \overset{(2.1)}{\leqslant} \frac{\eta n}{2} .$$

Thus

$$|\mathbb{L}_{\eta/2,k}(G_{j^*})| \geqslant |\mathbb{L}_{\eta,k}(G)| - |\mathbb{L}_{\eta,k}(G) \cap \mathbb{S}_{\eta/2,k}(G_{j^*})| \geqslant (1/2 + \eta/2)n ,$$

and consequently, $G_{j^*} \in \mathbf{LKS}(n, k, \eta/2)$.

Last, we obtain the graph G' by successively deleting any edge from G_{j^*} which connects a vertex from $\mathbb{S}_{\eta/2,k}(G_{j^*})$ with a vertex whose degree is not exactly $\lceil(1+\frac{\eta}{2})k\rceil$. This does not affect the already obtained Property (ii), since we could not apply (T2) to G_{j^*}. We claim that for the resulting graph G' we have $G' \in \mathbf{LKSsmall}(n, k, \eta/2)$. Indeed, $\mathbb{L}_{\eta/2,k}(G') = \mathbb{L}_{\eta/2,k}(G_{j^*})$, and thus $G' \in \mathbf{LKS}(n, k, \eta/2)$. Property 2 of Definition 2.6 follows from the last step of the construction of G'. To see Property 1 of Definition 2.6 we use Fact 2.5(2) for G (which by assumption is in $\mathbf{LKSmin}(n, k, \eta)$). □

4.2 Decomposition of graphs with moderate maximum degree

First we introduce some useful notions. We start with dense spots which indicate an accumulation of edges in a sparse graph.

Definition 4.2 (**(m, γ)-dense spot, (m, γ)-nowhere-dense**). *An* (m, γ)-dense spot *in a graph G is a non-empty bipartite subgraph $D = (U, W; F)$ of G with* $\mathrm{d}(D) > \gamma$ *and* $\deg^{\min}(D) > m$. *We call G* (m, γ)-nowhere-dense *if it does not contain any (m, γ)-dense spot.*

We remark that dense spots as bipartite graphs do not have a specified orientation, that is, we view $(U, W; F)$ and $(W, U; F)$ as the same object.

Fact 4.3. *Let $(U, W; F)$ be a $(\gamma k, \gamma)$-dense spot in a graph G of maximum degree at most Ωk. Then* $\max\{|U|, |W|\} \leqslant \frac{\Omega}{\gamma}k$.

Proof. It suffices to observe that

$$\gamma|U||W| \leqslant e(U, W) \leqslant \deg^{\max}(G) \cdot \min\{|U|, |W|\} \leqslant \Omega k \cdot \min\{|U|, |W|\}.$$

□

The next fact asserts that in a bounded degree graph there cannot be too many edge-disjoint dense spots containing a given vertex.

Fact 4.4. *Let H be a graph of maximum degree at most Ωk, let $v \in V(H)$, and let $\mathcal{D}$ be a family of edge-disjoint $(\gamma k, \gamma)$-dense spots. Then less than $\frac{\Omega}{\gamma}$ dense spots from $\mathcal{D}$ contain v.*

Proof. This follows as v sends more than γk edges to each dense spot from $\mathcal{D}$ it is incident with, the dense spots $\mathcal{D}$ are edge-disjoint, and $\deg(v) \leqslant \Omega k$. □

Last, we include a bound concerning the total size of dense spots intersecting substantially a given set.

Fact 4.5. *Let H be a graph of maximum degree at most Ωk. Let $Y \subseteq V(H)$ be a set of size at most Ak, and $\mathcal{D}$ a family of edge-disjoint $(\gamma k, \gamma)$-dense spots. Define $\mathcal{D}' := \{D \in \mathcal{D} : |V(D) \cap Y| \geqslant \beta k\}$. Then for the set $X := \bigcup_{D \in \mathcal{D}'} V(D)$ we have $|X| \leqslant \frac{2A\Omega^2}{\beta\gamma^2} k$.*

Proof. Let us count the number of certain pairs (y, D) in two different ways.

$$\beta k|\mathcal{D}'| \leqslant \left|\{(y, D) \ : \ y \in Y, D \in \mathcal{D}', y \in V(\mathcal{D}')\}\right| \overset{\text{F4.4}}{\leqslant} |Y| \frac{\Omega}{\gamma} \,.$$

Put together, $|\mathcal{D}'| \leqslant \frac{A\Omega}{\beta\gamma}$. The fact now follows from Fact 4.3. □

Our second definition of this section might seem less intuitive at first sight. It describes a property for finding dense spots outside some "forbidden" set U, which in later applications will be the set of vertices already used for a partial embedding of a tree $T_{\triangleright \text{T1.3}} \in \mathbf{trees}(k)$ in Theorem 1.3 during our sequential embedding procedure.

Definition 4.6 (**$(\Lambda, \varepsilon, \gamma, k)$-avoiding set**)**.** *Suppose that G is a graph and $\mathcal{D}$ is a family of dense spots in G. A set $\mathfrak{A} \subseteq \bigcup_{D \in \mathcal{D}} V(D)$ is* $(\Lambda, \varepsilon, \gamma, k)$-avoiding *with respect to $\mathcal{D}$ if for every $\bar{U} \subseteq V(G)$ with $|\bar{U}| \leqslant \Lambda k$ the following holds that for all but at most εk vertices $v \in \mathfrak{A}$. There is a dense spot $D \in \mathcal{D}$ with $|\bar{U} \cap V(D)| \leqslant \gamma^2 k$ that contains v.*

Note that a subset of a $(\Lambda, \varepsilon, \gamma, k)$-avoiding set is also $(\Lambda, \varepsilon, \gamma, k)$-avoiding.

We now come to the main concepts of this section, the bounded and the sparse decompositions. These notions in a way correspond to the partition structure from the Regularity Lemma, although naturally more complex since we deal with (possibly) sparse graphs here. Lemma 4.13 is then a corresponding regularization result.

Definition 4.7 (**$(k, \Lambda, \gamma, \varepsilon, \nu, \rho)$-bounded decomposition**)**.** *Let $\mathcal{V} = \{V_1, V_2, \ldots, V_s\}$ be a partition of the vertex set of a graph G. We say that $(\mathbf{V}, \mathcal{D}, G_{\text{reg}}, G_{\text{exp}}, \mathfrak{A})$ is a* $(k, \Lambda, \gamma, \varepsilon, \nu, \rho)$-bounded decomposition *of G with respect to $\mathcal{V}$ if the following properties are satisfied:*

1. *The elements of $\mathbf{V}$ are disjoint subsets of $V(G)$.*

2. *G_{reg} is a subgraph of $G - G_{\text{exp}}$ on the vertex set $\bigcup \mathbf{V}$. For each edge $xy \in E(G_{\text{reg}})$ there are distinct $C_x \ni x$ and $C_y \ni y$ from $\mathbf{V}$, and $G[C_x, C_y] = G_{\text{reg}}[C_x, C_y]$. Furthermore, $G[C_x, C_y]$ forms an ε-regular pair of density least γ^2.*

3. We have $\nu k \leqslant |C| = |C'| \leqslant \varepsilon k$ for all $C, C' \in \mathbf{V}$.

4. $\mathcal{D}$ is a family of edge-disjoint $(\gamma k, \gamma)$-dense spots in $G - G_{\mathrm{exp}}$. For each $D = (U, W; F) \in \mathcal{D}$ all the edges of $G[U, W]$ are covered by $\mathcal{D}$ (but not necessarily by D).

5. If G_{reg} contains at least one edge between $C_1, C_2 \in \mathbf{V}$ then there exists a dense spot $D = (U, W; F) \in \mathcal{D}$ such that $C_1 \subseteq U$ and $C_2 \subseteq W$.

6. For all $C \in \mathbf{V}$ there is $V \in \mathcal{V}$ so that either $C \subseteq V \cap V(G_{\mathrm{exp}})$ or $C \subseteq V \setminus V(G_{\mathrm{exp}})$. For all $C \in \mathbf{V}$ and $D = (U, W; F) \in \mathcal{D}$ we have $C \cap U \in \{\emptyset, C\}$.

7. G_{exp} is a $(\gamma k, \gamma)$-nowhere-dense subgraph of G with $\mathrm{deg}^{\mathrm{min}}(G_{\mathrm{exp}}) > \rho k$.

8. $\mathfrak{A}$ is a $(\Lambda, \varepsilon, \gamma, k)$-avoiding subset of $V(G) \setminus \bigcup \mathbf{V}$ with respect to dense spots $\mathcal{D}$.

We say that the bounded decomposition $(\mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ respects the avoiding threshold b *if for each $C \in \mathbf{V}$ we either have* $\mathrm{deg^{max}}_G(C, \mathfrak{A}) \leqslant b$, *or* $\mathrm{deg^{min}}_G(C, \mathfrak{A}) > b$.

Let us remark that "exp" in G_{exp} stands for "expander" and "reg" in G_{reg} stands for "regular(ity)".

The members of $\mathbf{V}$ are called *clusters*. Define the *cluster graph* $\mathbf{G}_{\mathrm{reg}}$ as the graph on the vertex set $\mathbf{V}$ that has an edge C_1C_2 for each pair (C_1, C_2) which has density at least γ^2 in the graph G_{reg}.

Property 6 tells us that the clusters may be prepartitioned, just as it is the case in the classic Regularity Lemma. When classifying the graph $G_{\triangleright\mathrm{T1.3}}$ in Lemma 4.14 below we shall use the prepartition into (roughly) $\mathbb{S}_{\alpha_{\triangleright\mathrm{T1.3}},k}(G_{\triangleright\mathrm{T1.3}})$ and $\mathbb{L}_{\alpha_{\triangleright\mathrm{T1.3}},k}(G_{\triangleright\mathrm{T1.3}})$.

As said above, the notion of bounded decomposition is needed for our Regularity Lemma type decomposition given in Lemma 4.13. It turns out that such a decomposition is possible only when the graph is of moderate maximum degree. On the other hand, Lemma 4.1 tells us that the vertex set of any graph[viii] can be decomposed into vertices of enormous degree and moderate degree. The graph induced by the latter type of vertices then admits the decomposition from Lemma 4.13. Thus, it makes sense to enhance the structure of bounded decomposition by vertices of unbounded degree. This is done in the next definition.

Definition 4.8 ($(k, \Omega^{**}, \Omega^*, \Lambda, \gamma, \varepsilon, \nu, \rho)$**-sparse decomposition).** *Let $\mathcal{V} = \{V_1, V_2, \ldots, V_s\}$ be a partition of the vertex set of a graph G. We say that $\nabla = (\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ is a*

[viii]Lemma 4.1 is stated only for graphs from $\mathbf{LKSmin}(n, k, \eta)$, but a similar statement can be made about any graph.

$(k, \Omega^{**}, \Omega^*, \Lambda, \gamma, \varepsilon, \nu, \rho)$-sparse decomposition *of G with respect to $V_1, V_2, \ldots, V_s$ if the following holds.*

1. *$\mathbf{\Psi} \subseteq V(G)$, $\mathrm{deg}^{\mathrm{min}}{}_G(\mathbf{\Psi}) \geqslant \Omega^{**}k$, $\mathrm{deg}^{\mathrm{max}}{}_H(V(G) \setminus \mathbf{\Psi}) \leqslant \Omega^* k$, where H is spanned by the edges of $\bigcup \mathcal{D}$, G_{exp}, and edges incident with $\mathbf{\Psi}$,*
2. *$(\mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ is a $(k, \Lambda, \gamma, \varepsilon, \nu, \rho)$-bounded decomposition of $G - \mathbf{\Psi}$ with respect to $V_1 \setminus \mathbf{\Psi}, V_2 \setminus \mathbf{\Psi}, \ldots, V_s \setminus \mathbf{\Psi}$.*

If the parameters do not matter, we call ∇ simply a *sparse decomposition*, and similarly we speak about a *bounded decomposition*.

Definition 4.9 (captured edges). *In the situation of Definition 4.8, we refer to the edges in $E(G_{\mathrm{reg}}) \cup E(G_{\mathrm{exp}}) \cup E_G(\mathbf{\Psi}, V(G)) \cup E_G(\mathfrak{A}, \mathfrak{A} \cup \bigcup \mathbf{V})$ as* captured *by the sparse decomposition. We write G_∇ for the subgraph of G on the same vertex set which consists of the captured edges. Likewise, the captured edges of a bounded decomposition $(\mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ of a graph G are those in $E(G_{\mathrm{reg}}) \cup E(G_{\mathrm{exp}}) \cup E_G(\mathfrak{A}, \mathfrak{A} \cup \bigcup \mathbf{V})$.*

Throughout the paper we write $G_{\mathcal{D}}$ for the subgraph of G which consists of the edges contained in $\mathcal{D}$. We now include an easy fact about the relation of $G_{\mathcal{D}}$ and G_{reg}.

Fact 4.10. *Let $\nabla = (\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ be a sparse decomposition of a graph G. Then each edge $xy \in E(G_{\mathcal{D}})$ with $x, y \in \bigcup \mathbf{V}$ is either contained in G_{reg}, or is not captured.*

Proof. Indeed, suppose that $xy \in E(G_{\mathcal{D}})$, $x, y \in \bigcup \mathbf{V}$, and $xy \notin E(G_{\mathrm{reg}})$. Property 2 of Definition 4.8 says that $x, y \notin \mathbf{\Psi}$. Further, by Property 8 of Definition 4.7, we have $x, y \notin \mathfrak{A}$. Last, Property 4 of Definition 4.7 implies that $xy \notin E(G_{\mathrm{exp}})$. Hence xy is not captured, as desired. □

We now give a bound on the number of clusters reachable through edges of the dense spots from a fixed vertex outside $\mathbf{\Psi}$.

Fact 4.11. *Let $\nabla = (\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ be a $(k, \Omega^{**}, \Omega^*, \Lambda, \gamma, \varepsilon, \nu, \rho)$-sparse decomposition of a graph G. Let $x \in V(G) \setminus \mathbf{\Psi}$. Assume that $\mathbf{V} \neq \emptyset$, and let $\mathfrak{c}$ be the size of any member of $\mathbf{V}$. Then there are less than*

$$\frac{2(\Omega^*)^2 k}{\gamma^2 \mathfrak{c}} \leqslant \frac{2(\Omega^*)^2}{\gamma^2 \nu}$$

clusters $C \in \mathbf{V}$ with $\mathrm{deg}_{G_{\mathcal{D}}}(x, C) > 0$.

Proof. Property 1 of Definition 4.8 says that $\mathrm{deg}_{G_{\mathcal{D}}}(x) \leqslant \Omega^* k$. For each $D \in \mathcal{D}$ with $x \in V(D)$ we have that $\mathrm{deg}_D(x) > \gamma k$, since D is a $(\gamma k, \gamma)$-dense spot. By Fact 4.4

$$|\{D \in \mathcal{D} : \mathrm{deg}_D(x) > 0\}| < \frac{\Omega^*}{\gamma}. \tag{4.3}$$

Furthermore, by Fact 4.3, and using Property 3 of Definition 4.7, we see that for a fixed $D \in \mathcal{D}$, we have

$$|\{C \in \mathbf{V} \,:\, C \subseteq V(D)\}| \leqslant \frac{2\Omega^* k}{\gamma} \cdot \frac{1}{\mathfrak{c}} \leqslant \frac{2\Omega^*}{\gamma\nu} \,.$$

Together with (4.3) this gives that the number of clusters $C \in \mathbf{V}$ with $\deg_{G_{\mathcal{D}}}(x, C) > 0$ is less than

$$\frac{\Omega^*}{\gamma} \cdot \frac{2\Omega^* k}{\gamma\mathfrak{c}} \leqslant \frac{\Omega^*}{\gamma} \cdot \frac{2\Omega^*}{\gamma\nu} \,,$$

as desired. □

As a last step before we state the main result of this section we show that the cluster graph $\mathbf{G}_{\mathrm{reg}}$ corresponding to a $(k, \Omega^{**}, \Omega^*, \Lambda, \gamma, \varepsilon, \nu, \rho)$-sparse decomposition $(\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ has bounded degree.

Fact 4.12. *Let* $\nabla = (\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ *be a* $(k, \Omega^{**}, \Omega^*, \Lambda, \gamma, \varepsilon, \nu, \rho)$*-sparse decomposition of a graph* G*, and let* $\mathbf{G}_{\mathrm{reg}}$ *be the corresponding cluster graph. Let* $\mathfrak{c}$ *be the size of any cluster in* $\mathbf{V}$*. Then* $\deg^{\max}(\mathbf{G}_{\mathrm{reg}}) \leqslant \frac{\Omega^* k}{\gamma^2 \mathfrak{c}} \leqslant \frac{\Omega^*}{\gamma^2 \nu}$*.*

Proof. Let $C \in \mathbf{V}$. Then by the definition of $\mathbf{G}_{\mathrm{reg}}$, and by the properties of Definitions 4.7 and 4.8, we get

$$\deg_{\mathbf{G}_{\mathrm{reg}}}(C) \leqslant \sum_{C' \in \mathrm{N}_{\mathbf{G}_{\mathrm{reg}}}(C)} \frac{e_{G_{\mathrm{reg}}}(C, C')}{\gamma^2 |C||C'|} \leqslant \frac{\Omega^* k |C|}{\gamma^2 |C| \mathfrak{c}} \leqslant \frac{\Omega^*}{\gamma^2 \nu},$$

as desired. □

We now state the most important lemma of this section. It says that any graph of bounded degree has a bounded decomposition which captures almost all its edges. This lemma can be considered as a sort of Regularity Lemma for sparse graphs.

Lemma 4.13 (Decomposition lemma)**.** *For each* $\Lambda, \Omega, s \in \mathbb{N}$ *and each* $\gamma, \varepsilon, \rho > 0$ *there exist* $k_0 \in \mathbb{N}$, $\nu > 0$ *such that for every* $k \geqslant k_0$ *and every* n*-vertex graph* G *with* $e(G) \leqslant kn$, $\deg^{\max}(G) \leqslant \Omega k$*, and with a given partition* $\mathcal{V}$ *of its vertex set into at most* s *sets, there exists a* $(k, \Lambda, \gamma, \varepsilon, \nu, \rho)$*-bounded decomposition* $(\mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ *with respect to* $\mathcal{V}$*, which captures all but at most* $(\frac{4\varepsilon}{\gamma} + \varepsilon\Omega + \gamma + \rho)kn$ *edges of* G*. Furthermore, this bounded decomposition respects any given avoiding threshold* b *and we have*

$$|E(G_{\mathcal{D}} - \mathfrak{A}) \setminus E(G_{\mathrm{reg}})| \leqslant (\varepsilon\Omega + \gamma^2)kn \,. \tag{4.4}$$

A proof of Lemma 4.13 is given in Section 4.6.

4.3 Decomposition of LKS graphs

Lemma 4.1 and Lemma 4.13 enable us to decompose graphs in $\mathbf{LKS}(n,k,\eta)$ in a particular manner.

Lemma 4.14. *For every $\eta,\Lambda,\gamma,\varepsilon,\rho>0$ there are $\nu>0$ and $k_0\in\mathbb{N}$ such that for every $k>k_0$ the following holds. For every sequence $(\Omega_j)_{j\in\mathbb{N}}$ of positive numbers with $\Omega_j/\Omega_{j+1}\leqslant\eta^2/100$ for all $j\in\mathbb{N}$ and for every $G\in\mathbf{LKS}(n,k,\eta)$ there are an index i and a subgraph G' of G with the following properties for every number b:*

(a) $G'\in\mathbf{LKSsmall}(n,k,\eta/2)$,

(b) $i\leqslant 100\eta^{-2}$,

(c) G' has a $(k,\Omega_{i+1},\Omega_i,\Lambda,\gamma,\varepsilon,\nu,\rho)$-sparse decomposition $(\mathbf{\Psi},\mathbf{V},\mathcal{D},G'_{\mathrm{reg}},G'_{\mathrm{exp}},\mathfrak{A})$ with respect to the partition $\{V_1,V_2\}:=\{\mathbb{S}_{\eta/2,k}(G'),\mathbb{L}_{\eta/2,k}(G')\}$, and with respect to avoiding threshold b,

(d) $(\mathbf{\Psi},\mathbf{V},\mathcal{D},G'_{\mathrm{reg}},G'_{\mathrm{exp}},\mathfrak{A})$ captures all but at most $(\frac{4\varepsilon}{\gamma}+\varepsilon\Omega_{\lfloor 100\eta^{-2}\rfloor}+\gamma+\rho)kn$ edges of G', and

(e) $|E(G_{\mathcal{D}})\setminus E(G_{\mathrm{reg}})|\leqslant(\varepsilon\Omega_{\lfloor 100\eta^{-2}\rfloor}+\gamma^2)kn$.

Proof. Let ν and k_0 be given by Lemma 4.13 for input parameters $\Omega_{\triangleright\mathrm{L}4.13}:=\Omega_{\lfloor 100\eta^{-2}\rfloor}$, $\Lambda_{\triangleright\mathrm{L}4.13}:=\Lambda$, $\gamma_{\triangleright\mathrm{L}4.13}:=\gamma$, $\varepsilon_{\triangleright\mathrm{L}4.13}:=\varepsilon$, $\rho_{\triangleright\mathrm{L}4.13}:=\rho$, $b_{\triangleright\mathrm{L}4.13}:=b$, and $s_{\triangleright\mathrm{L}4.13}:=2$. Now, given G, let us consider a subgraph $\tilde{G}$ of G such that $\tilde{G}\in\mathbf{LKSmin}(n,k,\eta)$. Lemma 4.1 applied to the sequence $(\Omega_j)_j$ and $\tilde{G}$ yields a graph $G'\in\mathbf{LKSsmall}(n,k,\eta/2)$ and an index $i\leqslant 100\eta^{-2}$. We set $\mathbf{\Psi}:=\{v\in V(G)\,:\,\deg_{G'}(v)\geqslant\Omega_{i+1}k\}$.

Observe that by (2.1), $e(G')<kn$. Let $(\mathbf{\Psi},\mathcal{D},G'_{\mathrm{reg}},G'_{\mathrm{exp}},\mathfrak{A})$ be the $(k,\Lambda,\gamma,\varepsilon,\nu,\rho)$-bounded decomposition of the graph $G'-\mathbf{\Psi}$ with respect to $\{\mathbb{S}_{\eta/2,k}(G'),\mathbb{L}_{\eta/2,k}(G')\setminus\mathbf{\Psi}\}$ that is given by Lemma 4.13. Clearly, $(\mathbf{\Psi},\mathbf{V},\mathcal{D},G'_{\mathrm{reg}},G'_{\mathrm{exp}},\mathfrak{A})$ is a $(k,\Omega_{i+1},\Omega_i,\Lambda,\gamma,\varepsilon,\nu,\rho)$-sparse decomposition of G' capturing at least as many edges as promised in the statement of the lemma. □

A version of Lemma 4.14 could be formulated for a general n-vertex graph with $\Theta(kn)$ edges. It would assert that such a graph has a sparse classification which captures all but at most $o(kn)$ edges. Such a lemma could be used to attack other problems. However, our feeling is that such a decomposition lemma is limited in applications to tree-containment problems. The reason is that two of the features of the sparse decomposition, the nowhere-dense graph G_{exp} and the avoiding set $\mathfrak{A}$, seem to be useful only for embedding trees. See Section 4.4 and Section 4.5 for a discussion of the respective embedding strategies.

The process of embedding a given tree $T_{\triangleright T1.3} \in \mathbf{trees}(k)$ into $G_{\triangleright T1.3}$ is based on the sparse decomposition $\nabla = (\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ of a graph G from Lemma 4.14 and is much more complex than in approaches based on the standard Regularity Lemma. The embedding ingredient in the classic (dense) Regularity Method inheres in Blow-up Lemma type statements which roughly tell that regular pairs of positive density in some sense behave like complete bipartite graphs. In our setting, in addition to regular pairs[ix] we shall use three other components of ∇: the vertices of huge degree $\mathbf{\Psi}$, the nowhere-dense graph G_{exp}, and the avoiding set $\mathfrak{A}$. Each of these components requires a different strategy for embedding (parts of) $T_{\triangleright T1.3}$. Let us mention that rather major technicalities arise when combining these strategies; for example, for traversing between $\mathbf{\Psi}$ and the rest of the graph we have to introduce a certain "cleaned" structure in Lemma 7.32.

These strategies are described precisely and in detail in Section 8. A lighter informal account on the role of $\mathfrak{A}$ is given in Section 4.4. We discuss the use of G_{exp} in Section 4.5. Only very little can be said about the set $\mathbf{\Psi}$ at an intuitive level: these vertices have huge degrees but are very unstructured otherwise. If only $o(kn)$ edges are incident with $\mathbf{\Psi}$ then we can neglect them. If, on the other hand, there are $\Omega(kn)$ edges incident with $\mathbf{\Psi}$, then we have no choice but to use them for our embedding. Very roughly speaking, in that case we find sets $\mathbf{\Psi}' \subseteq \mathbf{\Psi}$ and $V' \subseteq V(G) \setminus \mathbf{\Psi}$ such that still $\mathrm{deg}^{\min}(\mathbf{\Psi}', V') \gg k$, and $\mathrm{deg}^{\min}(V', \mathbf{\Psi}') = \Omega(k)$, and then use $\mathbf{\Psi}'$ and V' in our embedding.

Last, let us note that when $G_{\triangleright T1.3}$ is close to the extremal graph (depicted in Figure 1.1) then all the structure in $G_{\triangleright T1.3}$ captured by Lemma 4.14 accumulates in the cluster graph G'_{reg}, i.e., $\mathbf{\Psi}$, G'_{exp} and $\mathfrak{A}$ are all almost empty. For that reason, when some of $\mathbf{\Psi}$, G'_{exp} or $\mathfrak{A}$ is substantial we gain some extra aid. In comparison, one of the almost extremal graphs for the Erdős-Sós Conjecture 1.1 has a substantial $\mathbf{\Psi}$-component (see Figure 1.2).

4.4 The role of the avoiding set $\mathfrak{A}$

Let us explain the role of the avoiding set $\mathfrak{A}$ in Lemma 4.13. As said above, our aim in Lemma 4.13 will be to locally regularize parts of the input graph G. Of course, first we try to regularize as large a part of the G as possible. The avoiding set arises as a result of the impossibility to regularize certain parts of the graph. Indeed, it is one of the most surprising steps in our proof of Theorem 1.3 that the set $\mathfrak{A}$ is initially defined as – very loosely speaking – "those vertices where the Regularity Lemma fails to work

[ix]Some of the regular pairs we shall use are already present in G_{reg}, and there are some additional regular pairs hidden in $\mathcal{D}$ which we shall extract and make use of in a form of so-called semiregular matchings (Definition 5.4) in Sections 5 and 6.

properly", and only then we prove[x] that $\mathfrak{A}$ actually satisfies the useful conditions of Definition 4.6.

We now sketch how to utilize avoiding sets for the purpose of embedding trees. In our proof of Theorem 1.3 we preprocess the tree $T = T_{\triangleright T1.3} \in \mathbf{trees}(k)$ by considering its (τk)-fine partition, and then sequentially embed its shrubs (and knags). Thus embedding techniques for embedding a single shrub are the building blocks of our embedding machinery; and $\mathfrak{A}$ is one of the enviroments which provides us with such a technique. Let us discuss here the simpler case of end shrubs. More precisely, we show how to extend a partial embedding of a tree by one end-shrub. To this end, let us suppose that ϕ is a partial embedding of a tree T, and $v \in V(T)$ is its *active vertex*, i.e., a vertex which is embedded, but not all its children are. We write $U \subseteq V(G)$ for the current image of ϕ. Let $T' \subseteq T$ be an end-shrub which is not embedded yet, and suppose $u \in V(T')$ is adjacent to v. We have $v(T') \leqslant \tau k$.

We now show how to extend the partial embedding ϕ to T', assuming that $\deg_G\big(\phi(v), \mathfrak{A} \setminus U\big) \geqslant \gamma k$ for some $(1, \varepsilon, \gamma, k)$-avoiding set $\mathfrak{A}$ (where $\tau \ll \varepsilon \ll \gamma \ll 1$). Let X be the set of at most εk exceptional vertices from Definition 4.6 corresponding to the set U. We now embed T' into G, starting by embedding u in a vertex of $\mathfrak{A} \setminus (U \cup X)$ in the neighborhood of $\phi(v)$. By Definition 4.6, there is a dense spot $D = (A_D, B_D; F) \in \mathcal{D}$ such that $\phi(u) \in V(D)$ and $|U \cap V(D)| \leqslant \gamma^2 k$. As D is a dense spot, we have $\deg_G(\phi(u), V(D)) > \gamma k$. It is now easy to embed T' into D using the minimum degree in D. See Figure 4.1 for an illustration, and Lemma 8.3 for a precise formulation.

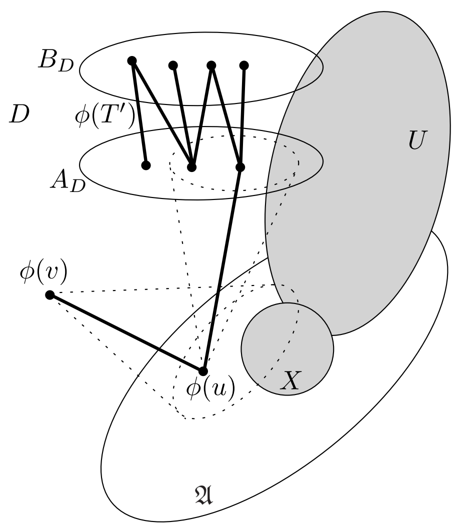

Figure 4.1: Embedding using the set $\mathfrak{A}$.

We indeed use the avoiding set for embedding shrubs of a fine partition of T as above. The major simplification we made in the exposition is that we only discussed

[x]See the last step of the proof of Lemma 4.13.

the case when T' is an end shrub. To cover embedding of an internal shrub T' as well, one needs to have a more detailed control over the embedding, i.e., one must be able to extend the embedding from leaves of T' to the neighboring cut-vertices of the fine partition, is such a way that one can then continue embedding of the shrubs below these cut-vertices.

Last, let us remark, that unlike our baby-example above, we use an $(\Lambda, \varepsilon, \gamma, k)$-avoiding set with $\Lambda \gg 1$. This is because in the actual proof one has to avoid more vertices than just the current image of the embedding.

4.5 The role of the nowhere-dense graph G_{exp} and using the (τk)-fine partition

In this section we shall give some intuition on how the $(\gamma k, \gamma)$-nowhere-dense graph G_{exp} from the $(k, \Omega^{**}, \Omega^*, \Lambda, \gamma, \varepsilon', \nu, \rho)$-sparse decomposition[xi] $(\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ of a graph G is useful for embedding a given tree $T \in \mathbf{trees}(k)$. We start out with the rather simple case when T is a path. We then point out an issue with this approach for trees with many branching vertices and show how to overcome this problem using the (τk)-fine partition from Lemma 3.4.

Embedding a path in G_{exp}. Assume we are given a path $T = u_1 u_2 \cdots u_k \in \mathbf{trees}(k)$ and we wish to embed it into G_{exp}. The naive idea is to apply a one-step look-ahead strategy. We first embed u_1 in an arbitrary vertex $v \in V(G_{\mathrm{exp}})$. Then, we extend our embedding ϕ_ℓ of the path $u_1 \cdots u_\ell$ in G_{exp} in step ℓ by embedding $u_{\ell+1}$ in a (yet unused) neighbour w of the image of the *active* vertex u_ℓ, requiring that

$$\deg_{G_{\mathrm{exp}}}\big(w, \phi_\ell(u_1 \cdots u_\ell)\big) < \sqrt{\gamma} k \ . \tag{4.5}$$

Let us argue that such a vertex w exists. First, observe that Property 7 of Definition 4.7 implies that $\phi_\ell(u_\ell)$ has at least ρk neighbours. By (4.5) applied to $\ell - 1$, at most $\sqrt{\gamma}k$ of these neighbours lie inside $\phi_\ell(u_1 \cdots u_{\ell-1})$; this property is also trivially satisfied when $\ell = 1$. Further, an easy calculation shows that at most $16\sqrt{\gamma}k$ of them have degree more than $\sqrt{\gamma}k$ in G_{exp} into the set $\phi_\ell(u_1 \cdots u_\ell)$, otherwise we would get a contradiction to G_{exp} being $(\gamma k, \gamma)$-nowhere-dense. Since we assumed $\rho > 17\sqrt{\gamma}$ we can find a vertex w as desired and thus embed all of T.

Embedding trees with many branching points and the role of fine partitions. We certainly cannot hope that a nonempty graph G_{exp} alone will provide us with

[xi] We shall assume that $17\sqrt{\gamma} < \rho$; this will be the setting of the sparse decomposition we shall work with in the proof of Theorem 1.3.

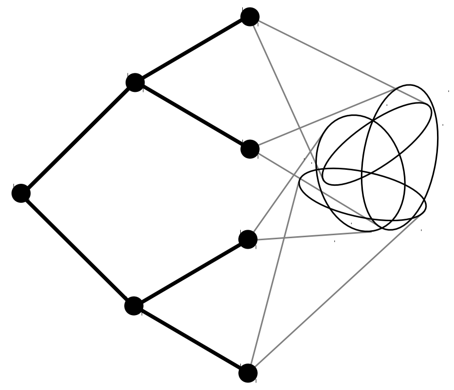

Figure 4.2: Embedded part of the binary tree in bold. The neighbourhoods of active vertices may overlap.

embeddings of all trees $T \in \mathbf{trees}(k)$ from Theorem 1.3. For instance, if T is a star, then we need in G a vertex of degree $k-1$, which G_{exp} might not have. In order to run into a problem with the method described above, we do not even need to have such a large degree in our tree T.

Consider a binary tree $T \in \mathbf{trees}(k)$, rooted at its central vertex r. Now if we try to embed T sequentially as above we will arrive at a moment when there are many (as many as $k/2$) active vertices; regardless in which order we embed. Now, the neighbourhoods of the images of the active vertices cannot be controlled much, i.e., they may be intersecting considerably. Hence, embedding children of active vertices we might block available space in the neighbourhoods of other active vertices. See Figure 4.2 for an illustration.

To rescue the situation we use the (τk)-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of T (for some $0 < \tau \ll \gamma$) given by Lemma 3.4. Recall the structure of this partition, as shown in Figure 3.1: the first q levels of T from the root r comprise the sole knag. All other vertices make up the end shrubs $T_1^*, \ldots, T_h^*$.

We first embed the knag, which consists of the cut vertices $W_A \cup W_B$, and so has size at most $O(\frac{1}{\tau})$. As ρk will be much larger than that, following a strategy similar to the one above we ensure that all of $W_A \cup W_B$ gets correctly embedded, we even have a (limited) choice for its images. The next step is to make the transitions at the q-th level from embedding cut vertices $W_A \cup W_B$ to embedding shrubs $T_1^*, \ldots, T_h^*$. But since this step requires to exploit the structure of LKS graphs, we skip the details in the high-level overview here. We just remark that one needs to put the cut vertices $W_A \cup W_B$ in the sets $\mathbb{XA}$ and $\mathbb{XB}$ from Lemma 6.1; these vertices are powerful enough to allow such a transition.

For the point we wish to make here, it is more relevant to see how to complete the last part of our embedding, that is, how to embed a tree T_i^* whose root r_i is already embedded in a vertex $\phi(r_i) \in V(G_{\mathrm{exp}})$. Let $\mathrm{im}_i := \mathrm{im}(\phi)$ be the current (partial) image of ϕ at this stage. We emphasize that at this moment we are working exclusively with the tree T_i^*, i.e., any other tree T_j^* is either completely embedded, or will be embedded only after we finish the embedding of T_i^*. Suppose we are about to embed a vertex $v \in V(T_i^*)$ whose ancestor $v' \in V(T_i^*)$ is already embedded in $V(G_{\mathrm{exp}})$. We choose for the image of v any (yet unused) vertex w in the neighbourhood of $\varphi(v')$, requiring that

$$\deg_{G_{\mathrm{exp}}}(w, \mathrm{im}_i) < \rho k/100. \tag{4.6}$$

This condition is very similar to our path-embedding procedure above, and can be proved in exactly the same way, using the fact that G_{exp} is $(\gamma k, \gamma)$-nowhere-dense. Note that during our embedding $|\mathrm{im}(\phi) \setminus \mathrm{im}_i|$ will grow, but however is at most $v(T_i^*) \leqslant \tau k$. Thus, for every vertex $v'' \in V(T_i^*)$, when its time comes to be embedded, we still have $\deg_{G_{\mathrm{exp}}}\big(\phi(v'), \mathrm{im}(\phi)\big) \leqslant \rho k/100 + \tau k < \rho k/99$, and thus v'' can be embedded.

Note that the trick here was to keep on working on one subtree T_i^*, whose size is small enough to be negligible in comparison to the degree of a vertex in G_{exp} so that it does not matter that the set we wish to avoid having a considerable degree into $(\mathrm{im}(\phi))$ is not the same as the one we can actually avoid having a considerable degree into (im_i). (Observe that since $\mathrm{im}(\phi)$ keeps changing during the procedure, we cannot have direct control over it.) Thus, breaking up the tree into tiny shrubs in the (τk)-fine partition was the key to successfully embedding it in this case.

4.6 Proof of Lemma 4.13

This subsection is devoted to the proof of Lemma 4.13. We give an overview of our decomposition procedure. We start by extracting the edges of as many $(\gamma k, k)$-dense spots from G as possible; these together with the incident vertices will form the auxiliary graph $G_{\mathcal{D}}$. Most of the remaining edges will form the edge set of the graph G_{exp}. Next, we consider the intersections of the dense spots captured in $G_{\mathcal{D}}$. To the subgraph of $G_{\mathcal{D}}$ that is spanned by the large intersections we apply the Regularity Lemma for locally dense graphs (Lemma 2.13), and thus obtain G_{reg}. The other part of $V(G_{\mathcal{D}})$ will be taken as the $(\Lambda, \varepsilon, \gamma, k)$-avoiding set $\mathfrak{A}$.

Setting up the parameters. We start by setting

$$\tilde{\nu} := \varepsilon \cdot 3^{-\frac{\Omega\Lambda}{\gamma^3}}.$$

Let q_{MAXCL} be given by Lemma 2.13 for input parameters

$$m_{\triangleright \mathrm{L}2.13} := \frac{\Omega}{\gamma\tilde{\nu}} \quad , \quad z_{\triangleright \mathrm{L}2.13} := 4s \quad \text{and} \quad \varepsilon_{\triangleright \mathrm{L}2.13} := \varepsilon\ . \tag{4.7}$$

Define an auxiliary parameter $q := \max\{q_{\mathrm{MAXCL}}, \varepsilon^{-1}\}$ and choose the output parameters of Lemma 4.13 as

$$k_0 := \left\lceil \frac{q_{\mathrm{MAXCL}}}{\tilde{\nu}} \right\rceil \qquad \text{and} \qquad \nu := \frac{\tilde{\nu}}{q}.$$

Defining $\mathcal{D}$ and G_{exp}. Given a graph G, take a set $\mathcal{D}$ of edge-disjoint $(\gamma k, \gamma)$-dense spots such that the resulting graph $G_{\mathcal{D}} \subseteq G$ (which contains those vertices and edges that are contained in $\bigcup \mathcal{D}$) has a maximal number of edges.

Then by Lemma 2.3 there exists a graph $G_{\mathrm{exp}} \subseteq G - G_{\mathcal{D}}$ with $\deg^{\min}(G_{\mathrm{exp}}) > \rho k$ and such that

$$|E(G) \setminus (E(G_{\mathrm{exp}}) \cup E(G_{\mathcal{D}}))| \leqslant \rho k n \; . \tag{4.8}$$

This choice of $\mathcal{D}$ and G_{exp} already satisfies Properties 4 and 7 of Definition 4.7.

Preparing for an application of the Regularity Lemma. Let

$$\mathcal{X} := \boxplus_D \{U, W, V(G) \setminus V(D)\} \; ,$$

where the partition refinement ranges over all $D = (U, W; F) \in \mathcal{D}$. Let $\mathcal{B} := \{X \in \mathcal{X} \; : \; X \subseteq V(G_{\mathcal{D}})\}$, $\tilde{\mathcal{B}} := \{B \in \mathcal{B} \; : \; |B| > 2\tilde{\nu}k\}$, and $\tilde{\mathcal{C}} := \mathcal{B} \setminus \tilde{\mathcal{B}}$. Furthermore let $\tilde{B} := \bigcup_{B \in \tilde{\mathcal{B}}} B$ and $\mathfrak{A} := \bigcup_{C \in \tilde{\mathcal{C}}} C$. Let $V_{\leadsto \mathfrak{A}} := \{v \in V(G) \; : \; \deg(v, \mathfrak{A}) > b\}$.

Now, partition each set $B \in \tilde{\mathcal{B}}$ into $c_B := \lceil |B|/2\tilde{\nu}k \rceil$ sets $B_1, \ldots, B_{c_B}$ of cardinalities differing by at most one, and let $\mathcal{B}'$ be the set containing all the sets B_i (for all $B \in \tilde{\mathcal{B}}$). Then for each $B \in \mathcal{B}'$ we have that

$$\tilde{\nu}k \leqslant |B| \leqslant 2\tilde{\nu}k \leqslant \varepsilon k \; . \tag{4.9}$$

Construct a graph H on $\mathcal{B}'$ by making two vertices $A_1, A_2 \in \mathcal{B}'$ adjacent in H if

(A) there is a dense spot $D = (U, W; F) \in \mathcal{D}$ such that $A_1 \subseteq U$ and $A_2 \subseteq W$, and

(B) $\mathrm{d}_G(A_1, A_2) \geqslant \gamma$.

Note that it follows from the way $\mathcal{D}$ was chosen that if $A_1A_2 \in E(H)$ then $G[A_1, A_2] = G_{\mathcal{D}}[A_1, A_2]$. But on the other hand note that we do not necessarily have $G[A_1, A_2] = D[A_1, A_2]$ for the dense spot D appearing in (A); just because there may be several such dense spots D.

By assumption of Lemma 4.13, $\deg^{\max}(G) \leqslant \Omega k$. So, for each $B \in \mathcal{B}'$ we have $e_G(B, \tilde{B} \setminus B) \leqslant \Omega k|B|$. On the other hand, (4.9) and (B) imply that $\gamma\tilde{\nu}k|B| \deg_H(B) \leqslant e_G(B, \tilde{B} \setminus B)$. We conclude that

$$\deg^{\max}(H) \leqslant \frac{\Omega}{\gamma\tilde{\nu}} = m_{\rhd \mathrm{L2.13}} \; . \tag{4.10}$$

Regularising the dense spots in $\tilde{B}$. We use Lemma 2.13 with parameters $m_{\triangleright \text{L2.13}}, z_{\triangleright \text{L2.13}}$ and $\varepsilon_{\triangleright \text{L2.13}}$ as defined by (4.7) on the graphs $H_{\triangleright \text{L2.13}} := G_{\mathcal{D}}$ and $F_{\triangleright \text{L2.13}} := H$, together with the ensemble $\mathcal{B}'$ in the role of the sets W_i, and partition of $V(G_{\mathcal{D}})$ induced by

$$\mathcal{Z}_{\triangleright \text{L2.13}} := \mathcal{V} \boxplus \left\{V(G_{\text{exp}}), V(G) \setminus V(G_{\text{exp}})\right\} \boxplus \left\{V_{\leadsto \mathfrak{A}}, V(G) \setminus V_{\leadsto \mathfrak{A}}\right\} .$$

Observe that $\mathcal{B}'$ is an $(\tilde{\nu}k)$-ensemble satisfying condition (2.3) of Lemma 2.13, by (4.9), by the choice of k_0, and by (4.10). We thus obtain integers $\{p_A\}_{A \in \mathcal{B}'}$ and a family $\mathbf{V} = \{W_A^{(0)}, \ldots, W_A^{(p_A)}\}_{A \in \mathcal{B}'}$ such that in particular we have the following.

(I) We have $\varepsilon^{-1} \leqslant p_A \leqslant q_{\text{MAXCL}}$ for all $A \in \mathcal{B}'$.

(II) We have $|W_A^{(a)}| = |W_B^{(b)}|$ for any $A, B \in \mathcal{B}'$ and for any $a \in [p_A]$, $b \in [p_B]$.

(III) For any $A \in \mathcal{B}'$ and any $a \in [p_A]$, there is $V \in \mathcal{V}$ such that $W_A^{(a)} \subseteq V$. We either have that $W_A^{(a)} \subseteq V(G_{\text{exp}})$, or $W_A^{(a)} \cap V(G_{\text{exp}}) = \emptyset$ and $W_A^{(a)} \subseteq V_{\leadsto \mathfrak{A}}$, or $W_A^{(a)} \cap V_{\leadsto \mathfrak{A}} = \emptyset$.

(IV) $\sum_{e \in E(H)} |\text{irreg}(e)| \leqslant \varepsilon \sum_{AB \in E(H)} |A||B|$, where $\text{irreg}(AB)$ is the set of all edges of the graph G contained in an ε-irregular pair $(W_A^{(a)}, W_B^{(b)})$, with $a \in [p_A]$, $b \in [p_B]$, $AB \in E(H)$.

Let G_{reg} be obtained from $G_{\mathcal{D}}$ by erasing all vertices in $\bigcup_A W_A^{(0)}$, and all edges that lie in pairs $(W_A^{(a)}, W_B^{(b)})$ which are irregular or of density at most γ^2. Then Properties 1, 2, 5 and 6 of Definition 4.7 are satisfied. Further, (4.4) is satisfied.

Note that Properties (I), (II) and (4.9) imply that for all $A \in \mathcal{B}'$ and for any $a \in [p_A]$ we have that

$$\varepsilon k \geqslant |A| \geqslant |W_A^{(a)}| \geqslant \frac{\tilde{\nu}k}{q_{\text{MAXCL}}} \geqslant \frac{\tilde{\nu}k}{q} = \nu k.$$

Thus also Property 3 of Definition 4.7 holds.

Furthermore, by (4.8), and by Lemma 2.14, the number of edges that are not captured by $(\mathbf{V}, \mathcal{D}, G_{\text{reg}}, G_{\text{exp}}, \mathfrak{A})$ is at most $(\frac{4\varepsilon}{\gamma} + \varepsilon\Omega + \gamma + \rho)kn$.

So, it only remains to see Property 8 of Definition 4.7.

The avoiding property of $\mathfrak{A}$. In order to see Property 8 of Definition 4.7, we have to show that $\mathfrak{A}$ is $(\Lambda, \varepsilon, \gamma, k)$-avoiding with respect to $\mathcal{D}$. For this, let $\bar{U} \subseteq V(G)$ be such that $|\bar{U}| \leqslant \Lambda k$. Let X be the set of those vertices $v \in \mathfrak{A}$ that are not contained in any dense spot $D \in \mathcal{D}$ for which $|\bar{U} \cap V(D)| \leqslant \gamma^2 k$. Our aim is to see that $|X| \leqslant \varepsilon k$.

Let $\mathcal{D}_X \subseteq \mathcal{D}$ be the set of all dense spots D with $X \cap V(D) \neq \emptyset$. Setting $\mathcal{A} := \{A \in \tilde{\mathcal{C}} : A \cap X \neq \emptyset\}$, the definition of $\mathfrak{A}$ trivially implies that $\frac{|X|}{2\tilde{\nu}k} \leqslant |\mathcal{A}|$. Now, by the definition of $\mathcal{B}$, we know that there are at most $3^{|\mathcal{D}_X|}$ sets $A \in \mathcal{A}$. Indeed, for each $D = (U, W; F) \in \mathcal{D}_X$, either A is a subset of U, or of W, or of $V(G) \setminus V(D)$. Thus,

$$3^{|\mathcal{D}_X|} \geqslant |\mathcal{A}| \geqslant \frac{|X|}{\tilde{\nu}k} \,. \tag{4.11}$$

By Fact 4.4, each vertex of $V(G)$ lies in at most Ω/γ of the $(\gamma k, \gamma)$-dense spots from $\mathcal{D}$. Hence

$$\frac{\Omega}{\gamma}|\bar{U}| \geqslant \sum_{D \in \mathcal{D}_X} |V(D) \cap \bar{U}| \geqslant |\mathcal{D}_X|\gamma^2 k \overset{(4.11)}{\geqslant} \log_3\left(\frac{|X|}{\tilde{\nu}k}\right)\gamma^2 k \,,$$

where the second inequality holds by the definition of X. Thus

$$|X| \leqslant 3^{\frac{\Omega\Lambda}{\gamma^3}} \cdot \tilde{\nu}k = \varepsilon k \,,$$

as desired. This finishes the proof of Lemma 4.13.

Remark 4.15. *The bounded decomposition given by Lemma 4.13 is not uniquely determined, and can actually vary vastly. This is caused by the arbitrariness in the choice of the dense spots from which we obtain the cluster graph G_{reg}.*

This situation is an acute contrast with the situation of decomposition of dense graphs (which is given by the Szemerédi Regularity Lemma). Indeed, in the dense setting the structure of the cluster graph is essentially unique, cf. [ASS09].[xii]

Of course, the ambiguity of the bounded decomposition of G propagates to Lemma 4.14. We will have to deal with implications of this ambiguity in Section 6.

4.7 Lemma 4.13 algorithmically

Let us look back at the proof of Lemma 4.13 and see that we can get a bounded decomposition of any bounded-degree graph algorithmically in quasipolynomial time (in the order of the graph). Note that this in turn provides efficiently a sparse classification of any graph since the initial step of splitting the graph into huge degree vertices and bounded degree (cf. Lemma 4.1) can be done in polynomial time.

There are only two steps in the proof of Lemma 4.13 which need to be done algorithmically: the extraction of dense spots, and the simultaneous regularization of some dense pairs.

It will be more convenient to work with a relaxation of the notion of dense spots. We call a graph H *(d, ℓ)-thick* if $v(H) \geqslant \ell$, and $e(H) \geqslant dv(H)^2$. Thick graphs are a relaxation of dense spots, where the minimum degree condition is replaced by imposing

[xii] The setting needs to be somewhat strengthened as otherwise there are counterexamples to uniqueness; compare Theorem 1 and Theorem 2 in [ASS09]. However morally this is true because of the uniqueness of graph limits [BCL09].

a lower bound on the order, and the bipartiteness requirement is dropped. It can be verified that in our proof it is not important that the dense spots $\mathcal{D}$ and the nowhere-dense graph G_{exp} are parametrized by the same constants, i.e., the entire proof would go through even if the spots in $\mathcal{D}$ were $(\gamma k, \gamma)$-dense, and G_{exp} was $(\beta k, \beta)$-nowhere-dense for some $\beta \gg \gamma$. Each $(\beta k, \beta)$-thick graph gives (algorithmically) a $(\beta k/4, \beta/4)$-dense spot, and thus it is enough to extract thick graphs.

For the extraction of thick graphs we would need to efficiently answer the following: Given a number $\beta > 0$ find a number $\gamma > 0$ such that for an input number h and an N-vertex graph we can localize in G a (γ, h)-thick graph if it contains a (β, h)-thick graph, or output NO otherwise.[xiii] Employing techniques from a deep paper of Arora, Frieze and Kaplan [AFK02], one can solve this problem in quasipolynomial time $O(N^{c \cdot \log N})$. This was communicated to us by Maxim Sviridenko. On the negative side, a truly polynomial algorithm seems to be out of reach as Alon, Arora, Manokaran, Moshovitz, and Weinstein [AAM$^+$] reduced the problem to the notorious hidden clique problem whose tractability has been open for twenty years.

Theorem 4.16 (Alon et al. [AAM$^+$])**.** *If there is no polynomial time algorithm for solving the clique problem for a planted clique of size $n^{1/3}$ then for any $\varepsilon \in (0,1)$ and $\delta > 0$ there is no polynomial time algorithm that distinguishes between a graph G on N vertices containing a clique of size $\kappa = N^{\varepsilon}$ and a graph G' on N vertices in which the densest subgraph on κ vertices has density at most δ.*[xiv]

Of course, Theorem 4.16 leaves some hope for a polynomial time algorithm when $h = N^{o(1)}$ (which corresponds to $k_{\triangleright \text{L4.13}} = n_{\triangleright \text{L4.13}}^{o(1)})$).

The regularity lemma can be made algorithmic [ADL$^+$94]. The algorithm from [ADL$^+$94] is based on index pumping-up, and thus applies even to the locally dense setting of Lemma 2.13.

It will turn out that the extraction of dense spots is the only obstruction to a polynomial time algorithm for Theorem 1.3. In Section 10.1 we sketch a truly polynomial time algorithm which avoids this step. It seems that the method sketched there is generally applicable for problems which employ sparse classifications.

[xiii] We could additionally assume that $\deg^{\max}(G) \leqslant O(h)$ due to the previous step of removing the set $\mathbf{\Psi}$ of huge degree vertices.

[xiv] The result as stated in [AAM$^+$] covers only the range $\varepsilon \in (\frac{1}{3}, 1)$. However there is a simple reduction by taking many disjoint copies of the general range to the restricted one.

5 Augmenting a matching

In previous papers [AKS95, Zha11, PS12, Coo09, HP] concerning the LKS Conjecture in the dense setting the crucial turn was to find a matching in the cluster graph of the host graph possessing certain properties. We will prove a similar "structural result" in Section 6. In the present section, we prove the main tool for Section 6, namely Lemma 5.10. All preceding statements are only preparatory. The only exception is (the easy) Lemma 5.6 which is recycled later, in Section 7.

5.1 Dense spots and semiregular matchings

We need two definitions concerning graphs covered by dense spots.

Definition 5.1 (**(m,γ)-dense cover).** *A* (m,γ)-dense cover *of a graph G is a family $\mathcal{D}$ of edge-disjoint (m,γ)-dense spots such that $E(G)=\bigcup_{D\in\mathcal{D}}E(D)$.*

Definition 5.2 ($\mathcal{G}(n,k,\Omega,\rho,\nu,\tau)$ and $\bar{\mathcal{G}}(n,k,\Omega,\rho,\nu)$)**.** *We define $\mathcal{G}(n,k,\Omega,\rho,\nu,\tau)$ to be the class of all tuples $(G,\mathcal{D},H,\mathcal{A})$ with the following properties:*

(i) G is a graph of order n with $\deg^{\max}(G)\leqslant\Omega k$,

(ii) H is a bipartite subgraph of G with colour classes A_H and B_H and with $e(H)\geqslant\tau kn$,

(iii) $\mathcal{D}$ is a $(\rho k,\rho)$-dense cover of G,

(iv) $\mathcal{A}$ is a (νk)-ensemble in G, and $A_H\subseteq\bigcup\mathcal{A}$,

(v) $A\cap U\in\{\emptyset,A\}$ for each $A\in\mathcal{A}$ and for each $D=(U,W;F)\in\mathcal{D}$.

Those G, $\mathcal{D}$ and $\mathcal{A}$ for which all conditions but (ii) *and the last part of* (iv) *hold will make up the triples $(G,\mathcal{D},\mathcal{A})$ of the class $\bar{\mathcal{G}}(n,k,\Omega,\rho,\nu)$.*

We now prove our first auxiliary lemma on our way towards Lemma 5.10.

Lemma 5.3. *For every $\Omega\in\mathbb{N}$ and $\varepsilon,\rho,\tau>0$ there is a number $\alpha>0$ such that for every $\nu\in(0,1)$ there exists a number $k_0\in\mathbb{N}$ such that for each $k>k_0$ the following holds.*

For every $(G,\mathcal{D},H,\mathcal{A})\in\mathcal{G}(n,k,\Omega,\rho,\nu,\tau)$ there are $(U,W;F)\in\mathcal{D}$, $A\in\mathcal{A}$ and $X,Y\subseteq V(G)$ such that

1) $|X|=|Y|>\alpha\nu k$,

2) $X\subseteq A\cap U\cap A_H$ and $Y\subseteq W\cap B_H$, where A_H and B_H are the colour classes of H, and

3) (X,Y) *is an* ε*-regular pair in* G *of density* $\mathrm{d}(X,Y) \geqslant \frac{\tau\rho}{4\Omega}$.

Proof. Let Ω, ε, ρ and τ be given. Applying Lemma 2.12 to $\varepsilon_{\triangleright \mathrm{L}2.12} := \min\{\varepsilon, \frac{\rho^2}{8\Omega}\}$ and $\ell_{\triangleright \mathrm{L}2.12} := 2$, we obtain numbers n_0 and M. We set

$$\alpha := \frac{\tau\rho}{\Omega^2 M}, \tag{5.1}$$

and given $\nu \in (0,1)$, we set

$$k_0 := \frac{2n_0}{\alpha\nu M}.$$

Now suppose we are given $k > k_0$ and $(G, \mathcal{D}, H, \mathcal{A}) \in \mathcal{G}(n, k, \Omega, \rho, \nu, \tau)$.

Property (i) of Definition 5.2 gives that $e(G) \leqslant \Omega kn/2$, and Property (ii) says that $e(H) \geqslant \tau kn$. So $e(H)/e(G) \geqslant 2\tau/\Omega$. Averaging, we find a dense spot $D = (U, W; F) \in \mathcal{D}$ such that

$$e_D(A_H, B_H) = |F \cap E(H)| \geqslant \frac{e(H)}{e(G)}|F| \geqslant \frac{2\tau|F|}{\Omega}\ . \tag{5.2}$$

Without loss of generality, we assume that

$$e_D(U \cap A_H, W \cap B_H) \geqslant \frac{1}{2} \cdot e_D(A_H, B_H) \geqslant e_D(U \cap B_H, W \cap A_H)\ , \tag{5.3}$$

as otherwise one can just interchange the roles of U and W. Then,

$$e_G(U \cap A_H, W \cap B_H) \overset{(5.3)}{\geqslant} \frac{1}{2} \cdot e_D(A_H, B_H) \overset{(5.2)}{\geqslant} \frac{\tau}{\Omega} \cdot |F|. \tag{5.4}$$

Let $\mathcal{A}' \subseteq \mathcal{A}$ denote the set of those $A \in \mathcal{A}$ with $0 < e_G(A \cap U \cap A_H, W \cap B_H) < \frac{\tau}{\Omega} \cdot |F| \cdot \frac{|A|}{|U|}$. Note that for each $A \in \mathcal{A}'$ we have $A \subseteq U$ by Definition 5.2 (v). Therefore,

$$e_G\left(\bigcup \mathcal{A}' \cap U \cap A_H, W \cap B_H\right) < \frac{\tau}{\Omega} \cdot |F| \cdot \frac{|\mathcal{A}'|}{|U|} \leqslant \frac{\tau}{\Omega} \cdot |F| \overset{(5.4)}{\leqslant} e_G(U \cap A_H, W \cap B_H)\ .$$

As $\mathcal{A}$ covers A_H, G has an edge xy with $x \in U \cap A_H \cap A$ for some $A \in \mathcal{A} \setminus \mathcal{A}'$ and $y \in W \cap B_H$. Set $X' := A \cap U \cap A_H = A \cap A_H$ and $Y' := W \cap B_H$. Then directly from the definition of $\mathcal{A}'$ and since D is a $(\rho k, \rho)$-dense spot, we obtain that

$$\mathrm{d}_G(X', Y') = \frac{e_G(X', Y')}{|X'||Y'|} \geqslant \frac{\frac{\tau}{\Omega} \cdot |F| \cdot \frac{|A|}{|U|}}{|A||W|} > \frac{\tau\rho}{\Omega}. \tag{5.5}$$

Also, since $(U, W; F) \in \mathcal{D}$, we have

$$|F| \geqslant \rho k |U|\ . \tag{5.6}$$

This enables us to bound the size of X' as follows.

$$\begin{aligned} |X'| &\geqslant \frac{e_G(X',Y')}{\deg^{\max}(G)} \\ \text{(as } A \notin \mathcal{A}' \text{ and by D5.2(i))} \quad &\geqslant \frac{\frac{\tau}{\Omega}\cdot\frac{|F|}{|U|}\cdot|A|}{\Omega k} \\ \text{(by (5.6))} \quad &\geqslant \frac{\tau\cdot\rho k\cdot|A|}{\Omega^2 k} \\ &\geqslant \frac{\tau\rho\nu k}{\Omega^2} \\ &\overset{(5.1)}{=} \alpha\nu k M\ . \end{aligned} \tag{5.7}$$

In the same way we see that

$$|Y'| \geqslant \ \alpha\nu k M\ . \tag{5.8}$$

Applying Lemma 2.12 to $G[X',Y']$ with prepartition $\{X',Y'\}$ we obtain a collection of sets $\mathcal{C}=\{C_i\}_{i=0}^{p}$, with $p<M$. By (5.7), and (5.8), we have that $|C_i|\geqslant\alpha\nu k$ for every $i\in[p]$. It is easy to deduce from (5.5) that there is at least one $\varepsilon_{\triangleright\mathrm{L}2.12}$-regular (and thus ε-regular) pair (X,Y), $X,Y\in\mathcal{C}\setminus\{C_0\}$, $X\subseteq X'$, $Y\subseteq Y'$ with $\mathrm{d}(X,Y)\geqslant\frac{\tau\rho}{4\Omega}$. Indeed, it suffices to count the number of edges incident with C_0, lying in $\varepsilon_{\mathrm{L}2.12}$-irregular pairs or belonging to too sparse pairs. These are strictly less than

$$(\varepsilon_{\triangleright\mathrm{L}2.12}+\varepsilon_{\triangleright\mathrm{L}2.12}+\frac{\rho^2}{4\Omega})|X||Y|\leqslant\frac{\rho^2}{2\Omega}|X||Y|\overset{(5.5)}{\leqslant}e(X',Y')$$

many, and thus not all edges between X' and Y'. This finishes the proof of Lemma 5.3. □

Instead of just one pair (X,Y), as it is given by Lemma 5.3, we shall later need several disjoint pairs. This motivates the following definition.

Definition 5.4 (**(ε,d,ℓ)-semiregular matching**)**.** *A collection $\mathcal{N}$ of pairs (A,B) with $A,B\subseteq V(H)$ is called an (ε,d,ℓ)-*semiregular matching *of a graph H if*

(i) $|A|=|B|\geqslant\ell$ for each $(A,B)\in\mathcal{N}$,

(ii) (A,B) induces in H an ε-regular pair of density at least d, for each $(A,B)\in\mathcal{N}$, and

(iii) all involved sets A and B are pairwise disjoint.

Sometimes, when the parameters do not matter (as for instance in Definition 5.7 below) we write lazily semiregular matching.

For a semiregular matching $\mathcal{N}$, we shall write $\mathcal{V}_1(\mathcal{N}) := \{A : (A,B) \in \mathcal{N}\}$, $\mathcal{V}_2(\mathcal{N}) := \{B : (A,B) \in \mathcal{N}\}$ and $\mathcal{V}(\mathcal{N}) := \mathcal{V}_1(\mathcal{N}) \cup \mathcal{V}_2(\mathcal{N})$. Furthermore, we set $V_1(\mathcal{N}) := \bigcup \mathcal{V}_1(\mathcal{N})$, $V_2(\mathcal{N}) := \bigcup \mathcal{V}_2(\mathcal{N})$ and $V(\mathcal{N}) := V_1(\mathcal{N}) \cup V_2(\mathcal{N}) = \bigcup \mathcal{V}(\mathcal{N})$. As these definitions suggest, the orientations of the pairs $(A,B) \in \mathcal{N}$ are important. The sets A and B are called *$\mathcal{N}$-vertices* and the pair (A,B) is a *$\mathcal{N}$-edge*.

We say that a semiregular matching $\mathcal{N}$ *absorbes* a semiregular matching $\mathcal{M}$ if for every $(S,T) \in \mathcal{M}$ there exists $(X,Y) \in \mathcal{N}$ such that $S \subseteq X$ and $T \subseteq Y$. In the same way, we say that a family of dense spots $\mathcal{D}$ *absorbes* a semiregular matching $\mathcal{M}$ if for every $(S,T) \in \mathcal{M}$ there exists $(U,W;F) \in \mathcal{D}$ such that $S \subseteq U$ and $T \subseteq W$.

We later need the following easy bound on the size of the elements of $\mathcal{V}(\mathcal{M})$.

Fact 5.5. *Suppose that $\mathcal{M}$ is an (ε, d, ℓ)-semiregular matching in a graph H. Then $|C| \leqslant \frac{\deg^{\max}(H)}{d}$ for each $C \in \mathcal{V}(\mathcal{M})$.*

Proof. Let for example $(C,D) \in \mathcal{M}$. The maximum degree of H is at least as large as the average degree of the vertices in D, which is at least $d|C|$. □

The next lemma, Lemma 5.6, is a second step towards Lemma 5.10. Whereas Lemma 5.3 gives one dense regular pair, in the same setting Lemma 5.6 provides us with a dense semiregular matching.

Lemma 5.6. *For every $\Omega \in \mathbb{N}$ and $\rho, \varepsilon, \tau \in (0,1)$ there exists $\alpha > 0$ such that for every $\nu \in (0,1)$ there is a number $k_0 \in \mathbb{N}$ such that the following holds for every $k > k_0$.*

For each $(G, \mathcal{D}, H, \mathcal{A}) \in \mathcal{G}(n,k,\Omega,\rho,\nu,\tau)$ there exists an $(\varepsilon, \frac{\tau\rho}{8\Omega}, \alpha\nu k)$-semiregular matching $\mathcal{M}$ of G such that

(1) for each $(X,Y) \in \mathcal{M}$ there are $A \in \mathcal{A}$, and $D = (U,W;F) \in \mathcal{D}$ such that $X \subseteq U \cap A \cap A_H$ and $Y \subseteq W \cap B_H$, and

(2) $|V(\mathcal{M})| \geqslant \frac{\tau}{2\Omega} n$.

Proof. Let $\alpha := \alpha_{\triangleright \mathrm{L5.3}} > 0$ be given by Lemma 5.3 for the input parameters $\Omega_{\triangleright \mathrm{L5.3}} := \Omega$, $\varepsilon_{\triangleright \mathrm{L5.3}} := \varepsilon$, $\tau_{\triangleright \mathrm{L5.3}} := \tau/2$ and $\rho_{\triangleright \mathrm{L5.3}} := \rho$. Now, for $\nu_{\triangleright \mathrm{L5.3}} := \nu$, Lemma 5.3 yields a number $k_0 \in \mathbb{N}$.

Now let $(G, \mathcal{D}, H, \mathcal{A}) \in \mathcal{G}(n,k,\Omega,\rho,\nu,\tau)$. Let $\mathcal{M}$ be an inclusion-maximal $(\varepsilon\rho, \frac{\tau\rho}{8\Omega}, \alpha\nu k)$-semiregular matching with property (1). We claim that

$$e_G(A_H \setminus V_1(\mathcal{M}), B_H \setminus V_2(\mathcal{M})) < \frac{\tau}{2} kn. \tag{5.9}$$

Indeed, suppose otherwise. Then the bipartite subgraph H' of G induced by the sets $A_H \setminus V_1(\mathcal{M}) = A_H \setminus V(\mathcal{M})$ and $B_H \setminus V_2(\mathcal{M}) = B_H \setminus V(\mathcal{M})$ satisfies Property (ii) of Definition 5.2, with $\tau_{\triangleright \mathrm{D5.2}} := \tau/2$. So, we have that $(G, \mathcal{D}, H', \mathcal{A}) \in \mathcal{G}(n,k,\Omega,\rho,\nu,\tau/2)$.

Thus Lemma 5.3 for $(G, \mathcal{D}, H', \mathcal{A})$ yields a dense spot $D = (U, W; F) \in \mathcal{D}$ and a set $A \in \mathcal{A}$, together with two sets $X \subseteq U \cap A \cap (A_H \setminus V(\mathcal{M}))$, $Y \subseteq W \cap (B_H \setminus V(\mathcal{M}))$ such that $|X| = |Y| > \alpha_{\triangleright L5.3} \nu k = \alpha \nu k$, and such that (X, Y) is $\varepsilon_{\triangleright L5.3}$-regular and has density at least

$$\frac{\tau_{\triangleright L5.3} \rho_{\triangleright L5.3}}{4\Omega_{\triangleright L5.3}} = \frac{\tau \rho}{8\Omega}.$$

As this contradicts the maximality of $\mathcal{M}$, we have shown (5.9).

In order to see (2), it suffices to observe that by (5.9) and by Property (ii) of Definition 5.2, the set $V(\mathcal{M})$ is incident with at least $\tau kn - \frac{\tau}{2} kn = \frac{\tau}{2} kn$ edges. By Definition 5.2 (i), it follows that $|V(\mathcal{M})| \geqslant \frac{\tau}{2} kn \cdot \frac{1}{\Omega k} \geqslant \frac{\tau}{2\Omega} n$, as desired. $\square$

5.2 Augmenting paths for matchings

We now prove the main lemma of Section 5, namely Lemma 5.10. We will use an augmenting path technique for our semiregular matchings, similar to the augmenting paths commonly used for traditional matching theorems. For this, we need the following definitions.

Definition 5.7 (**Alternating path, augmenting path**). *Given an n-vertex graph G, and a semiregular matching $\mathcal{M}$, we call a sequence $\mathfrak{S} = (Y_0, \mathcal{A}_1, Y_1, \mathcal{A}_2, Y_2, \ldots, \mathcal{A}_h, Y_h)$ $(h \geqslant 0)$ an (δ, s)*-alternating path for $\mathcal{M}$ from Y_0 *if for all $i \in [h]$ we have*

(i) $\mathcal{A}_i \subseteq \mathcal{V}_1(\mathcal{M})$ and the sets $\mathcal{A}_i$ are pairwise disjoint,

(ii) $Y_0 \subseteq V(G) \setminus V(\mathcal{M})$ and $Y_i = \bigcup_{(A,B) \in \mathcal{M}, A \in \mathcal{A}_i} B$,

(iii) $|Y_{i-1}| \geqslant \delta n$, and

(iv) $e(A, Y_{i-1}) \geqslant s \cdot |A|$, for each $A \in \mathcal{A}_i$.

If in addition there is a set $\mathcal{C}$ of disjoint subsets of $V(G) \setminus (Y_0 \cup V(\mathcal{M}))$ such that

(v) $e(\bigcup \mathcal{C}, Y_h) \geqslant t \cdot n$,

then we say that $\mathfrak{S}' = (Y_0, \mathcal{A}_1, Y_1, \mathcal{A}_2, Y_2, \ldots, \mathcal{A}_h, Y_h, \mathcal{C})$ is an (δ, s, t)-augmenting path for $\mathcal{M}$ from Y_0 to $\mathcal{C}$.

The number h is called the the length *of $\mathfrak{S}$ (or of $\mathfrak{S}'$).*

Next, we show that a semiregular matching either has an augmenting path or admits a partition into two parts so that there are only few edges which cross these parts in a certain way.

Lemma 5.8. *Given an n-vertex graph G with $\deg^{\max}(G) \leqslant \Omega k$, a number $\tau \in (0, 1)$, a semiregular matching $\mathcal{M}$, a set $Y_0 \subseteq V(G) \setminus V(\mathcal{M})$, and a set $\mathcal{C}$ of disjoint subsets of $V(G) \setminus (V(\mathcal{M}) \cup Y_0)$, one of the following holds:*

(M1) *There is a semiregular matching* $\mathcal{M}'' \subseteq \mathcal{M}$ *with* $e\left(\bigcup \mathcal{C} \cup V_1(\mathcal{M} \setminus \mathcal{M}''), Y_0 \cup V_2(\mathcal{M}'')\right) < \tau nk$*,*

(M2) $\mathcal{M}$ *has an* $(\frac{\tau}{2\Omega}, \frac{\tau^2}{8\Omega}k, \frac{\tau^2}{16\Omega}k)$*-augmenting path of length at most* $2\Omega/\tau$ *from* Y_0 *to* $\mathcal{C}$*.*

Proof. If $|Y_0| \leqslant \frac{\tau}{2\Omega}n$ then **(M1)** is satisfied for $\mathcal{M}'' := \emptyset$. Let us therefore assume otherwise.

Choose a $(\frac{\tau}{2\Omega}, \frac{\tau^2}{8\Omega}k)$-alternating path $\mathfrak{S} = (Y_0, \mathcal{A}_1, Y_1, \mathcal{A}_2, Y_2, \dots, \mathcal{A}_h, Y_h)$ for $\mathcal{M}$ with $|\bigcup_{\ell=1}^{h} \mathcal{A}_\ell|$ maximal.

Now, let $\ell^* \in \{0, 1, \dots, h\}$ be maximal with $|Y_{\ell^*}| \geqslant \frac{\tau}{2\Omega}n$. Then $\ell^* \in \{h, h-1\}$. Moreover, as $|Y_\ell| \geqslant \frac{\tau}{2\Omega}n$ for all $\ell \leqslant \ell^*$, we have that $(\ell^*+1) \cdot \frac{\tau}{2\Omega}n \leqslant |\bigcup_{\ell \leqslant \ell^*} Y_\ell| \leqslant n$ and thus

$$\ell^* + 1 \leqslant \frac{2\Omega}{\tau}. \tag{5.10}$$

Let $\mathcal{M}'' \subseteq \mathcal{M}$ consist of all $\mathcal{M}$-edges $(A, B) \in \mathcal{M}$ with $A \in \bigcup_{\ell \in [h]} \mathcal{A}_\ell$. Then, by the choice of $\mathfrak{S}$,

$$\begin{aligned} e\left(V_1(\mathcal{M} \setminus \mathcal{M}''), \bigcup_{\ell=0}^{\ell^*} Y_\ell\right) &= \sum_{\ell=0}^{\ell^*} e\left(V_1(\mathcal{M} \setminus \mathcal{M}''), Y_\ell\right) \\ &< (\ell^*+1) \cdot \frac{\tau^2}{8\Omega}k \cdot |V_1(\mathcal{M} \setminus \mathcal{M}'')| \overset{(5.10)}{\leqslant} \frac{\tau}{4}kn. \end{aligned} \tag{5.11}$$

Furthermore, if $\ell^* = h-1$ (that is, if $|Y_h| < \frac{\tau}{2\Omega}n$) then

$$e\left(V_1(\mathcal{M} \setminus \mathcal{M}'') \cup \bigcup \mathcal{C}, Y_h\right) \;<\; \frac{\tau}{2\Omega}n \cdot \deg^{\max}(G) \;\leqslant\; \frac{\tau}{2\Omega}\Omega kn \;=\; \frac{\tau}{2}kn. \tag{5.12}$$

So, regardless whether $h = \ell^*$ or $h = \ell^*+1$, we get from (5.11) and (5.12) that

$$e\left(V_1(\mathcal{M} \setminus \mathcal{M}'') \cup \bigcup \mathcal{C}, Y_0 \cup V_2(\mathcal{M}'')\right) < \frac{3}{4}\tau kn + e\left(\bigcup \mathcal{C}, \bigcup_{\ell=0}^{\ell^*} Y_\ell\right).$$

Thus, if $e(\bigcup \mathcal{C}, \bigcup_{\ell=0}^{\ell^*} Y_\ell) \leqslant \frac{\tau}{4}kn$, we see that (**M1**) satisfied for $\mathcal{M}''$. So, assume otherwise. Then, by (5.10), there is an index $j \in \{0, 1, \dots, \ell^*\}$ so that

$$e\left(\bigcup \mathcal{C}, Y_j\right) \;>\; \frac{\tau^2}{16\Omega}kn,$$

and thus, $(Y_0, \mathcal{A}_1, Y_1, \mathcal{A}_2, Y_2, \dots, \mathcal{A}_h, Y_h, \mathcal{C})$ is an $(\frac{\tau}{2\Omega}, \frac{\tau^2}{8\Omega}k, \frac{\tau^2}{16\Omega}k)$-augmenting path for $\mathcal{M}$. This shows **(M2)**. $\square$

Building on Lemma 5.6 and Lemma 5.8 we prove the following.

Lemma 5.9. *For every* $\Omega \in \mathbb{N}$ *and* $\tau \in (0, \frac{1}{2\Omega})$ *there is a number* $\tau' \in (0, \tau)$ *such that for every* $\rho \in (0,1)$ *there is a number* $\alpha \in (0, \tau'/2)$ *such that for every* $\varepsilon \in (0, \alpha)$ *there is a number* $\pi > 0$ *such that for every* $\gamma > 0$ *there is* $k_0 \in \mathbb{N}$ *such that the following holds for every* $k > k_0$ *and every* $h \in (\gamma k, k/2)$*.*

Let G be a graph of order n with $\deg^{\max}(G) \leqslant \Omega k$, with an (ε^3, ρ, h)-semiregular matching $\mathcal{M}$ and with a $(\rho k, \rho)$-dense cover $\mathcal{D}$ that absorbs $\mathcal{M}$. Let $Y \subseteq V(G) \setminus V(\mathcal{M})$, and let $\mathcal{C}$ be an h-ensemble in G outside $V(\mathcal{M}) \cup Y$. Assume that $U \cap C \in \{\emptyset, C\}$ for each $D = (U, W; F) \in \mathcal{D}$ and each $C \in \mathcal{C} \cup \mathcal{V}_1(\mathcal{M})$.

Then one of the following holds.

(I) There is a semiregular matching $\mathcal{M}'' \subseteq \mathcal{M}$ such that

$$e\left(\bigcup \mathcal{C} \cup V_1(\mathcal{M} \setminus \mathcal{M}''), Y \cup V_2(\mathcal{M}'')\right) < \tau n k.$$

(II) There is an $(\varepsilon, \alpha, \pi h)$-semiregular matching $\mathcal{M}'$ such that

- **(C1)** *$|V(\mathcal{M}) \setminus V(\mathcal{M}')| \leqslant \varepsilon n$, and $|V(\mathcal{M}')| \geqslant |V(\mathcal{M})| + \frac{\tau'}{2} n$, and*
- **(C2)** *for each $(T, Q) \in \mathcal{M}'$ there are sets $C_1 \in \mathcal{V}_1(\mathcal{M}) \cup \mathcal{C}$, $C_2 \in \mathcal{V}_2(\mathcal{M}) \cup \{Y\}$ and a dense spot $D = (U, W; F) \in \mathcal{D}$ such that $T \subseteq C_1 \cap U$ and $Q \subseteq C_2 \cap W$.*

Proof. We divide the proof into five steps.

Step 1: Setting up the parameters. Suppose that Ω and τ are given. For $\ell = 0, 1, \ldots, \lceil 2\Omega/\tau \rceil$, we define the auxiliary parameters

$$\tau^{(\ell)} := \left(\frac{\tau^2}{32\Omega}\right)^{\lceil \frac{2\Omega}{\tau} \rceil - \ell + 2}, \tag{5.13}$$

and set

$$\tau' := \frac{\tau^{(0)}}{2\Omega}.$$

Given ρ, we define

$$\alpha := \frac{\tau' \rho}{16\Omega}.$$

Then, given ε, for $\ell = 0, 1, \ldots, \lceil 2\Omega/\tau \rceil$, we define the further auxiliary parameters

$$\mu^{(\ell)} := \alpha_{\triangleright \text{L5.6}}\big(\Omega, \rho, \varepsilon^3, \tau^{(\ell)}\big)$$

which are given by Lemma 5.6 for input parameters $\Omega_{\triangleright \text{L5.6}} := \Omega$, $\rho_{\triangleright \text{L5.6}} := \rho$, $\varepsilon_{\triangleright \text{L5.6}} := \varepsilon^3$, and $\tau_{\triangleright \text{L5.6}} := \tau^{(\ell)}$. Set

$$\pi := \frac{\varepsilon}{2} \cdot \min\left\{\mu^{(\ell)} \ : \ \ell = 0, \ldots, \lceil 2\Omega/\tau \rceil\right\},$$

Given the next input parameter γ, Lemma 5.6 for parameters as above and the final input $\nu_{\triangleright \text{L5.6}} := \gamma$ yields $k_{0_{\triangleright \text{L5.6}}} =: k_0^{(\ell)}$. Set

$$k_0 := \max\left\{k_0^{(\ell)} \ : \ \ell = 0, \ldots, \lceil 2\Omega/\tau \rceil\right\}.$$

Step 2: Finding an augmenting path. We apply Lemma 5.8 to G, τ, $\mathcal{M}$, Y and $\mathcal{C}$. Since (**M1**) corresponds to (I), let us assume that the outcome of the lemma is (**M2**). Then there is a $(\frac{\tau}{2\Omega}, \frac{\tau^2}{8\Omega}k, \frac{\tau^2}{16\Omega}k)$-augmenting path $\mathfrak{S}' = (Y_0, \mathcal{A}_1, Y_1, \mathcal{A}_2, Y_2, \ldots, \mathcal{A}_{j^*}, Y_{j^*}, \mathcal{C})$ for $\mathcal{M}$ starting from $Y_0 := Y$ such that $j^* \leqslant 2\Omega/\tau$.

Our aim is now to show that (II) holds.

Step 3: Creating parallel matchings. Inductively, for $\ell = j^*, j^*-1, \ldots, 0$ we shall define auxiliary bipartite induced subgraphs $H^{(\ell)} \subseteq G$ with colour classes $P^{(\ell)}$ and Y_ℓ that satisfy

(a) $e(H^{(\ell)}) \geqslant \tau^{(\ell)} kn$,

and $(\varepsilon^3, 2\alpha, \mu^{(\ell)}h)$-semiregular matchings $\mathcal{M}^{(\ell)}$ that satisfy

(b) $V_1(\mathcal{M}^{(\ell)}) \subseteq P^{(\ell)}$,

(c) for each $(A', B') \in \mathcal{M}^{(\ell)}$ there are a dense spot $(U, W; F) \in \mathcal{D}$ and a set $A \in \mathcal{V}_1(\mathcal{M})$ (or a set $A \in \mathcal{C}$ if $\ell = j^*$) such that $A' \subseteq U \cap A$ and $B' \subseteq W \cap Y_\ell$,

(d) $|V(\mathcal{M}^{(\ell)})| \geqslant \frac{\tau^{(\ell)}}{2\Omega} n$, and

(e) $|B \cap V_2(\mathcal{M}^{(\ell)})| = |A \cap P^{(\ell-1)}|$ for each edge $(A, B) \in \mathcal{M}$, if $\ell > 0$.

We take $H^{(j^*)}$ as the induced bipartite subgraph of G with colour classes $P^{(j^*)} := \bigcup \mathcal{C}$ and Y_{j^*}. Definition 5.7 (v) together with (5.13) ensures (a) for $\ell = j^*$. Now, for $\ell \leqslant j^*$, suppose $H^{(\ell)}$ is defined already. Further, if $\ell < j^*$ suppose also that $\mathcal{M}^{(\ell+1)}$ is defined already. We shall define $\mathcal{M}^{(\ell)}$, and, if $\ell > 0$, we shall also define $H^{(\ell-1)}$.

Observe that $(G, \mathcal{D}, H^{(\ell)}, \mathcal{A}_\ell) \in \mathcal{G}(n, k, \Omega, \rho, \frac{h}{k}, \tau^{(\ell)})$, because of (a) and the assumptions of the lemma. So, applying Lemma 5.6 to $(G, \mathcal{D}, H^{(\ell)}, \mathcal{A}_\ell)$ and noting that $\frac{\tau^{(\ell)}\rho}{8\Omega} \geqslant 2\alpha$ we obtain an $(\varepsilon^3, 2\alpha, \mu^{(\ell)}h)$-semiregular matching $\mathcal{M}^{(\ell)}$ that satisfies conditions (b)–(d).

If $\ell > 0$, we define $H^{(\ell-1)}$ as follows. For each $(A, B) \in \mathcal{M}$ take a set $\tilde{A} \subseteq A$ of cardinality $|\tilde{A}| = |B \cap V(\mathcal{M}^{(\ell)})|$ so that

$$e(\tilde{A}, Y_{\ell-1}) \;\geqslant\; \frac{\tau^2}{8\Omega} k \cdot |\tilde{A}| \,. \tag{5.14}$$

This is possible by Definition 5.7 (iv): just choose those vertices from A for $\tilde{A}$ that send most edges to $Y_{\ell-1}$. Let $P^{(\ell-1)}$ be the union of all the sets $\tilde{A}$. Then, (e) is satisfied. Furthermore,

$$|P^{(\ell-1)}| \;=\; |V_2(\mathcal{M}^{(\ell)})| \;\overset{\text{(d)}}{\geqslant}\; \frac{\tau^{(\ell)}}{4\Omega} n.$$

So, by (5.14),

$$e(P^{(\ell-1)}, Y_{\ell-1}) \;\geqslant\; \frac{\tau^2}{8\Omega} k \cdot |P^{(\ell-1)}| \;\geqslant\; \frac{\tau^2 \cdot \tau^{(\ell)}}{32\Omega^2} kn \;\overset{(5.13)}{=}\; \tau^{(\ell-1)} kn \,. \tag{5.15}$$

We let $H^{(\ell-1)}$ be the bipartite subgraph of G induced by the colour classes $P^{(\ell-1)}$ and $Y_{\ell-1}$. Then (5.15) establishes (a) for $H^{(\ell-1)}$. This finishes step ℓ.[xv]

Step 4: Harmonising the matchings. Our semiregular matchings $\mathcal{M}^{(0)},\dots,\mathcal{M}^{(j^*)}$ will be a good base for constructing the semiregular matching $\mathcal{M}'$ we are after. However, we do not know anything about $|B\cap V_2(\mathcal{M}^{(\ell)})|-|A\cap V_1(\mathcal{M}^{(\ell-1)})|$ for the $\mathcal{M}$-edges $(A,B)\in\mathcal{M}$. But this term will be crucial in determining how much of $V(\mathcal{M})$ gets lost when we replace some of its $\mathcal{M}$-edges with $\bigcup\mathcal{M}^{(\ell)}$-edges. For this reason, we refine $\mathcal{M}^{(\ell)}$ in a way that its $\mathcal{M}^{(\ell)}$-edges become almost equal-sized.

Formally, we shall inductively construct semiregular matchings $\mathcal{N}^{(0)},\dots,\mathcal{N}^{(j^*)}$ such that for $\ell=0,\dots,j^*$ we have

(A) $\mathcal{N}^{(\ell)}$ is an $(\varepsilon,\alpha,\pi h)$-semiregular matching,

(B) $\mathcal{M}^{(\ell)}$ absorbes $\mathcal{N}^{(\ell)}$,

(C) if $\ell>0$ and $(A,B)\in\mathcal{M}$ with $A\in\mathcal{A}_\ell$ then $|A\cap V(\mathcal{N}^{(\ell-1)})|\geqslant|B\cap V(\mathcal{N}^{(\ell)})|$, and

(D) $|V_2(\mathcal{N}^{(\ell)})|\geqslant|V_1(\mathcal{N}^{(\ell-1)})|-\frac{\varepsilon}{2}\cdot|V_2(\mathcal{M}^{(\ell)})|$ if $\ell>0$ and $|V_2(\mathcal{N}^{(0)})|\geqslant\frac{\tau^{(0)}}{2\Omega}n=\tau' n$.

Set $\mathcal{N}^{(0)}:=\mathcal{M}^{(0)}$. Clearly (B) holds for $\ell=0$, (A) is easy to check, and (C) is void. Finally, Property (D) holds because of (d). Suppose now $\ell>0$ and that we already constructed matchings $\mathcal{N}^{(0)},\dots,\mathcal{N}^{(\ell-1)}$ satisfying Conditions (A)–(D).

Observe that for any $(A,B)\in\mathcal{M}$ we have that

$$|B\cap V_2(\mathcal{M}^{(\ell)})|\ \overset{(b),(e)}{\geqslant}\ |A\cap V_1(\mathcal{M}^{(\ell-1)})|\ \geqslant\ |A\cap V_1(\mathcal{N}^{(\ell-1)})|, \tag{5.16}$$

where the last inequality holds because of (B) for $\ell-1$.

So, we can choose a subset $X^{(\ell)}\subseteq V_2(\mathcal{M}^{(\ell)})$ such that $|B\cap X^{(\ell)}|=|A\cap V(\mathcal{N}^{(\ell-1)})|$ for each $(A,B)\in\mathcal{M}$. Now, for each $(S,T)\in\mathcal{M}^{(\ell)}$ write $\widehat{T}:=T\cap X^{(\ell)}$, and choose a subset $\widehat{S}$ of S of size $|\widehat{T}|$. Set

$$\mathcal{N}^{(\ell)}:=\left\{(\widehat{S},\widehat{T})\ :\ (S,T)\in\mathcal{M}^{(\ell)},|\widehat{T}|\geqslant\frac{\varepsilon}{2}\cdot|T|\right\}.$$

Then (B) and (C) hold for ℓ.

For (A), note that Fact 2.7 implies that $\mathcal{N}^{(\ell)}$ is an $\left(\varepsilon,2\alpha-\varepsilon^3,\frac{\varepsilon}{2}\mu^{(\ell)}h\right)$-semiregular matching.

[xv] Recall that the matching $\mathcal{M}^{(\ell-1)}$ is only to be defined in step $\ell-1$.

In order to see (D), it suffices to observe that

$$\begin{aligned}
|V_2(\mathcal{N}^{(\ell)})| &= \sum_{(\widehat{S},\widehat{T})\in\mathcal{N}^{(\ell)}} |\widehat{T}| \\
&\geqslant |X^{(\ell)}| \;-\; \sum_{(S,T)\in\mathcal{M}^{(\ell)}} \frac{\varepsilon}{2}\cdot|T| \\
&\geqslant \sum_{(A,B)\in\mathcal{M}} |A\cap V_1(\mathcal{N}^{(\ell-1)})| \;-\; \frac{\varepsilon}{2}\cdot|V_2(\mathcal{M}^{(\ell)})| \\
&= |V_1(\mathcal{N}^{(\ell-1)})| \;-\; \frac{\varepsilon}{2}\cdot|V_2(\mathcal{M}^{(\ell)})|.
\end{aligned}$$

Step 5: The final matching. For each $\ell = 1,2,\ldots,j^*$ let $\mathcal{L}$ denote the set of all $\mathcal{M}$-edges $(A,B)\in\mathcal{M}$ with $|A'| > \frac{\varepsilon}{2}\cdot|A|$, where $A' := A\setminus V_1(\mathcal{N}^{(\ell-1)})$. Further, for each $(A,B)\in\mathcal{M}$, choose a set $B'\subseteq B\setminus V_2(\mathcal{N}^{(\ell)})$ of cardinality $|A'|$. This is possible by (C). Set

$$\mathcal{K} := \{(A',B') : (A,B)\in\mathcal{L}\}.$$

By the assumption of the lemma, for every $(A',B')\in\mathcal{K}$ there are an edge $(A,B)\in\mathcal{M}$ and a dense spot $D=(U,W;F)\in\mathcal{D}$ such that

$$A'\subseteq A\subseteq U \text{ and } B'\subseteq B\subseteq W. \tag{5.17}$$

Since $\mathcal{M}$ is (ε^3,ρ,h)-semiregular we have by Fact 2.7 that $\mathcal{K}$ is a $(\varepsilon,\rho-\varepsilon^3,\frac{\varepsilon}{2}h)$-semiregular matching. Set

$$\mathcal{M}' := \mathcal{K}\cup\bigcup_{\ell=0}^{j^*}\mathcal{N}^{(\ell)},$$

now it is easy to check that $\mathcal{M}'$ is an $(\varepsilon,\alpha,\pi h)$-semiregular matching. Using (5.17) together with (B) and (c), we see that **(C2)** holds for $\mathcal{M}'$.

In order to see **(C1)**, we calculate

$$\begin{aligned}
|V(\mathcal{M})\setminus V(\mathcal{M}')| &\leqslant \sum_{(A,B)\in\mathcal{M}\setminus\mathcal{L}} |A'\cup B'| \;+\; \sum_{(A,B)\in\mathcal{L}}\sum_{\ell=1}^{j^*}\Big(|A\cap V_1(\mathcal{N}^{(\ell-1)})| - |B\cap V_2(\mathcal{N}^{(\ell)})|\Big) \\
&\leqslant \frac{\varepsilon}{2}\cdot\sum_{(A,B)\in\mathcal{M}\setminus\mathcal{L}} |A\cup B| \;+\; \sum_{\ell=1}^{j^*}\Big(|V_1(\mathcal{N}^{(\ell-1)})| - |V_2(\mathcal{N}^{(\ell)})|\Big) \\
&\overset{\text{(D)}}{\leqslant} \frac{\varepsilon}{2}n \;+\; \sum_{\ell=1}^{j^*}\frac{\varepsilon}{2}\cdot|V_2(\mathcal{M}^{(\ell)})| \\
&\leqslant \varepsilon n\,.
\end{aligned}\tag{5.18}$$

Using the fact that $V_2(\mathcal{N}^{(0)}) \subseteq V(\mathcal{M}') \setminus V(\mathcal{M})$ the last calculation also implies that

$$|V(\mathcal{M}')| - |V(\mathcal{M})| \geqslant |V_2(\mathcal{N}^{(0)})| - |V(\mathcal{M}) \setminus V(\mathcal{M}')|$$
$$\overset{\text{(D)}}{\geqslant} \tau' n - \varepsilon n$$
$$> \frac{\tau'}{2} n\,,$$

since $\varepsilon < \alpha \leqslant \tau'/2$ by assumption. □

Iterating Lemma 5.9 we prove the main result of the section.

Lemma 5.10. *For every $\Omega \in \mathbb{N}$, $\rho \in (0, 1/\Omega)$ there exists a number $\beta > 0$ such that for every $\varepsilon \in (0, \beta)$, there are $\varepsilon', \pi > 0$ such that for each $\gamma > 0$ there exists $k_0 \in \mathbb{N}$ such that the following holds for every $k > k_0$ and $c \in (\gamma k, k/2)$.*

Let G be a graph of order n, with $\deg^{\max}(G) \leqslant \Omega k$. Let $\mathcal{D}$ be a $(\rho k, \rho)$-dense cover of G, and let $\mathcal{M}$ be an (ε', ρ, c)-semiregular matching that is absorbed by $\mathcal{D}$. Let $\mathcal{C}$ be a c-ensemble in G outside $V(\mathcal{M})$. Let $Y \subseteq V(G) \setminus (V(\mathcal{M}) \cup \bigcup \mathcal{C})$. Assume that for each $(U, W; F) \in \mathcal{D}$, and for each $C \in \mathcal{V}_1(\mathcal{M}) \cup \mathcal{C}$ we have that

$$U \cap C \in \{\emptyset, C\}\,. \tag{5.19}$$

Then there exists an $(\varepsilon, \beta, \pi c)$-semiregular matching $\mathcal{M}'$ such that

(i) $|V(\mathcal{M}) \setminus V(\mathcal{M}')| \leqslant \varepsilon n$,

(ii) for each $(T, Q) \in \mathcal{M}'$ there are sets $C_1 \in \mathcal{V}_1(\mathcal{M}) \cup \mathcal{C}$, $C_2 \in \mathcal{V}_2(\mathcal{M}) \cup \{Y\}$ and a dense spot $D = (U, W; F) \in \mathcal{D}$ such that $T \subseteq C_1 \cap U$ and $Q \subseteq C_2 \cap W$, and

(iii) $\mathcal{M}'$ can be partitioned into $\mathcal{M}_1$ and $\mathcal{M}_2$ so that

$$e\Big((\bigcup \mathcal{C} \cup V_1(\mathcal{M})) \setminus V_1(\mathcal{M}_1)\ ,\ (Y \cup V_2(\mathcal{M})) \setminus V_2(\mathcal{M}_2)\Big)\ <\ \rho k n\,.$$

Proof. Let Ω and ρ be given. Let $\tau' := \tau'_{\triangleright\text{L5.9}}$ be the output given by Lemma 5.9 for input parameters $\Omega_{\triangleright\text{L5.9}} := \Omega$ and $\tau_{\triangleright\text{L5.9}} := \rho/2$.

Set $\rho^{(0)} := \rho$, set $L := \lceil 2/\tau' \rceil + 1$, and for $\ell \in [L]$, inductively define $\rho^{(\ell)}$ to be the output $\alpha_{\triangleright\text{L5.9}}$ given by Lemma 5.9 for the further input parameter $\rho_{\triangleright\text{L5.9}} := \rho^{(\ell-1)}$ (keeping $\Omega_{\triangleright\text{L5.9}} = \Omega$ and $\tau_{\triangleright\text{L5.9}} = \rho/2$ fixed). Then $\rho^{(\ell+1)} \leqslant \rho^{(\ell)}$ for all ℓ. Set $\beta := \rho^{(L)}$.

Given $\varepsilon < \beta$ we set $\varepsilon^{(\ell)} := (\varepsilon/2)^{3^{L-\ell}}$ for $\ell \in [L] \cup \{0\}$, and set $\varepsilon' := \varepsilon^{(0)}$. Clearly,

$$\sum_{\ell=0}^{L} \varepsilon^{(\ell)}\ \leqslant\ \varepsilon. \tag{5.20}$$

Now, for $\ell+1 \in [L]$, let $\pi^{(\ell)} := \pi_{\triangleright L5.9}$ be given by Lemma 5.9 for input parameters $\Omega_{\triangleright L5.9} := \Omega$, $\tau_{\triangleright L5.9} := \rho/2$, $\rho_{\triangleright L5.9} := \rho^{(\ell)}$ and $\varepsilon_{\triangleright L5.9} := \varepsilon^{(\ell+1)}$. For $\ell \in [L] \cup \{0\}$, set $\Pi^{(\ell)} := \frac{\rho}{2\Omega} \prod_{j=0}^{\ell-1} \pi^{(j)}$. Let $\pi := \Pi^{(L)}$.

Given γ, let k_0 be the maximum of the lower bounds $k_{0_{\triangleright L5.9}}$ given by Lemma 5.9 for input parameters $\Omega_{\triangleright L5.9} := \Omega$, $\tau_{\triangleright L5.9} := \rho/2$, $\rho_{\triangleright L5.9} := \rho^{(\ell-1)}$, $\varepsilon_{\triangleright L5.9} := \varepsilon^{(\ell)}$, $\gamma_{\triangleright L5.9} := \gamma\Pi^{(\ell)}$, for $\ell \in [L]$.

Suppose now we are given G, $\mathcal{D}$, $\mathcal{C}$, Y and $\mathcal{M}$. Suppose further that $c > \gamma k > \gamma k_0$. Let $\ell \in \{0, 1, \ldots, L\}$ be maximal such that there is a matching $\mathcal{M}^{(\ell)}$ with the following properties:

(a) $\mathcal{M}^{(\ell)}$ is an $(\varepsilon^{(\ell)}, \rho^{(\ell)}, \Pi^{(\ell)}c)$-semiregular matching,

(b) $|V(\mathcal{M}^{(\ell)})| \geqslant \ell \cdot \frac{\tau'}{2} n$,

(c) $|V(\mathcal{M}) \setminus V(\mathcal{M}^{(\ell)})| \leqslant \sum_{i=0}^{\ell} \varepsilon^{(\ell)} n$, and

(d) for each $(T, Q) \in \mathcal{M}^{(\ell)}$ there are sets $C_1 \in \mathcal{V}_1(\mathcal{M}) \cup \mathcal{C}$, $C_2 \in \mathcal{V}_2(\mathcal{M}) \cup \{Y\}$ and a dense spot $D = (U, W; F) \in \mathcal{D}$ such that $T \subseteq C_1 \cap U$ and $Q \subseteq C_2 \cap W$.

Observe that such a number ℓ exists, as for $\ell = 0$ we may take $\mathcal{M}^{(0)} = \mathcal{M}$. Also note that $\ell \leqslant 2/\tau' < L$ because of (b).

We now apply Lemma 5.9 with input parameters $\Omega_{\triangleright L5.9} := \Omega$, $\tau_{\triangleright L5.9} := \rho/2$, $\rho_{\triangleright L5.9} := \rho^{(\ell)}$, $\varepsilon_{\triangleright L5.9} := \varepsilon^{(\ell+1)} < \beta \leqslant \rho^{(\ell+1)} = \alpha_{\triangleright L5.9}$, $\gamma_{\triangleright L5.9} := \gamma\Pi^{(\ell)}$ to the graph G with the $(\rho^{(\ell)}k, \rho^{(\ell)})$-dense cover $\mathcal{D}$, the $(\varepsilon^{(\ell)}, \rho^{(\ell)}, \Pi^{(\ell)}c)$-semiregular matching $\mathcal{M}^{(\ell)}$, the set

$$\tilde{Y} := (Y \cup V_2(\mathcal{M})) \setminus V_2(\mathcal{M}^{(\ell)}),$$

and the $(\Pi^{(\ell)}c)$-ensemble

$$\tilde{\mathcal{C}} := \left\{ C \setminus V(\mathcal{M}^{(\ell)}) \ : \ C \in \mathcal{V}_1(\mathcal{M}) \cup \mathcal{C}, \ |C \setminus V_1(\mathcal{M}^{(\ell)})| \geqslant \Pi^{(\ell)}c \right\}.$$

Lemma 5.9 yields a semiregular matching which either corresponds to $\mathcal{M}''$ as in Assertion (I) or to $\mathcal{M}'$ as in Assertion (II). Note that in the latter case, the matching $\mathcal{M}'$ actually constitutes an $(\varepsilon^{(\ell+1)}, \rho^{(\ell+1)}, \Pi^{(\ell+1)}c)$-semiregular matching $\mathcal{M}^{(\ell+1)}$ fulfilling all the above properties for $\ell + 1 \leqslant L$. In fact, (b) and (c) hold for $\mathcal{M}^{(\ell+1)}$ because of (**C1**), and it is not difficult to deduce (d) from (**C2**) and from (d) for ℓ. But this contradicts the choice of ℓ. We conclude that we obtained a semiregular matching $\mathcal{M}'' \subseteq \mathcal{M}^{(\ell)}$ as in Assertion (I) of Lemma 5.9.

Thus, in other words, $\mathcal{M}^{(\ell)}$ can be partitioned into $\mathcal{M}_1$ and $\mathcal{M}_2$ so that

$$e\Big(\bigcup \tilde{\mathcal{C}} \cup V_1(\mathcal{M}_2)\ ,\ \tilde{Y} \cup V_2(\mathcal{M}_1)\Big) \ < \ \tau_{\triangleright L5.9} kn \ = \ \rho kn/2. \tag{5.21}$$

Set $\mathcal{M}' := \mathcal{M}^{(\ell)}$. Then $\mathcal{M}'$ is $(\varepsilon, \beta, \pi c)$-semiregular by (a). Note that Assertion (i) of the lemma holds by (5.20) and by (c). Assertion (ii) holds because of (d).

Since

$$(Y \cup V_2(\mathcal{M})) \setminus V_2(\mathcal{M}_2) \subseteq \tilde{Y} \cup V_2(\mathcal{M}_1),$$

and because of (5.21) we know that in order to prove Assertion (iii) it suffices to show that the set

$$\begin{aligned} X &:= ((\bigcup \mathcal{C} \cup V_1(\mathcal{M})) \setminus V_1(\mathcal{M}_1)) \setminus (\bigcup \tilde{\mathcal{C}} \cup V_1(\mathcal{M}_2)) \\ &= (\bigcup \mathcal{C} \cup V_1(\mathcal{M})) \setminus (\bigcup \tilde{\mathcal{C}} \cup V_1(\mathcal{M}^{(\ell)})) \end{aligned}$$

sends at most $\rho kn/2$ edges to the rest of the graph. For this, it would be enough to see that $|X| \leqslant \frac{\rho}{2\Omega} n$, as by assumption, G has maximum degree Ωk.

To this end, note that by assumption, $|\mathcal{V}_1(\mathcal{M}) \cup \mathcal{C}| \leqslant \frac{n}{c}$. Further, the definition of $\tilde{\mathcal{C}}$ implies that for each $A \in \mathcal{C} \cup \mathcal{V}_1(\mathcal{M})$ we have that $|A \setminus (\bigcup \tilde{\mathcal{C}} \cup V_1(\mathcal{M}^{(\ell)})| \leqslant \Pi^{(\ell)} c$. Combining these two observations, we obtain that

$$|X| < \Pi^{(\ell)} n \leqslant \frac{\rho}{2\Omega} n ,$$

as desired.

□

6 Structure of LKS graphs

In this section we give a structural result for graphs $G \in \mathbf{LKSsmall}(n, k, \eta)$, stated in Lemma 6.1. Similar structural results were essential also for proving Conjecture 1.2 in the dense setting in [AKS95, PS12]. There, a certain matching structure was proved to exist in the cluster graph of the host graph. This matching structure then allowed to embed a given tree into the host graph.

Naturally, in our possibly sparse setting the sparse decomposition ∇ of G will enter the picture (instead of just the cluster graph of G). There is an important subtlety though: we need to "re-regularize" the cluster graph $\mathbf{G}_{\mathrm{reg}}$ of ∇. The necessity of this step arises from the ambiguity of the sparse decomposition ∇ given by Lemma 4.14, see Remark 4.15. Consequently, the cluster graph $\mathbf{G}_{\mathrm{reg}}$ given by a sparse decomposition $(\boldsymbol{\Psi}, \mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ of G might not be suitable for locating a matching structure in analogue to the dense setting. In this case, we have to find another regularization of parts of G, partially based on G_{reg}. Lemma 5.10 is the main tool to this end. The re-regularization is captured by the semiregular matchings $\mathcal{M}_A$ and $\mathcal{M}_B$.

Let us note that this step is one of the biggest differences between our approach and the announced solution of the Erdős-Sós Conjecture by Ajtai, Komlós, Simonovits and

Szemerédi. In other words, the nature of the graphs arising in the Erdős-Sós Conjecture allows a less careful approach with respect to regularization, still yielding a structure suitable for embedding trees. We discuss the necessity of this step in further detail in Section 6.2, after proving the main result of this section, Lemma 6.1, in Section 6.1.

6.1 Finding the structure

We now introduce some notation we need in order to state Lemma 6.1. Suppose that G is a graph with a $(k, \Omega^{**}, \Omega^*, \Lambda, \gamma, \varepsilon, \nu, \rho)$-sparse decomposition

$$\nabla = (\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$$

with respect to $\mathbb{L}_{\eta,k}(G)$ and $\mathbb{S}_{\eta,k}(G)$. Suppose further that $\mathcal{M}_A, \mathcal{M}_B$ are $(\varepsilon', d, \gamma k)$-semiregular matchings in $G_{\mathcal{D}}$. We then define the triple

$$(\mathbb{XA}, \mathbb{XB}, \mathbb{XC}) = (\mathbb{XA}, \mathbb{XB}, \mathbb{XC})(\eta, \nabla, \mathcal{M}_A, \mathcal{M}_B)$$

by setting

$$\begin{aligned}
\mathbb{XA} &:= \mathbb{L}_{\eta,k}(G) \setminus V(\mathcal{M}_B) , \\
\mathbb{XB} &:= \left\{ v \in V(\mathcal{M}_B) \cap \mathbb{L}_{\eta,k}(G) \ : \ \widehat{\deg}(v) < (1+\eta)\frac{k}{2} \right\} , \\
\mathbb{XC} &:= \mathbb{L}_{\eta,k}(G) \setminus (\mathbb{XA} \cup \mathbb{XB}) ,
\end{aligned}$$

where $\widehat{\deg}(v)$ on the second line is defined by

$$\widehat{\deg}(v) := \deg_G\left(v, \mathbb{S}_{\eta,k}(G) \setminus (V(G_{\mathrm{exp}}) \cup \mathfrak{A} \cup V(\mathcal{M}_A \cup \mathcal{M}_B))\right) . \tag{6.1}$$

Clearly, $\{\mathbb{XA}, \mathbb{XB}, \mathbb{XC}\}$ is a partition of $\mathbb{L}_{\eta,k}(G)$.

We now give the main and only lemma of this section, a structural result for graphs from **LKSsmall**(n, k, η).

Lemma 6.1. *For every $\eta > 0, \Omega > 0, \gamma \in (0, \eta/3)$ there is $\beta > 0$ so that for every $\varepsilon \in (0, \frac{\gamma^2\eta}{12})$ there exist $\varepsilon', \pi > 0$ such that for every $\nu > 0$ there exists $k_0 \in \mathbb{N}$ such that for every Ω^* with $\Omega^* < \Omega$ and every k with $k > k_0$ the following holds.*

*Suppose $\nabla = (\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$ is a $(k, \Omega^{**}, \Omega^*, \Lambda, \gamma, \varepsilon', \nu, \rho)$-sparse decomposition of a graph $G \in$ **LKSsmall**(n, k, η) with respect to $S := \mathbb{S}_{\eta,k}(G)$ and $L := \mathbb{L}_{\eta,k}(G)$ which captures all but at most $\eta kn/6$ edges of G. Let $\mathfrak{c}$ be the size of the clusters $\mathbf{V}$.*[xvi] *Write*

$$S^0 := S \setminus (V(G_{\mathrm{exp}}) \cup \mathfrak{A}) . \tag{6.2}$$

Then $G_{\mathcal{D}}$ contains two $(\varepsilon, \beta, \pi\mathfrak{c})$-semiregular matchings $\mathcal{M}_A$ and $\mathcal{M}_B$ such that for the triple $(\mathbb{XA}, \mathbb{XB}, \mathbb{XC}) := (\mathbb{XA}, \mathbb{XB}, \mathbb{XC})(\eta, \nabla, \mathcal{M}_A, \mathcal{M}_B)$ we have

[xvi] The number $\mathfrak{c}$ is irrelevant when $\mathbf{V} = \emptyset$. In particular, note that in that case we necessarily have $\mathcal{M}_A = \mathcal{M}_B = \emptyset$ for the semiregular matchings given by the lemma.

(a) $V(\mathcal{M}_A) \cap V(\mathcal{M}_B) = \emptyset$,

(b) $V_1(\mathcal{M}_B) \subseteq S^0$,

(c) *for each* $(T, Q) \in \mathcal{M}_A \cup \mathcal{M}_B$*, there is a dense spot* $(A_D, B_D; E_D) \in \mathcal{D}$ *with* $T \subseteq A_D$*,* $Q \subseteq B_D$*, and furthermore, either* $T \subseteq S$ *or* $T \subseteq L$*, and* $Q \subseteq S$ *or* $Q \subseteq L$*,*

(d) *for each* $X_1 \in \mathcal{V}_1(\mathcal{M}_A \cup \mathcal{M}_B)$ *there exists a cluster* $C_1 \in \mathbf{V}$ *such that* $X_1 \subseteq C_1$*, and for each* $X_2 \in \mathcal{V}_2(\mathcal{M}_A \cup \mathcal{M}_B)$ *there exists* $C_2 \in \mathbf{V} \cup \{L \cap \mathfrak{A}\}$ *such that* $X_2 \subseteq C_2$*,*

(e) $e_{G_\nabla}\big(\mathbb{XA}, S^0 \setminus V(\mathcal{M}_A)\big) \leqslant \gamma kn$,

(f) $e_{G_{\mathrm{reg}}}(V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) \leqslant \varepsilon\Omega^* kn$,

(g) *for the semiregular matching* $\mathcal{N}_{\mathfrak{A}} := \{(X, Y) \in \mathcal{M}_A \cup \mathcal{M}_B : (X \cup Y) \cap \mathfrak{A} \neq \emptyset\}$ *we have* $e_{G_{\mathrm{reg}}}\big(V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), V(\mathcal{N}_{\mathfrak{A}})\big) \leqslant \varepsilon\Omega^* kn$*,*

(h) *for* $\mathcal{M}_{\mathrm{good}} := \{(A, B) \in \mathcal{M}_A : A \cup B \subseteq \mathbb{XA}\}$ *we have that each* $\mathcal{M}_{\mathrm{good}}$*-edge is an edge of* $\mathbf{G}_{\mathrm{reg}}$*, and at least one of the following conditions holds*

(K1) $2e_G(\mathbb{XA}) + e_G(\mathbb{XA}, \mathbb{XB}) \geqslant \eta kn/3$,

(K2) $|V(\mathcal{M}_{\mathrm{good}})| \geqslant \eta n/3$.

Remark 6.2. *In some sense, property* (h) *is the most important part of Lemma 6.1. Note that the assertion* **(K2)** *implies a quantitatively weaker version of* **(K1)**. *Indeed, consider* $(C, D) \in \mathcal{M}_A$*. An average vertex* $v \in C$ *sends at least* $\beta \cdot \pi\mathfrak{c} \geqslant \beta \cdot \pi\nu k$ *edges to* D*. Thus, if* $|V(\mathcal{M}_{\mathrm{good}})| \geqslant \eta n/3$ *then* $\mathcal{M}_{\mathrm{good}}$ *induces at least* $(\eta n/6) \cdot \beta \cdot \pi\nu k = \Theta(kn)$ *edges in* $\mathbb{XA}$*. Such a bound, however, would be insufficient for our purposes as later* $\eta \gg \pi, \nu$*.*

Proof of Lemma 6.1. The idea of the proof is to first obtain some information about the structure of the graph $\mathbf{G}_{\mathrm{reg}}$ with the help of the Gallai-Edmonds Matching Theorem (Theorem 2.4). Then this rough structure is refined by Lemma 5.10 to yield the assertions of the lemma.

Let us begin with setting the parameters. Let $\beta := \beta_{\triangleright \mathrm{L5.10}}$ be given by Lemma 5.10 for input parameters $\Omega_{\triangleright \mathrm{L5.10}} := \Omega$, $\rho_{\triangleright \mathrm{L5.10}} := \gamma^2$, and let ε' and π be given by Lemma 5.10 for further input parameter $\varepsilon_{\triangleright \mathrm{L5.10}} := \varepsilon$. Last, let k_0 be given by Lemma 5.10 with the above parameters and $\gamma_{\triangleright \mathrm{L5.10}} := \nu$.

Without loss of generality we assume that $\varepsilon' \leqslant \varepsilon$ and $\beta < \gamma^2$. We write $\mathbf{S} := \{C \in \mathbf{V} : C \subseteq S\}$ and $\mathbf{L} := \{C \in \mathbf{V} : C \subseteq L\}$. Further, let $\mathbf{S}^0 := \{C \in \mathbf{S} : C \subseteq S^0\}$.

Let $\mathbf{Q}$ be a separator and N_0 a matching given by Theorem 2.4 applied to the graph $\mathbf{G}_{\mathrm{reg}}$. We will presume that the pair $(\mathbf{Q}, N_0)$ is chosen among all the possible choices so that the number of vertices of $\mathbf{S}^0$ that are isolated in $\mathbf{G}_{\mathrm{reg}} - \mathbf{Q}$ and are not covered by N_0 is minimized. Let $\mathbf{S}^{\mathrm{I}}$ denote the set of vertices in $\mathbf{S}^0$ that are isolated in $\mathbf{G}_{\mathrm{reg}} - \mathbf{Q}$. Recall that the components of $\mathbf{G}_{\mathrm{reg}} - \mathbf{Q}$ are factor critical.

Define $\mathbf{S}^{\mathrm{R}} \subseteq V(\mathbf{G}_{\mathrm{reg}})$ as a minimal set such that

- $\mathbf{S}^{\mathrm{I}} \setminus V(N_0) \subseteq \mathbf{S}^{\mathrm{R}}$, and
- if $C \in \mathbf{S}$ and there is an edge $DZ \in E(\mathbf{G}_{\mathrm{reg}})$ with $Z \in \mathbf{S}^{\mathrm{R}}$, $D \in \mathbf{Q}$, $CD \in N_0$ then $C \in \mathbf{S}^{\mathrm{R}}$.

Then each vertex from $\mathbf{S}^{\mathrm{R}}$ is reachable from $\mathbf{S}^{\mathrm{I}} \setminus V(N_0)$ by a path in $\mathbf{G}_{\mathrm{reg}}$ that alternates between $\mathbf{S}^{\mathrm{R}}$ and $\mathbf{Q}$, and has every second edge in N_0. Also note that for all $CD \in N_0$ with $C \in \mathbf{Q}$ and $D \in \mathbf{S}^0 \setminus \mathbf{S}^{\mathrm{R}}$ we have

$$\deg_{\mathbf{G}_{\mathrm{reg}}}(C, \mathbf{S}^{\mathrm{R}}) = 0 \,. \tag{6.3}$$

Let us show another property of $\mathbf{S}^{\mathrm{R}}$.

Claim 6.1.1. $\mathbf{S}^{\mathrm{R}} \subseteq \mathbf{S}^{\mathrm{I}} \subseteq \mathbf{S}^{\mathrm{R}} \cup V(N_0)$. In particular, $\mathbf{S}^{\mathrm{R}} \subseteq \mathbf{S}^0$.

Proof of Claim 6.1.1. By the definition of $\mathbf{S}^{\mathrm{R}}$, we only need to show that $\mathbf{S}^{\mathrm{R}} \subseteq \mathbf{S}^{\mathrm{I}}$. So suppose there is a vertex $C \in \mathbf{S}^{\mathrm{R}} \setminus \mathbf{S}^{\mathrm{I}}$. By the definition of $\mathbf{S}^{\mathrm{R}}$ there is a non-trivial path R going from C to $\mathbf{S}^{\mathrm{I}} \setminus V(N_0)$, that alternates between $\mathbf{S}^{\mathrm{R}}$ and $\mathbf{Q}$, and has every second edge in N_0. Then, the matching $N_0' := N_0 \triangle E(R)$ covers more vertices of $\mathbf{S}^{\mathrm{I}}$ than N_0 does. Further, it is straightforward to check that the separator $\mathbf{Q}$ together with the matching N_0' satisfies the assertions of Theorem 2.4. This is a contradiction, as desired. □

Using a very similar alternating path argument we see the following.

Claim 6.1.2. If $CD \in N_0$ with $C \in \mathbf{Q}$ and $D \notin \mathbf{S}^{\mathrm{I}}$ then $\deg_{\mathbf{G}_{\mathrm{reg}}}(C, \mathbf{S}^{\mathrm{R}}) = 0$.

Using the factor-criticality of the components of $\mathbf{G}_{\mathrm{reg}} - \mathbf{Q}$ we extend N_0 to a matching N_1 as follows. For each component K of $\mathbf{G}_{\mathrm{reg}} - \mathbf{Q}$ which meets $V(N_0)$, we add a perfect matching of $K - V(N_0)$. Furthermore, for each non-singleton component K of $\mathbf{G}_{\mathrm{reg}} - \mathbf{Q}$ which does not meet $V(N_0)$, we add a matching which meets all but exactly one vertex of $\mathbf{L} \cap V(K)$. This is possible as by the definition of the class $\mathbf{LKSsmall}(n, k, \eta)$ we have that $\mathbf{G}_{\mathrm{reg}} - \mathbf{L}$ is independent, and so $\mathbf{L} \cap V(K) \neq \emptyset$. This choice of N_1 guarantees that

$$e_{\mathbf{G}_{\mathrm{reg}}}(\mathbf{V} \setminus V(N_1)) = 0 \,. \tag{6.4}$$

We set

$$M := \left\{ AB \in N_0 \,:\, A \in \mathbf{S}^{\mathrm{R}}, B \in \mathbf{Q} \right\} \,.$$

We have that

$$e_{\mathbf{G}_{\mathrm{reg}}}\big(\mathbf{V} \setminus V(N_1), V(M) \cap \mathbf{S}^{\mathrm{R}}\big) = 0 \,. \tag{6.5}$$

As $\mathbf{S}$ is an independent set in $\mathbf{G}_{\mathrm{reg}}$, we have that

$$\mathbf{Q}_M := V(M) \cap \mathbf{Q} \subseteq \mathbf{L} \,. \tag{6.6}$$

The matching M in $\mathbf{G}_{\mathrm{reg}}$ corresponds to an $(\varepsilon', \gamma^2, \mathfrak{c})$-semiregular matching $\mathcal{M}$ in the underlying graph G_{reg}, with $V_2(\mathcal{M}) \subseteq \bigcup \mathbf{Q}$ (recall that semiregular matchings have orientations on their edges). Likewise, we define $\mathcal{N}_1$ as the $(\varepsilon', \gamma^2, \mathfrak{c})$-regular matching corresponding to N_1. The $\mathcal{N}_1$-edges are oriented so that $V_1(\mathcal{N}_1) \cap \bigcup \mathbf{Q} = \emptyset$; this condition does not specify orientations of all the $\mathcal{N}_1$-edges and we orient the remaining ones in an arbitrary fashion. We write $S^{\mathrm{R}} := \bigcup \mathbf{S}^{\mathrm{R}}$.

Claim 6.1.3. $e_{G_\nabla}\big(L \setminus (\mathfrak{A} \cup V(\mathcal{M})), S^{\mathrm{R}}\big) = 0$.

Proof of Claim 6.1.3. We start by showing that for every cluster $C \in \mathbf{L} \setminus V(M)$ we have

$$\deg_{\mathbf{G}_{\mathrm{reg}}}(C, \mathbf{S}^{\mathrm{R}}) = 0 \,. \tag{6.7}$$

First, if $C \notin \mathbf{Q}$, then (6.7) is true since $\mathbf{S}^{\mathrm{R}} \subseteq \mathbf{S}^{\mathrm{I}}$ by Claim 6.1.1. So suppose that $C \in \mathbf{Q}$, and let $D \in V(\mathbf{G}_{\mathrm{reg}})$ be such that $DC \in N_0$. Now if $D \notin \mathbf{S}^{\mathrm{I}}$ then (6.7) follows from Claim 6.1.2. On the other hand, suppose $D \in \mathbf{S}^{\mathrm{I}} \subseteq \mathbf{S}^0$. As $C \notin V(M)$, we know that $D \notin \mathbf{S}^{\mathrm{R}}$, and thus, (6.7) follows from (6.3).

Now, by (6.7), G_{reg} has no edges between $L \setminus (\mathfrak{A} \cup V(\mathcal{M}))$ and S^{R}. Also, no such edges can be in G_{exp} or incident with $\mathfrak{A}$, since $\mathbf{S}^{\mathrm{R}} \subseteq \mathbf{S}^0$ by Claim 6.1.1. Finally, since $G \in \mathbf{LKSsmall}(n, k, \eta)$, there are no edges between $\mathbf{\Psi}$ and S. This proves the claim. □

We prepare ourselves for an application of Lemma 5.10. The numerical parameters of the lemma are $\Omega_{\triangleright L5.10}, \rho_{\triangleright L5.10}, \varepsilon_{\triangleright L5.10}$ and $\gamma_{\triangleright L5.10}$ as above. The input objects for the lemma are the graph $G_{\mathcal{D}}$ of order $n' \leqslant n$, the collection of $(\gamma k, \gamma)$-dense spots $\mathcal{D}$, the matching $\mathcal{M}$, the (νk)-ensemble $\mathcal{C}_{\triangleright L5.10} := \mathbf{S}^{\mathrm{R}} \setminus V(N_1)$, and the set $Y_{\triangleright L5.10} := L \cap \mathfrak{A}$. Note that Definition 4.7, item 5, implies that $\mathcal{D}$ absorbes $\mathcal{M}$. Further, (5.19) is satisfied by Definition 4.7, item 6.

The output of Lemma 5.10 is an $(\varepsilon, \beta, \pi\mathfrak{c})$-semiregular matching $\mathcal{M}'$ with the following properties.

(I) $|V(\mathcal{M}) \setminus V(\mathcal{M}')| < \varepsilon n' \leqslant \varepsilon n$.

(II) For each $(T, U) \in \mathcal{M}'$ there are sets $C \in \mathbf{S}^{\mathrm{R}}$ and $D = (A_D, B_D; E_D) \in \mathcal{D}$ such that $T \subseteq C \cap A_D$ and $U \subseteq ((L \cap \mathfrak{A}) \cup \bigcup \mathbf{Q}_M) \cap B_D$.

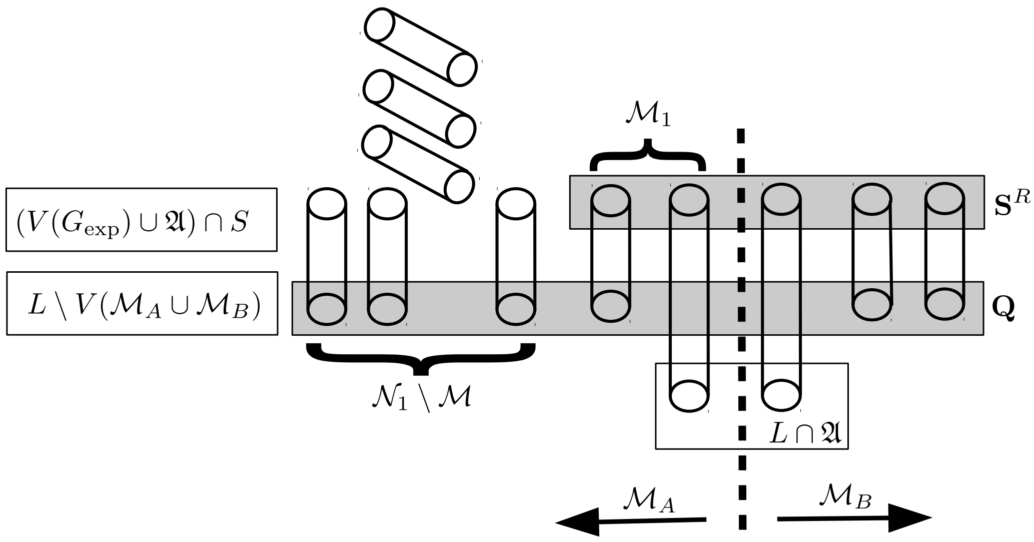

Figure 6.1: The situation in G after applying Lemma 5.10. The dotted line illustrates the separation as in (III).

(Indeed, to see this we use that $\mathcal{V}_1(\mathcal{M}) \subseteq \mathbf{S}^{\mathrm{R}}$ and that $V_2(\mathcal{M}) \subseteq \bigcup \mathbf{Q}_M$ by the definition of $\mathcal{M}$.)

(III) There is a partition of $\mathcal{M}'$ into $\mathcal{M}_1$ and $\mathcal{M}_B$ such that

$$e_{G_{\mathcal{D}}}\left(\left((S^{\mathrm{R}} \setminus V(\mathcal{N}_1)) \cup V_1(\mathcal{M}) \right) \setminus V_1(\mathcal{M}_1) \, , \, ((L \cap \mathfrak{A}) \cup V_2(\mathcal{M})) \setminus V_2(\mathcal{M}_B) \right) < \gamma k n' \, .$$

We claim that also

(IV) $V(\mathcal{M}') \cap V(\mathcal{N}_1 \setminus \mathcal{M}) = \emptyset$.

Indeed, let $(T, U) \in \mathcal{M}'$ be arbitrary. Then by (II) there is $C \in \mathbf{S}^{\mathrm{R}}$ such that $T \subseteq C$. By Claim 6.1.1, C is a singleton component of $\mathbf{G}_{\mathrm{reg}} - \mathbf{Q}$. In particular, if C is covered by N_1 then $C \in V(M)$. It follows that $T \cap V(\mathcal{N}_1 \setminus \mathcal{M}) = \emptyset$. In a similar spirit, the easy fact that $(Y \cup \bigcup \mathbf{Q}_M) \cap V(\mathcal{N}_1 \setminus \mathcal{M}) = \emptyset$ together with (II) gives $U \cap V(\mathcal{N}_1 \setminus \mathcal{M}) = \emptyset$. This establishes (IV).

Observe that (II) implies that $V_1(\mathcal{M}') \subseteq S^{\mathrm{R}}$, and so, by Claim 6.1.1 we know that

$$V_1(\mathcal{M}_B) \subseteq S^{\mathrm{R}} \subseteq \bigcup \mathbf{S}^{\mathrm{I}} \subseteq S^0. \tag{6.8}$$

Set

$$\mathcal{M}_A := (\mathcal{N}_1 \setminus \mathcal{M}) \cup \mathcal{M}_1 \, . \tag{6.9}$$

Then $\mathcal{M}_A$ is an $(\varepsilon, \beta, \pi\mathfrak{c})$-semiregular matching. Note that from now on, the sets $\mathbb{XA}, \mathbb{XB}$ and $\mathbb{XC}$ are defined. The situtation is illustrated in Figure 6.1. By (IV), we have $V(\mathcal{M}_A) \cap V(\mathcal{M}_B) = \emptyset$, as required for Lemma 6.1(a). Lemma 6.1(b) follows from (6.8). Observe that by (II), also Lemma 6.1(c) and Lemma 6.1(d) are satisfied.

We now turn to Lemma 6.1(e). First we prove some auxiliary statements.

Claim 6.1.4. We have $\mathbf{S}^0 \setminus V(N_1 \setminus M) \subseteq \mathbf{S}^{\mathrm{R}}$.

Proof of Claim 6.1.4. Let $C \in \mathbf{S}^0 \setminus V(N_1 \setminus M)$. Note that if $C \notin \mathbf{S}^{\mathrm{I}}$, then $C \in V(N_1)$. On the other hand, if $C \in \mathbf{S}^{\mathrm{I}}$, then we use Claim 6.1.1 to see that $C \in \mathbf{S}^{\mathrm{R}} \cup V(N_1)$. We deduce that in either case $C \in \mathbf{S}^{\mathrm{R}} \cup V(N_1)$. The choice of C implies that thus $C \in \mathbf{S}^{\mathrm{R}} \cup V(M)$. Now, if $C \in V(M)$, then $C \in \mathbf{S}^{\mathrm{R}}$ by (6.6) and by the definition of M. Thus $C \in \mathbf{S}^{\mathrm{R}}$ as desired. □

It will be convenient to work with a set $\bar{S}^0 \subseteq S^0$, $\bar{S}^0 := (S \cap \bigcup \mathbf{V}) \setminus V(G_{\mathrm{exp}}) = \bigcup \mathbf{S}^0$. Note that $\bar{S}^0$ is essentially the same as S^0; the vertices in $S^0 \setminus \bar{S}^0$ are isolated in G_∇ and thus have very little effect on our considerations.

By Claim 6.1.4, we have

$$\bar{S}^0 \setminus V(\mathcal{M}_A) \subseteq \big(\bigcup \mathbf{S}^0 \setminus V(N_1 \setminus M)\big) \setminus V(\mathcal{M}_A) \subseteq S^{\mathrm{R}} \setminus V(\mathcal{M}_A). \tag{6.10}$$

As every edge incident to $S^0 \setminus \bar{S}^0$ is uncaptured, we see that

$$\begin{aligned} E_{G_\nabla}\big(\mathbb{XA} \cap \mathfrak{A}, S^0 \setminus V(\mathcal{M}_A)\big) &\subseteq E_{G_\mathcal{D}}\big((L \cap \mathfrak{A}) \setminus V(\mathcal{M}_B), \bar{S}^0 \setminus V(\mathcal{M}_A)\big) \\ {\scriptstyle (\text{by } (6.10))} \quad &\subseteq E_{G_\mathcal{D}}\big(\, (L \cap \mathfrak{A}) \setminus V(\mathcal{M}_B)\,,\ S^{\mathrm{R}} \setminus V(\mathcal{M}_A)\,\big). \end{aligned} \tag{6.11}$$

We claim that furthermore

$$E_{G_{\mathrm{reg}}}\big(\mathbb{XA} \cap \bigcup \mathbf{V}, S^0 \setminus V(\mathcal{M}_A)\big) \subseteq E_{G_\mathcal{D}}\left(((L \cap \mathfrak{A}) \cup V_2(\mathcal{M})) \setminus V_2(\mathcal{M}_B), S^{\mathrm{R}} \setminus V(\mathcal{M}_A)\right). \tag{6.12}$$

Before proving (6.12), let us see that it implies Lemma 6.1(e). As $G \in \mathbf{LKSsmall}(n,k,\eta)$, there are no edges between $\mathbf{\Psi}$ and S. That means that any captured edge from $\mathbb{XA}$ to $S^0 \setminus V(\mathcal{M}_A)$ must start in $\mathfrak{A}$ or in $\bigcup \mathbf{V}$. Thus Lemma 6.1(e) follows by plugging (III) into (6.11) and (6.12).

Let us now prove (6.12). First, observe that by the definition of $\mathbb{XA}$ and by the definition of $\mathcal{M}$ (and M) we have

$$\mathbb{XA} \cap \bigcup \mathbf{V} \subseteq (V_2(\mathcal{M}) \setminus V_2(\mathcal{M}_B)) \cup (L \setminus (\mathfrak{A} \cup V(\mathcal{M}))) . \tag{6.13}$$

Further, by applying (6.10) and Claim 6.1.3 we get

$$E_{G_{\mathrm{reg}}}\big(L \setminus (\mathfrak{A} \cup V(\mathcal{M})), \bar{S}^0 \setminus V(\mathcal{M}_A)\big) = \emptyset . \tag{6.14}$$

Therefore, we obtain

$$\begin{aligned} E_{G_{\mathrm{reg}}}\big(\mathbb{XA} \cap \bigcup \mathbf{V}, S^0 \setminus V(\mathcal{M}_A)\big) &\subseteq E_{G_{\mathrm{reg}}}\big(\mathbb{XA} \cap \bigcup \mathbf{V}, \bar{S}^0 \setminus V(\mathcal{M}_A)\big) \\ {\scriptstyle (\text{by } (6.13))} \quad &\subseteq E_{G_{\mathrm{reg}}}\big(V_2(\mathcal{M}) \setminus V_2(\mathcal{M}_B), \bar{S}^0 \setminus V(\mathcal{M}_A)\big) \\ &\quad \cup E_{G_{\mathrm{reg}}}\big(L \setminus (\mathfrak{A} \cup V(\mathcal{M})), \bar{S}^0 \setminus V(\mathcal{M}_A)\big) \\ {\scriptstyle (\text{by } (6.10),\, (6.14))} \quad &\subseteq E_{G_{\mathrm{reg}}}\big(V_2(\mathcal{M}) \setminus V_2(\mathcal{M}_B), S^{\mathrm{R}} \setminus V(\mathcal{M}_A)\big) , \end{aligned}$$

as needed for (6.12).

In order to prove (f) we first observe that

$$\begin{aligned}V(\mathcal{N}_1)\setminus V(\mathcal{M}_A\cup\mathcal{M}_B) &\overset{(6.9)}{=} V(\mathcal{N}_1)\setminus V\big((\mathcal{N}_1\setminus\mathcal{M})\cup\mathcal{M}_1\cup\mathcal{M}_B\big)\\ &= (V(\mathcal{N}_1)\cap V(\mathcal{M}))\setminus V(\mathcal{M}_B\cup\mathcal{M}_1)\\ &\overset{(III)}{=} (V(\mathcal{N}_1)\cap V(\mathcal{M}))\setminus V(\mathcal{M}')\\ &= V(\mathcal{M})\setminus V(\mathcal{M}')\,. \end{aligned} \tag{6.15}$$

Now, we have

$$\begin{aligned} e_{G_{\mathrm{reg}}}(V(G)\setminus V(\mathcal{M}_A\cup\mathcal{M}_B)) &\leqslant e_{G_{\mathrm{reg}}}(V(G)\setminus V(\mathcal{N}_1)) + \sum_{v\in V(\mathcal{N}_1)\setminus V(\mathcal{M}_A\cup\mathcal{M}_B)} \deg_{G_\nabla}(v)\\ {\scriptstyle\text{(by (6.4) and (6.15))}} \quad &\leqslant \sum_{v\in V(\mathcal{M})\setminus V(\mathcal{M}')} \deg_{G_\nabla}(v)\\ &\leqslant |V(\mathcal{M})\setminus V(\mathcal{M}')|\Omega^* k\\ {\scriptstyle\text{(by (I))}} \quad &< \varepsilon\Omega^* kn\ , \end{aligned}$$

which shows (f).

Let us turn to proving (g). We have

$$\begin{aligned} e_{G_{\mathrm{reg}}}(V(G)\setminus V(\mathcal{M}_A\cup\mathcal{M}_B), V(\mathcal{N}_{\mathfrak{A}})) &\leqslant e_{G_{\mathrm{reg}}}(V(G)\setminus V(\mathcal{N}_1), V(\mathcal{N}_{\mathfrak{A}}))\\ &\quad + e_{G_{\mathrm{reg}}}(V(\mathcal{N}_1)\setminus V(\mathcal{M}_A\cup\mathcal{M}_B), V(G))\\ {\scriptstyle\text{(by (6.5))}} \quad &\leqslant 0 + |V(\mathcal{N}_1)\setminus V(\mathcal{M}_A\cup\mathcal{M}_B)|\Omega^* k\\ {\scriptstyle\text{(by (6.15), (I))}} \quad &\leqslant \varepsilon\Omega^* kn\ , \end{aligned}$$

as needed.

We have thus shown Lemma 6.1(a)–(g). It only remains to prove Lemma 6.1(h), which we will do in the remainder of this section.

We first collect several properties of $\mathbb{XA}$ and $\mathbb{XC}$. The definitions of $\mathbb{XC}$ and S^0 give

$$|\mathbb{XC}|(1+\eta)\frac{k}{2} \leqslant e_G\big(\mathbb{XC}, S^0\setminus V(\mathcal{M}_A\cup\mathcal{M}_B)\big) \leqslant |S^0\setminus V(\mathcal{M}_A\cup\mathcal{M}_B)|(1+\eta)k\ . \tag{6.16}$$

Each $v\in\mathbb{XC}$ has neighbours in S. Thus, by 2. of Definition 2.6 we have

$$\deg_G(v) = \lceil (1+\eta)k\rceil \tag{6.17}$$

for each $v\in\mathbb{XC}$. Further, each vertex of $\mathbb{XC}$ has degree at least $(1+\eta)\frac{k}{2}$ into S, and so,

$$e_G(S,\mathbb{XC}) \geqslant |\mathbb{XC}| \left\lceil (1+\eta)\frac{k}{2}\right\rceil\ . \tag{6.18}$$

Consequently (using the elementary inequality $\lceil a\rceil - \lceil \frac{a}{2}\rceil \leqslant \frac{a}{2}$),

$$
\begin{aligned}
e_G\big(\mathbb{XA},\mathbb{XC}\big) &\overset{(6.17)}{\leqslant} |\mathbb{XC}|\lceil(1+\eta)k\rceil - e_G(S,\mathbb{XC}) \\
&\overset{(6.18)}{\leqslant} |\mathbb{XC}|(1+\eta)\frac{k}{2} &(6.19)\\
&\overset{(6.16)}{\leqslant} |S^0 \setminus V(\mathcal{M}_A\cup\mathcal{M}_B)|(1+\eta)k\ . &(6.20)
\end{aligned}
$$

Let $\mathcal{M}_{\mathrm{good}}$ be defined as in Lemma 6.1(h), that is, $\mathcal{M}_{\mathrm{good}} := \{(A,B)\in\mathcal{M}_A : A\cup B\subseteq \mathbb{XA}\}$. Note that (6.8) implies that $A\subseteq S$ for every $(A,B)\in\mathcal{M}_B$. Thus by the definition of $\mathbb{XA}$,

$$
\text{if } (A,B)\in\mathcal{M}_A\cup\mathcal{M}_B \text{ with } A\cup B\subseteq L \text{ then } (A,B)\in\mathcal{M}_{\mathrm{good}}. \qquad (6.21)
$$

We will now show the first part of Lemma 6.1(h), that is, we show that each $\mathcal{M}_{\mathrm{good}}$-edge is an edge of $\mathbf{G}_{\mathrm{reg}}$. Indeed, by (II), we have that $V_1(\mathcal{M}_1)\subseteq S$, so as $\mathbb{XA}\cap S=\emptyset$, it follows that $\mathcal{M}_1\cap\mathcal{M}_{\mathrm{good}}=\emptyset$. Thus $\mathcal{M}_{\mathrm{good}}\subseteq\mathcal{N}_1$. As $\mathcal{N}_1$ corresponds to a matching in $\mathbf{G}_{\mathrm{reg}}$, all is as desired.

Finally, let us assume that neither **(K1)** nor **(K2)** are fulfilled. After five preliminary observations (Claim 6.1.5–Claim 6.1.9), we will derive a contradiction from this assumption.

Claim 6.1.5. We have $|S\cap V(\mathcal{M}_A)|\leqslant|\mathbb{XA}\cap V(\mathcal{M}_A)|$.

Proof of Claim 6.1.5. To see this, recall that each $\mathcal{M}_A$-vertex $U\in\mathcal{V}(\mathcal{M}_A)$ is either contained in S, or in L. Further, if $U\subseteq S$ then its partner in $\mathcal{M}_A$ must be in L, as S is independent. Now, the claim follows after noticing that $L\cap V(\mathcal{M}_A)=\mathbb{XA}\cap V(\mathcal{M}_A)$. □

Claim 6.1.6. We have $|S\setminus V(\mathcal{M}_A\cup\mathcal{M}_B)|+2\eta n<|\mathbb{XA}\setminus V(\mathcal{M}_A)|+\eta n/3$.

Proof of Claim 6.1.6. As $G\in\mathbf{LKS}(n,k,\eta)$, we have $|S|+2\eta n\leqslant|L|$. Therefore,

$$
\begin{aligned}
|S\setminus V(\mathcal{M}_A\cup\mathcal{M}_B)|+2\eta n &\leqslant |L\setminus V(\mathcal{M}_A\cup\mathcal{M}_B)| + \sum_{\substack{(A,B)\in\mathcal{M}_A\cup\mathcal{M}_B\\ A\cup B\subseteq L}}|A\cup B| \\
&\overset{(6.21)}{=}|\mathbb{XA}\setminus V(\mathcal{M}_A)|+|V(\mathcal{M}_{\mathrm{good}})| \\
&\overset{\neg\mathbf{(K2)}}{<}|\mathbb{XA}\setminus V(\mathcal{M}_A)|+\eta n/3\ .
\end{aligned}
$$

□

Claim 6.1.7. We have $e_{G_\nabla}\big(\mathbb{XA}\cap(\mathfrak{A}\cup V(\mathcal{M})),S^{\mathrm{R}}\setminus V(\mathcal{M}_A)\big)<\eta kn/2$.

Proof of Claim 6.1.7. As

$$
\begin{aligned}
\mathbb{XA}\cap(\mathfrak{A}\cup V(\mathcal{M})) &\subseteq ((L\cap\mathfrak{A})\cup V_2(\mathcal{M}))\setminus V_2(\mathcal{M}_B) \quad \text{and}\\
S^{\mathrm{R}}\setminus V(\mathcal{M}_A) &\subseteq \big((S^{\mathrm{R}}\setminus V(\mathcal{N}_1))\cup V_1(\mathcal{M})\big)\setminus V_1(\mathcal{M}_1)\ ,
\end{aligned}
$$

we get from (III) that

$$e_{G_{\mathcal{D}}}\big(\mathbb{XA}\cap(\mathfrak{A}\cup V(\mathcal{M})),S^{\mathrm{R}}\setminus V(\mathcal{M}_A)\big)\leqslant\gamma kn\;. \tag{6.22}$$

Observe now that both sets $\mathbb{XA}\cap(\mathfrak{A}\cup V(\mathcal{M}))$ and $S^{\mathrm{R}}\setminus V(\mathcal{M}_A)$ avoid $\mathbf{\Psi}$. Further, no edges between them belong to G_{exp}, because Claim 6.1.1 implies that $S^{\mathrm{R}}\setminus V(\mathcal{M}_A)\subseteq S^0\subseteq V(G)\setminus V(G_{\mathrm{exp}})$. Therefore, we can pass from $G_{\mathcal{D}}$ to G_{∇} in (6.22) to get

$$e_{G_{\nabla}}\big(\mathbb{XA}\cap(\mathfrak{A}\cup V(\mathcal{M})),S^{\mathrm{R}}\setminus V(\mathcal{M}_A)\big)\leqslant\gamma kn<\eta kn/2\;.$$

□

Claim 6.1.8. We have $S\setminus(S^{\mathrm{R}}\cup V(\mathcal{M}_A))\subseteq S\setminus(\bar{S}^0\cup V(\mathcal{M}_A\cup\mathcal{M}_B))$.

Proof of Claim 6.1.8. The claim follows directly from the following two inclusions.

$$S^{\mathrm{R}}\cup V(\mathcal{M}_A)\supseteq S\cap V(\mathcal{M}_A\cup\mathcal{M}_B)\;,\text{ and} \tag{6.23}$$

$$S^{\mathrm{R}}\cup V(\mathcal{M}_A)\supseteq\bar{S}^0\;. \tag{6.24}$$

Now, (6.23) is trivial, as by (II) we have that $S^{\mathrm{R}}\supseteq S\cap V(\mathcal{M}_B)$. To see (6.24), it suffices by (6.9) to prove that $V(N_1\setminus M)\cup\mathbf{S}^{\mathrm{R}}\supseteq\mathbf{S}^0$. This is however the subject of Claim 6.1.4. □

Next, we bound $e_{G_{\nabla}}\big(\mathbb{XA},S\big)$.

Claim 6.1.9. We have

$$e_{G_{\nabla}}\big(\mathbb{XA},S\big)\leqslant|S\cap V(\mathcal{M}_A)|(1+\eta)k+|S\setminus(S^0\cup V(\mathcal{M}_A\cup\mathcal{M}_B))|(1+\eta)k+\frac{1}{2}\eta kn\;.$$

Proof of Claim 6.1.9. We have

$$\begin{aligned}e_{G_{\nabla}}\big(\mathbb{XA},S\big)=\;&e_{G_{\nabla}}\big(\mathbb{XA},S\cap V(\mathcal{M}_A)\big)\\&+e_{G_{\nabla}}\big(\mathbb{XA},S\setminus(S^{\mathrm{R}}\cup V(\mathcal{M}_A))\big)\\&+e_{G_{\nabla}}\big(\mathbb{XA}\setminus(\mathfrak{A}\cup V(\mathcal{M})),S^{\mathrm{R}}\setminus V(\mathcal{M}_A)\big)\\&+e_{G_{\nabla}}\big(\mathbb{XA}\cap(\mathfrak{A}\cup V(\mathcal{M})),S^{\mathrm{R}}\setminus V(\mathcal{M}_A)\big)\;.\end{aligned}$$

To bound the first term we use that the vertices in $S\cap V(\mathcal{M}_A)$ each have degree at most $(1+\eta)k$, and thus obtain $e_{G_{\nabla}}(\mathbb{XA},S\cap V(\mathcal{M}_A))\leqslant|S\cap V(\mathcal{M}_A)|(1+\eta)k$. To bound the second term, we again use a bound on degree of vertices of $S\setminus\big((S^{\mathrm{R}}\cup V(\mathcal{M}_A))\cup(S^0\setminus\bar{S}^0)\big)$, together with Claim 6.1.8. The third term is zero by Claim 6.1.3. The fourth term can be bounded by Claim 6.1.7. □

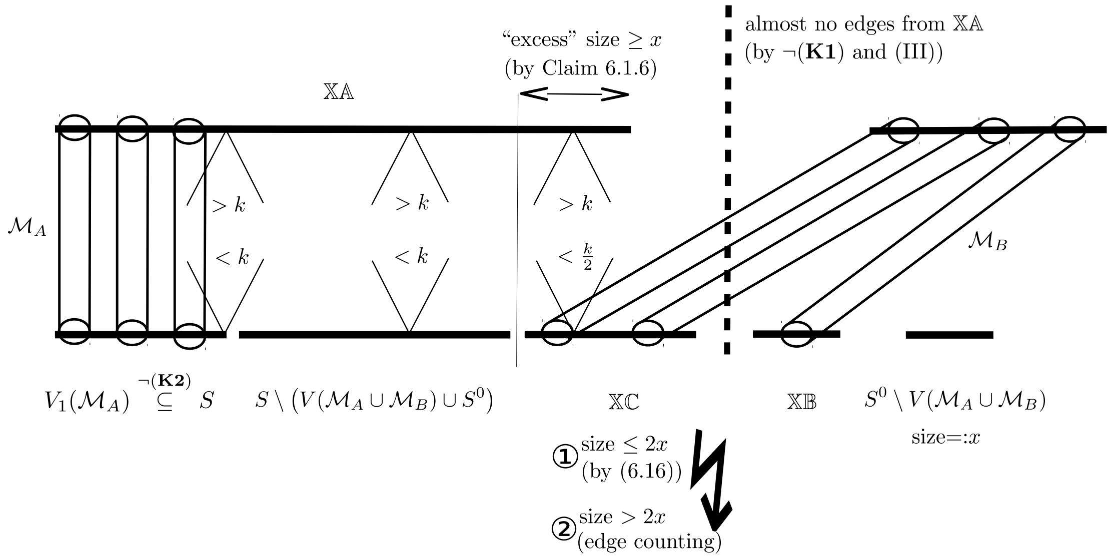

Figure 6.2: A simplified computation showing that $\neg$(**K1**), $\neg$(**K2**) leads to a contradiction. Denoting by x the size of $S^0 \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)$ we get on one hand ① $|\mathbb{XC}| \leqslant 2x$. On the other hand, each vertex of $\mathbb{XA}$ emanates $\gtrsim k$ edges which are absorbed by the sets $V_1(\mathcal{M}_A)$, $S \setminus (V(\mathcal{M}_A \cup \mathcal{M}_B) \cup S^0)$, and $\mathbb{XC}$. The vertices of $V_1(\mathcal{M}_A)$ and $S \setminus (V(\mathcal{M}_A \cup \mathcal{M}_B) \cup S^0)$ can absorb $\lesssim k$ edges. The vertices of $\mathbb{XC}$ receive $\lesssim \frac{k}{2}$ edges of $\mathbb{XA}$ by (6.19). This leads to ② doubling the size of the "excess" vertices of $\mathbb{XA}$, i.e., $|\mathbb{XC}| > 2x$.

A relatively short double counting below will lead to the final contradiction. The idea behind this computation is given in Figure 6.2.

$$
\begin{aligned}
|\mathbb{XA}|(1+\eta)k &\leqslant \sum_{v\in\mathbb{XA}} \deg_G(v) \\
&\leqslant \sum_{v\in\mathbb{XA}} \deg_{G_\nabla}(v) + 2\big(e(G)-e(G_\nabla)\big) \\
&\leqslant 2e_{G_\nabla}(\mathbb{XA}) + e_{G_\nabla}(\mathbb{XA},\mathbb{XB}) + e_{G_\nabla}\big(\mathbb{XA},\mathbb{XC}\big) \\
&\qquad + e_{G_\nabla}\big(\mathbb{XA},S\big) + \frac{\eta kn}{3} \\
\text{(by } \neg\textbf{(K1)}\text{, (6.20), C6.1.9)}\quad &\leqslant \frac{7}{6}\eta kn + \big|S^0 \setminus V(\mathcal{M}_A\cup\mathcal{M}_B)\big|(1+\eta)k \\
&\qquad + |S\cap V(\mathcal{M}_A)|(1+\eta)k \\
&\qquad + |S\setminus(S^0\cup V(\mathcal{M}_A\cup\mathcal{M}_B))|(1+\eta)k \\
\text{(by C6.1.5)}\quad &\leqslant \frac{7}{6}\eta kn + |S\setminus V(\mathcal{M}_A\cup\mathcal{M}_B)|(1+\eta)k \\
&\qquad + |\mathbb{XA}\cap V(\mathcal{M}_A)|(1+\eta)k \\
\text{(by C6.1.6)}\quad &\leqslant \frac{7}{6}\eta kn + \big(|\mathbb{XA}\setminus V(\mathcal{M}_A)| - \frac{5}{3}\eta n\big)(1+\eta)k \\
&\qquad + |\mathbb{XA}\cap V(\mathcal{M}_A)|(1+\eta)k \\
&< |\mathbb{XA}|(1+\eta)k - \frac{1}{2}\eta kn\ ,
\end{aligned}
\tag{6.25}
$$

a contradiction. This finishes the proof of Lemma 6.1. □

6.2 The role of Lemma 5.10 in the proof of Lemma 6.1

Let us explain the role of Lemma 5.10 in our proof of Lemma 6.1. First, let us attempt to use just the sparse decomposition ∇ to embed a tree $T \in \mathbf{trees}(k)$ in $G \in \mathbf{LKS}(n,k,\eta)$. We will eventually see that this is impossible and that we need to enhance ∇ by a semiregular matching (provided by Lemma 5.10).

We wish to find two sets $\mathbb{VA}$ and $\mathbb{VB}$ which are suitable for embedding the cut vertices W_A and W_B of a τk-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of T, respectively. In this sketch we just focus on finding $\mathbb{VA}$; the ideas behind finding a set suitable set $\mathbb{VB}$ are similar.

To accommodate all the shrubs from $\mathcal{S}_A$ — which might contain up to k vertices in total — we need $\mathbb{VA}$ to have degree at least $\sum_{T^*\in\mathcal{S}_A} v(T^*)$ into a suitable set of vertices we reserve for these shrubs. (The neighbourhood of a possible image of a vertex from W_A has to allow space for its children and for everything blocked by shrubs from $\mathcal{S}_A$ embedded earlier.)

Our methods of embedding in Section 8 determine which sets we find 'suitable' for $\mathcal{S}_A$: these are the large vertices $\mathbb{L}_{\eta,k}(G)$, the vertices of the nowhere-dense graph G_{exp}, the avoiding set $\mathfrak{A}$, and any matching consisting of regular pairs. This motivates us to look for a semiregular matching $\mathcal{M}$ which covers as much as possible of the set $S^0 := \mathbb{S}_{\eta,k}(G) \setminus (V(G_{\mathrm{exp}}) \cup \mathfrak{A})$ which consists of those vertices not utilizable by any other of the methods above. As a next step one would prove that there is a set $\mathbb{VA}$ with

$$\deg^{\min}\left(\mathbb{VA}, V(G)\setminus(S^0\setminus V(\mathcal{M}))\right)\gtrsim k\;.$$

In the dense setting [PS12], where the structure of G is determined by $\mathbf{G}_{\mathrm{reg}}$, and where $S^0=\mathbb{S}_{\eta,k}(G)$, such a matching $\mathcal{M}$ can be found inside $\mathbf{G}_{\mathrm{reg}}$ using the Gallai-Edmonds Matching Theorem. But here, just working with $\mathbf{G}_{\mathrm{reg}}$ is not enough for finding a suitable semiregular matching as the following example shows.

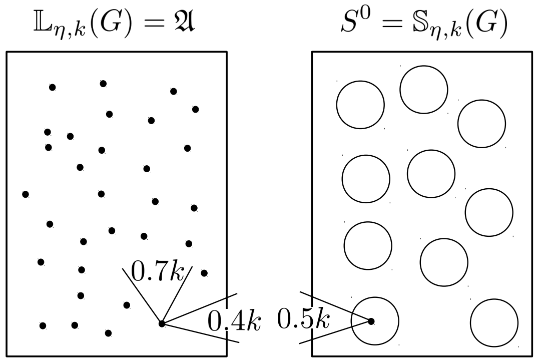

Figure 6.3: An example of a graph $G\in\mathbf{LKS}(n,k,\eta:=\frac{1}{10})$ in which $\mathbf{G}_{\mathrm{reg}}$ is empty, and yet there is no candidate set for $\mathbb{VA}$ of vertices which have degrees at least k outside the set S^0.

Figure 6.3 shows a graph G with $\mathbb{L}_{\eta,k}(G)\subseteq\mathfrak{A}$, and where the vertices in $S^0=\mathbb{S}_{\eta,k}(G)$ form clusters which do not induce any dense regular pairs. Each $\mathbb{L}_{\eta,k}(G)$-vertex sends $0.7k$ edges to $\mathbb{L}_{\eta,k}(G)$ and $0.4k$ edges to $\mathbb{S}_{\eta,k}(G)$, and each $\mathbb{S}_{\eta,k}(G)$-vertex receives $0.5k$ edges from $\mathbb{L}_{\eta,k}(G)$. The edges between $\mathbb{L}_{\eta,k}(G)$ and $\mathbb{S}_{\eta,k}(G)$ are contained in $\mathcal{D}$. No vertex has degree $\gtrsim k$ outside S^0, and the cluster graph $\mathbf{G}_{\mathrm{reg}}$ contains no matching.

However in this situation we can still find a large semiregular matching $\mathcal{M}$ between $\mathbb{L}_{\eta,k}(G)$ and $\mathbb{S}_{\eta,k}(G)$, by regularizing the crossing dense spots $\mathcal{D}$. (In general, obtaining a semiregular matching is of course more complicated.)

The example relates to Lemma 6.1 by setting $\mathbb{XA}:=\mathbb{VA}$, and $\mathcal{M}_A:=\mathcal{M}$. Indeed, (e) of Lemma 6.1 says that $\mathbb{XA}$-vertices send almost no edges to $S^0\setminus V(\mathcal{M}_A)$, and thus (since $\mathbb{XA}\subseteq\mathbb{L}_{\eta,k}(G)$), they have degree $\gtrsim k$ outside $S^0\setminus V(\mathcal{M}_A)$.

7 Configurations

In this section we introduce ten configurations — called ($\diamond$**1**)–($\diamond$**10**) — which may be found in a graph $G\in\mathbf{LKS}(n,k,\eta)$. We will be able to infer from the main results of this section (Lemmas 7.32–7.34) and from other structural results of this paper that each graph $G \in \mathbf{LKS}(n,k,\eta)$ contains at least one of these configurations. Lemmas 7.32–7.34 are based on the structure provided by Lemma 6.1 which itself is in a sense the most descriptive result of the structure of graphs from $\mathbf{LKS}(n,k,\eta)$. However, the structure given by Lemma 6.1 needs some burnishing. It will turn out in Section 8 that each of the configurations ($\diamond$**1**)–($\diamond$**10**) is suitable for the embedding of any tree from $\mathbf{trees}(k)$ as required for Theorem 1.3.

This section is organized as follows. In Section 7.1 we introduce an auxiliary notion of shadows and prove some simple properties of them. Section 7.2 introduces randomized splitting of the vertex set of an input graph. In Section 7.3 we define certain cleaned versions of the sets $\mathbb{XA}$ and $\mathbb{XB}$, and introduce other building blocks for the configurations ($\diamond$**1**)–($\diamond$**10**). In Section 7.4 we state some preliminary definitions and introduce the configurations ($\diamond$**1**)–($\diamond$**10**). In Section 7.6 we prove certain "cleaning lemmas". The main results are then stated and proved in Section 7.7. The results of Section 7.7 rely on the auxiliary lemmas of Section 7.2 and 7.6.

7.1 Shadows

We will find it convenient to work with the notion of a shadow. Given a graph H, a set $U \subseteq V(H)$, and a number ℓ we define inductively

$$\mathbf{shadow}_H^{(0)}(U,\ell) := U, \text{ and}$$
$$\mathbf{shadow}_H^{(i)}(U,\ell) := \{v \in V(H) \,:\, \deg_H(v, \mathbf{shadow}_H^{(i-1)}(U,\ell)) > \ell\} \text{ for } i \geqslant 1.$$

We abbreviate $\mathbf{shadow}_H^{(1)}(U,\ell)$ as $\mathbf{shadow}_H(U,\ell)$. Further, the graph H is omitted from the subscript if it is clear from the context. Note that the shadow of a set U might intersect U.

Below, we state two facts which bound the size of a shadow of a given set. Fact 7.1 gives a bound in general graphs of bounded maximum degree and Fact 7.2 gives a stronger bound for nowhere-dense graphs.

Fact 7.1. *Suppose H is a graph with $\deg^{\max}(H) \leqslant \Omega k$. Then for each $\alpha > 0, i \in \{0,1,\ldots\}$, and each set $U \subseteq V(H)$, we have*

$$|\mathbf{shadow}^{(i)}(U,\alpha k)| \leqslant \left(\frac{\Omega}{\alpha}\right)^i |U| \,.$$

Proof. Proceeding by induction on i it suffices to show that $|\mathbf{shadow}^{(1)}(U,\alpha k)| \leqslant \Omega|U|/\alpha$. To this end, observe that U sends out at most $\Omega k|U|$ edges while each vertex of $\mathbf{shadow}(U,\alpha k)$ receives at least αk edges from U. $\square$

Fact 7.2. *Let $\alpha, \gamma, Q > 0$ be three numbers such that $Q \geqslant 1$ and $16Q \leqslant \frac{\alpha}{\gamma}$. Suppose that H is a $(\gamma k, \gamma)$-nowhere-dense graph, and let $U \subseteq V(H)$ with $|U| \leqslant Qk$. Then we have*

$$|\mathbf{shadow}(U, \alpha k)| \leqslant \frac{16Q^2\gamma}{\alpha} k.$$

Proof. Suppose otherwise and let $W \subseteq \mathbf{shadow}(U, \alpha k)$ be of size $|W| = \frac{16Q^2\gamma}{\alpha} k \leqslant Qk$. Then $e_H(U \cup W) \geqslant \frac{1}{2} \sum_{v \in W} \deg_H(v, U) \geqslant 8\gamma Q^2 k^2$. Thus $H[U \cup W]$ has average degree at least

$$\frac{2e_H(U \cup W)}{|U| + |W|} \geqslant 8\gamma Qk \ ,$$

and therefore, by a well-known fact, contains a subgraph H' of minimum degree at least $4\gamma Qk$. Taking a maximal cut (A, B) in H', it is easy to see that $H'[A, B]$ has minimum degree at least $2\gamma Qk \geqslant \gamma k$. Further, $H'[A, B]$ has density at least $\frac{|A| \cdot 2\gamma Qk}{|A||B|} \geqslant \gamma$, contradicting the fact that H is $(\gamma k, \gamma)$-nowhere-dense. $\square$

7.2 Random splitting

Suppose a graph G (together with its bounded decomposition[xvii]) is given. In this section we split its vertex set in several classes in a given ratio. It is important that most vertices will have their degrees split obeying approximately this ratio. The corresponding statement is given in Lemma 7.3. It will be used to split the vertices of the host graph $G = G_{\triangleright T1.3}$ according to which part of the tree $T = T_{\triangleright T1.3} \in \mathbf{trees}(k)$ they will host. More precisely, suppose that $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ is an ℓ-fine partition of T (for a suitable number ℓ). Let t_{int} and t_{end} be the total sizes of the internal and end shrubs, respectively. We then want to partition $V(G)$ into three sets $\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2$ (which correspond to $\mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{U}_3$ in Lemma 7.3) in the ratio (approximately)

$$(|W_A| + |W_B|) \ : \ t_{\mathrm{int}} \ : \ t_{\mathrm{end}}$$

so that degrees of the vertices of $V(G)$ are split proportionally. This will allow us to embed the vertices of $W_A \cup W_B$ in $\mathfrak{P}_0$, the internal shrubs in $\mathfrak{P}_1$, and end shrubs in $\mathfrak{P}_2$. Actually, as our embedding procedure is more complex, we not only require the degrees to be split proportionally, but also to partition proportionally the objects from the bounded decomposition. In Section 7.5 we give some reasons why such a random splitting needs to be used.

Lemma 7.3 below is formulated in an abstract setting, without any reference to the tree T, and with a general number of classes in the partition.

[xvii] Note that in general we apply a *sparse* decomposition (as opposed to a *bounded* decomposition) on the graph $G = G_{\triangleright T1.3}$, cf. Lemma 4.14. However, it turns out that when the vertices $\mathbf{\Psi}$ of huge degrees form a substantial part of G (which is when the need of transition from bounded to sparse decomposition arises), the result of this section is not needed.

Lemma 7.3. *For each $p \in \mathbb{N}$ and $a > 0$ there exists $k_0 > 0$ such that for each $k > k_0$ we have the following.*

Suppose G is a graph of order $n \geqslant k_0$ and $\deg^{\max}(G) \leqslant \Omega^* k$ with its $(k, \Lambda, \gamma, \varepsilon, k^{-0.05}, \rho)$-bounded decomposition $(\mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$. As usual, we write G_∇ for the subgraph captured by $(\mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$, and $G_\mathcal{D}$ for the spanning subgraph of G consisting of the edges in $\mathcal{D}$. Let $\mathcal{M}$ be an $(\varepsilon, d, k^{0.95})$-semiregular matching in G, and $\mathfrak{U}_1, \ldots, \mathfrak{U}_p$ be subsets of $V(G)$. Suppose that $\Omega^* \geqslant 1$ and $\Omega^*/\gamma < k^{0.1}$.

Suppose that $\mathfrak{q}_1, \ldots, \mathfrak{q}_p \in \{0\} \cup [a, 1]$ are reals with $\sum \mathfrak{q}_i \leqslant 1$. Then there exists a partition $\mathfrak{Q}_1 \cup \ldots \cup \mathfrak{Q}_p = V(G)$, and sets $\bar{V} \subseteq V(G)$, $\bar{\mathcal{V}} \subseteq \mathcal{V}(\mathcal{M})$, $\bar{\mathbf{V}} \subseteq \mathbf{V}$ with the following properties.

(1) $|\bar{V}| \leqslant \exp(-k^{0.1})n$, $|\bigcup \bar{\mathcal{V}}| \leqslant \exp(-k^{0.1})n$, $|\bigcup \bar{\mathbf{V}}| < \exp(-k^{0.1})n$.

(2) For each $i \in [p]$ and each $C \in \mathbf{V} \setminus \bar{\mathbf{V}}$ we have $|C \cap \mathfrak{Q}_i| \geqslant \mathfrak{q}_i |\mathfrak{Q}_i| - k^{0.9}$.

(3) For each $i \in [p]$ and each $C \in \mathcal{V}(\mathcal{M}) \setminus \bar{\mathcal{V}}$ we have $|C \cap \mathfrak{Q}_i| \geqslant \mathfrak{q}_i |\mathfrak{Q}_i| - k^{0.9}$.

(4) For each $i \in [p]$, $D = (U, W; F) \in \mathcal{D}$ and $\deg^{\min}{}_D(U \setminus \bar{V}, W \cap \mathfrak{Q}_i) \geqslant \mathfrak{q}_i \gamma k - k^{0.9}$.

(5) For each $i, j \in [p]$ we have $|\mathfrak{Q}_i \cap \mathfrak{U}_j| \geqslant \mathfrak{q}_i |\mathfrak{U}_j| - n^{0.9}$.

(6) For each $i \in [p]$ each $J \subseteq [p]$ and each $v \in V(G) \setminus \bar{V}$ we have

$$\deg_H(v, \mathfrak{Q}_i \cap \mathfrak{U}_J) \geqslant \mathfrak{q}_i \deg_H(v, \mathfrak{U}_J) - 2^{-p} k^{0.9} ,$$

for each of the graphs $H \in \{G, G_\nabla, G_{\mathrm{exp}}, G_\mathcal{D}, G_\nabla \cup G_\mathcal{D}\}$. Here, $\mathfrak{U}_J$ is the set of those vertices which are present in all $\mathfrak{U}_j$ $(j \in J)$, and absent from all $\mathfrak{U}_j$ $(j \in [p] \setminus J)$.

(7) For each $i, i', j, j' \in [p]$ $(j \neq j')$, we have

$$e_H(\mathfrak{Q}_i \cap \mathfrak{U}_j, \mathfrak{Q}_{i'} \cap \mathfrak{U}_{j'}) \geqslant \mathfrak{q}_i \mathfrak{q}_{i'} e_H(\mathfrak{U}_j, \mathfrak{U}_{j'}) - k^{0.6} n^{0.6} ,$$
$$e_H(\mathfrak{Q}_i \cap \mathfrak{U}_j, \mathfrak{Q}_{i'} \cap \mathfrak{U}_j) \geqslant \mathfrak{q}_i \mathfrak{q}_{i'} e(H[\mathfrak{U}_j]) - k^{0.6} n^{0.6} \qquad \text{if } i \neq i', \text{ and}$$
$$e(H[\mathfrak{Q}_i \cap \mathfrak{U}_j]) \geqslant \mathfrak{q}_i^2 e(H[\mathfrak{U}_j]) - k^{0.6} n^{0.6} .$$

for each of the graphs $H \in \{G, G_\nabla, G_{\mathrm{exp}}, G_\mathcal{D}, G_\nabla \cup G_\mathcal{D}\}$.

(8) For each $i \in [p]$ if $\mathfrak{q}_i = 0$ then $\mathfrak{Q}_i = \emptyset$.

Proof. We can assume that $\sum \mathfrak{q}_i = 1$ as all bounds in (2)–(7) are lower bounds. Assume that k is large enough. We assign each vertex $v \in V(G)$ to one of the sets $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_p$ at random with respective probabilities $\mathfrak{q}_1, \ldots, \mathfrak{q}_p$. Let $\bar{V}_1$ and $\bar{V}_2$ be the vertices which do not satisfy (4) and (6), respectively. Let $\bar{\mathcal{V}}$ be the sets of $\mathcal{V}(\mathcal{M})$ which do not satisfy (3), and let $\bar{\mathbf{V}}$ be the clusters of $\mathbf{V}$ which do not satisfy (2). Setting $\bar{V} := \bar{V}_1 \cup \bar{V}_2$, we need to show that (1), (5) and (7) are fulfilled simultaneously with positive probability. Using the union bound, it suffices to show that each of the properties (1), (5) and (7) is violated with probability at most 0.2. The probability of each of these three properties can be controlled in a straightforward way by the Chernoff bound. We only give such a bound (with error probability at most 0.1) on the size of the set $\bar{V}_1$ (appearing in (1)), which is the most difficult one to control.

For $i \in [p]$, let $\bar{V}_{1,i}$ be the set of vertices v for which there exists $D = (U, W; F) \in \mathcal{D}$, $U \ni v$, such that $\deg_D(v, W \cap \mathfrak{Q}_i) < \mathfrak{q}_i \gamma k - k^{0.9}$. We aim to show that for each $i \in [p]$ the probability that $|\bar{V}_{1,i}| > \exp(-k^{0.2})n$ is at most $\frac{1}{10p}$. Indeed, summing such an error bound together with similar bounds for other properties will allow us to conclude the statement. This will in turn follow from the Markov Inequality provided that we show that

$$\mathbf{E}[|\bar{V}_{1,i}|] \leqslant \frac{1}{10p} \cdot \exp(-k^{0.2})n \ . \tag{7.1}$$

Indeed, let us consider an arbitrary vertex $v \in V(G)$. By Fact 4.3, v is contained in at most Ω^*/γ dense spots of $\mathcal{D}$. For a fixed dense spot $D = (U, W; F) \in \mathcal{D}$ with $v \in U$ let us bound the probability of the event $\mathcal{E}_{v,i,D}$ that $\deg_D(v, W \cap \mathfrak{Q}_i) < \mathfrak{q}_i \gamma k - k^{0.9}$. To this end, fix a set $N \subseteq W \cap \mathrm{N}_D(v)$ of size exactly γk before the random assignment is performed. Now, elements of $V(G)$ are distributed randomly into the sets $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_p$. In particular, the number $|\mathfrak{Q}_i \cap N|$ has binomial distribution with parameters γk and $\mathfrak{q}_i$. Using the Chernoff bound, we get

$$\mathbf{P}[\mathcal{E}_{v,i,D}] \leqslant \mathbf{P}\left[|\mathfrak{Q}_i \cap N| < \mathfrak{q}_i \gamma k - k^{0.9}\right] \leqslant \exp(-k^{0.3}) \ .$$

Thus, it follows by summing the tail over at most $\Omega^*/\gamma \leqslant k^{0.1}$ dense spots containing v, that

$$\mathbf{P}[v \in \bar{V}_{1,i}] \leqslant k^{0.1} \cdot \exp(-k^{0.3}) \ . \tag{7.2}$$

Now, (7.1) follows by linearity of expectation. □

Lemma 7.3 is utilized for the purpose of our proof of Theorem 1.3 using the notion of proportional partition introduced in Definition 7.6 below.

7.3 Common settings

Throughout Section 7 and Section 8 we shall be working with the setting that comes from Lemma 6.1. In order to keep statements of the subsequent lemmas reasonably short we introduce the following setting.

Setting 7.4. *We assume that the constants* $\Lambda, \Omega^*, \Omega^{**}, k_0$ *and* $\alpha_\odot, \gamma, \varepsilon, \varepsilon', \varepsilon_\odot, \eta, \pi, \rho, \tau, d$

satisfy

$$\eta \gg \frac{1}{\Omega^*} \gg \frac{1}{\Omega^{**}} \gg \rho \gg \gamma \gg d \geqslant \frac{1}{\Lambda} \geqslant \varepsilon \geqslant \pi \geqslant \varepsilon_\odot \geqslant \alpha_\odot \geqslant \varepsilon' \geqslant \nu \gg \tau \gg \frac{1}{k_0} > 0 \,, \tag{7.3}$$

and that $k \geqslant k_0$*. Here, by writing* $c > a_1 \gg a_2 \gg \ldots \gg a_\ell > 0$ *we mean that there exist non-decreasing functions* $f_i : (0,c)^i \to (0,c)$ $(i = 1, \ldots, \ell-1)$ *such that for each* $i \in [\ell-1]$ *we have* $a_{i+1} < f_i(a_1, \ldots, a_i)$*.*

Suppose that $G \in \mathbf{LKSsmall}(n,k,\eta)$ *is given together with its* $(k, \Omega^{**}, \Omega^*, \Lambda, \gamma, \varepsilon', \nu, \rho)$*-sparse decomposition*

$$\nabla = (\mathbf{\Psi}, \mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A}) \,,$$

with respect to the partition $\{\mathbb{S}_{\eta,k}(G), \mathbb{L}_{\eta,k}(G)\}$*, and with respect to the avoiding threshold* $\frac{\rho k}{100\Omega^*}$*. We write*

$$V_{\leadsto\mathfrak{A}} := \mathbf{shadow}_{G_\nabla - \mathbf{\Psi}}(\mathfrak{A}, \frac{\rho k}{100\Omega^*}) \quad and \quad \mathbf{V}_{\leadsto\mathfrak{A}} := \{C \in \mathbf{V} \,:\, C \subseteq V_{\leadsto\mathfrak{A}}\} \,. \tag{7.4}$$

The graph $\mathbf{G}_{\mathrm{reg}}$ *is the corresponding cluster graph. Let* $\mathfrak{c}$ *be the size of an arbitrary cluster in* $\mathbf{V}$*.*[xviii] *Let* G_∇ *be the spanning subgraph of* G *formed by the edges captured by* ∇*. There are two* $(\varepsilon, d, \pi\mathfrak{c})$*-semiregular matchings* $\mathcal{M}_A$ *and* $\mathcal{M}_B$ *in* $G_{\mathcal{D}}$*, with the following properties (we abbreviate* $\mathbb{XA} := \mathbb{XA}(\eta, \nabla, \mathcal{M}_A, \mathcal{M}_B)$*,* $\mathbb{XB} := \mathbb{XB}(\eta, \nabla, \mathcal{M}_A, \mathcal{M}_B)$*, and* $\mathbb{XC} := \mathbb{XC}(\eta, \nabla, \mathcal{M}_A, \mathcal{M}_B)$*):*

1. $V(\mathcal{M}_A) \cap V(\mathcal{M}_B) = \emptyset$*,*
2. $V_1(\mathcal{M}_B) \subseteq S^0$*, where*
$$S^0 := \mathbb{S}_{\eta,k}(G) \setminus (V(G_{\mathrm{exp}}) \cup \mathfrak{A}) \,, \tag{7.5}$$
3. *for each* $(X,Y) \in \mathcal{M}_A \cup \mathcal{M}_B$*, there is a dense spot* $(U,W;F) \in \mathcal{D}$ *with* $X \subseteq U$ *and* $Y \subseteq W$*, and further, either* $X \subseteq \mathbb{S}_{\eta,k}(G)$ *or* $X \subseteq \mathbb{L}_{\eta,k}(G)$*, and* $Y \subseteq \mathbb{S}_{\eta,k}(G)$ *or* $Y \subseteq \mathbb{L}_{\eta,k}(G)$*,*
4. *for each* $X_1 \in \mathcal{V}_1(\mathcal{M}_A \cup \mathcal{M}_B)$ *there exists a cluster* $C_1 \in \mathbf{V}$ *such that* $X_1 \subseteq C_1$*, and for each* $X_2 \in \mathcal{V}_2(\mathcal{M}_A \cup \mathcal{M}_B)$ *there exists* $C_2 \in \mathbf{V} \cup \{\mathbb{L}_{\eta,k}(G) \cap \mathfrak{A}\}$ *such that* $X_2 \subseteq C_2$*,*
5. *each pair of the semiregular matching* $\mathcal{M}_{\mathrm{good}} := \{(X_1, X_2) \in \mathcal{M}_A \,:\, X_1 \cup X_2 \subseteq \mathbb{XA}\}$ *corresponds to an edge in* $\mathbf{G}_{\mathrm{reg}}$*,*
6. $e_{G_\nabla}\big(\mathbb{XA}, S^0 \setminus V(\mathcal{M}_A)\big) \leqslant \gamma k n$*,*

[xviii] The number $\mathfrak{c}$ is not defined when $\mathbf{V} = \emptyset$. However in that case $\mathfrak{c}$ is never actually used.

7. $e_{G_{\text{reg}}}(V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) \leqslant \gamma^2 kn$,

8. *for the semiregular matching* $\mathcal{N}_{\mathfrak{A}} := \{(X,Y) \in \mathcal{M}_A \cup \mathcal{M}_B \ : \ (X \cup Y) \cap \mathfrak{A} \neq \emptyset\}$ *we have* $e_{G_{\text{reg}}}\big(V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), V(\mathcal{N}_{\mathfrak{A}})\big) \leqslant \gamma^2 kn$,

9. $|E(G) \setminus E(G_\nabla)| \leqslant 2\rho kn$,

10. $|E(G_{\mathcal{D}}) \setminus (E(G_{\text{reg}}) \cup E_G[\mathfrak{A} \cup \bigcup \mathbf{V}])| \leqslant \gamma kn$.

We write

$$
\begin{aligned}
V_+ &:= V(G) \setminus (S^0 \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) && (7.6)\\
&= \mathbb{L}_{\eta,k}(G) \cup V(G_{\text{exp}}) \cup \mathfrak{A} \cup V(\mathcal{M}_A \cup \mathcal{M}_B)\ , \\
L_\# &:= \mathbb{L}_{\eta,k}(G) \setminus \mathbb{L}_{\frac{9}{10}\eta,k}(G_\nabla)\ , \text{ and} && (7.7)\\
V_{\text{good}} &:= V_+ \setminus (\mathbf{\Psi} \cup L_\#)\ , && (7.8)\\
\mathbb{YA} &:= \mathbf{shadow}_{G_\nabla}\left(V_+ \setminus L_\#, (1+\frac{\eta}{10})k\right) \setminus \mathbf{shadow}_{G-G_\nabla}\left(V(G), \frac{\eta}{100}k\right)\ , && (7.9)\\
\mathbb{YB} &:= \mathbf{shadow}_{G_\nabla}\left(V_+ \setminus L_\#, (1+\frac{\eta}{10})\frac{k}{2}\right) \setminus \mathbf{shadow}_{G-G_\nabla}\left(V(G), \frac{\eta}{100}k\right)\ , && (7.10)\\
V_{\not\leadsto\mathbf{\Psi}} &:= (\mathbb{XA} \cup \mathbb{XB}) \cap \mathbf{shadow}_G\left(\mathbf{\Psi}, \frac{\eta}{100}k\right)\ , && (7.11)\\
\mathtt{P}_{\mathfrak{A}} &:= \mathbf{shadow}_{G_{\text{reg}}}(V(\mathcal{N}_{\mathfrak{A}}), \gamma k) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)\ , \\
\mathtt{P}_1 &:= \mathbf{shadow}_{G_{\text{reg}}}(V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), \gamma k) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)\ , \\
\mathtt{P} &:= (\mathbb{XA} \setminus \mathbb{YA}) \cup ((\mathbb{XA} \cup \mathbb{XB}) \setminus \mathbb{YB}) \cup V_{\not\leadsto\mathbf{\Psi}} \cup L_\# \\
&\qquad \cup \mathbf{shadow}_{G_{\mathcal{D}} \cup G_\nabla}(V_{\not\leadsto\mathbf{\Psi}} \cup L_\# \cup \mathtt{P}_{\mathfrak{A}} \cup \mathtt{P}_1, \frac{\eta^2 k}{10^5})\ , \\
\mathtt{P}_2 &:= \mathbb{XA} \cap \mathbf{shadow}_{G_\nabla}(S^0 \setminus V(\mathcal{M}_A), \sqrt{\gamma}k)\ , \\
\mathtt{P}_3 &:= \mathbb{XA} \cap \mathbf{shadow}_{G_\nabla}(\mathbb{XA}, \eta^3 k/10^3)\ , \\
\mathcal{F} &:= \{C \in \mathcal{V}(\mathcal{M}_A) : C \subseteq \mathbb{XA}\} \cup \mathcal{V}_1(\mathcal{M}_B)\ . && (7.12)
\end{aligned}
$$

The vertex set $\mathbb{YA}$ in Setting 7.4 should be regarded as $\mathbb{XA}$ cleaned from rare irregularities. Indeed, as it turns out most of the vertices from $\mathbb{XA}$ are contained in $\mathbb{YA}$. Likewise, $\mathbb{YB}$ should be regarded as a cleaned version of $\mathbb{XA} \cup \mathbb{XB}$. These properties are stated in Lemma 7.9 below.

On the interface between Lemma 7.31 and Lemma 7.34 we shall need to work with a semiregular matching which is formed of only those edges $E(\mathcal{D})$ which are either incident with $\mathfrak{A}$, or included in G_{reg}. The following lemma provides us with an appropriate "cleaned version of $\mathcal{D}$". The notion of being absorbed adapts in a straightforward way to two families of dense spots: a family of dense spots $\mathcal{D}_1$ *is absorbed* by another family $\mathcal{D}_2$ if for every $D_1 \in \mathcal{D}_2$ there exists $D_2 \in \mathcal{D}_2$ such that D_1 is contained in D_2 as a subgraph.

Lemma 7.5. *Assume Setting 7.4. Then there exists a family $\mathcal{D}_\nabla$ of edge-disjoint $(\gamma^3 k/4, \gamma/2)$-dense spots absorbed by $\mathcal{D}$ such that*

1. $|E(\mathcal{D}) \setminus E(\mathcal{D}_\nabla)| \leqslant 3\gamma kn$, *and*

2. $E(\mathcal{D}_\nabla) \subseteq E(G_{\mathrm{reg}}) \cup E(G[\mathfrak{A}, \mathfrak{A} \cup \bigcup \mathbf{V}])$.

The proof of Lemma 7.5 is a warm-up for proofs in Section 7.6.

Proof of Lemma 7.5. We discard those dense spots $D \in \mathcal{D}$ for which

$$\left|E(D) \setminus (E(G_{\mathrm{reg}}) \cup E(G[\mathfrak{A}, \mathfrak{A} \cup \bigcup \mathbf{V}])\right| \geqslant \frac{e(D)}{2} \ . \tag{7.13}$$

For each remaining dense spot $D \in \mathcal{D}$ we extract below a $(\gamma^3 k/4, \gamma/2)$-dense spot $D' \subseteq D$, with $e(D') \geqslant (1 - \frac{\gamma}{2})e(D)$. We include D' to $\mathcal{D}_\nabla$. This way we indeed get Property 1, as

$$\begin{aligned} |E(\mathcal{D}) \setminus E(\mathcal{D}_\nabla)| &\leqslant 2\left|E(\mathcal{D}) \setminus \left(E(G_{\mathrm{reg}}) \cup E(G[\mathfrak{A}, V(G)])\right)\right| + \frac{\gamma}{2} \cdot e(\mathcal{D}) \\ \text{(by S7.4(10), and as } e(\mathcal{D}) \leqslant e(G) \leqslant kn) \quad &\leqslant 3\gamma kn \ . \end{aligned}$$

We now show how to extract a $(\gamma^3 k/4, \gamma/2)$-dense spot $D' \subseteq D$, with $e(D') \geqslant (1 - \frac{\gamma}{2})e(D)$ from a spot $D \in \mathcal{D}$ which does not satisfy (7.13). Let $D = (A, B; F)$, and $a := |A|$, $b := |B|$. As D is $(\gamma k, \gamma)$-dense, we have $a, b \geqslant \gamma k$. We now start a sequential cleaning procedure. Discard from A any vertex whose current degree is less than $\gamma^2 b/4$, and discard from B any vertex whose current degree is less than $\gamma^2 a/4$. When the procedure terminates, we clearly have for the resulting graph $D' = (A', B'; F')$ that $\mathrm{deg^{min}}_{D'}(A') \geqslant \gamma^2 b/4 \geqslant \gamma^3 k/4$, and similarly, $\mathrm{deg^{min}}_{D'}(B') \geqslant \gamma^3 k/4$. Last, we deleted at most $a \times \gamma^2 b/4 + b \times \gamma^2 a/4$ edges out of at least γab original edges of D. This means that $\mathrm{d}_{D'}(A', B') \geqslant \gamma - \gamma^2/4 \geqslant \gamma/2$. Also, it gives that the proportion of deleted edges is at most $\gamma/2$. □

In some cases, we shall in addition partition the set $V(G)$ into three sets as in Lemma 7.3. This motivates the following definition.

Definition 7.6 (Proportional splitting). *Let $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2 > 0$ be three positive reals with $\sum_i \mathfrak{p}_i \leqslant 1$. Under Setting 7.4, suppose that $(\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2)$ is a partition of $V(G) \setminus \mathbf{\Psi}$ which satisfies assertions of Lemma 7.3 with parameter $p_{\triangleright \mathrm{L7.3}} := 10$ for graph $G^*_{\triangleright \mathrm{L7.3}} := (G_\nabla - \mathbf{\Psi}) \cup G_{\mathcal{D}}$ (here, by the union, we mean union of the edges), bounded decomposition $(\mathbf{V}, \mathcal{D}, G_{\mathrm{reg}}, G_{\mathrm{exp}}, \mathfrak{A})$, matching $\mathcal{M}_{\triangleright \mathrm{L7.3}} := \mathcal{M}_A \cup \mathcal{M}_B$, sets $\mathfrak{U}_1 := V_{\mathrm{good}}, \mathfrak{U}_2 := \mathbb{XA} \setminus (\mathbf{\Psi} \cup \mathtt{P})$, $\mathfrak{U}_3 := \mathbb{XB} \setminus \mathtt{P}$, $\mathfrak{U}_4 := V(G_{\mathrm{exp}})$, $\mathfrak{U}_5 := \mathfrak{A}$, $\mathfrak{U}_6 := V_{\rightsquigarrow \mathfrak{A}}$, $\mathfrak{U}_7 := \mathtt{P}_{\mathfrak{A}}$, $\mathfrak{U}_8 := \mathbb{L}_{\eta,k}(G)$, $\mathfrak{U}_9 := L_\#$, $\mathfrak{U}_{10} := V_{\not\rightsquigarrow \mathbf{\Psi}}$ and reals $\mathfrak{q}_1 := \mathfrak{p}_0, \mathfrak{q}_2 := \mathfrak{p}_1$, $\mathfrak{q}_3 := \mathfrak{p}_2$, $\mathfrak{q}_4 := \ldots \mathfrak{q}_{10} = 0$. Note* *that by Lemma 7.3*(8) *we have that* $(\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2)$ *is a partition of* $V(G) \setminus \mathbf{\Psi}$*. We call* $(\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2)$ proportional $(\mathfrak{p}_0 : \mathfrak{p}_1 : \mathfrak{p}_2)$ splitting.

We refer to properties of the proportional $(\mathfrak{p}_0 : \mathfrak{p}_1 : \mathfrak{p}_2)$ *splitting* $(\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2)$ *using the numbering of Lemma 7.3; for example, "Definition 7.6*(5)*" tells us among other things that* $|(\mathbb{XA} \setminus \mathtt{P}) \cap \mathfrak{P}_0| \geqslant \mathfrak{p}_0 |\mathbb{XA} \setminus (\mathtt{P} \cup \mathbf{\Psi})| - n^{0.9}$.

Setting 7.7. *Under Setting 7.4, suppose that we are given a proportional* $(\mathfrak{p}_0 : \mathfrak{p}_1 : \mathfrak{p}_2)$ *splitting* $(\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2)$ *of* $V(G) \setminus \mathbf{\Psi}$*. We assume that*

$$\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2 \geqslant \frac{\eta}{100} . \tag{7.14}$$

Let $\bar{V}, \bar{\mathcal{V}}, \bar{\mathbf{V}}$ *be the exceptional sets as in Definition 7.6*(1)*.*

We write

$$\mathbb{F} := \mathbf{shadow}_{G_{\mathcal{D}}} \left(\bigcup \bar{\mathcal{V}} \cup \bigcup \bar{\mathcal{V}}^* \cup \bigcup \bar{\mathbf{V}}, \frac{\eta^2 k}{10^{10}} \right) , \tag{7.15}$$

where $\bar{\mathcal{V}}^*$ *are the partners of* $\bar{\mathcal{V}}$ *in* $\mathcal{M}_A \cup \mathcal{M}_B$*.*

We have

$$|\mathbb{F}| \leqslant \varepsilon n . \tag{7.16}$$

For an arbitrary set $U \subseteq V(G)$ *and for* $i \in \{0, 1, 2\}$ *we write* $U^{\restriction i}$ *for the set* $U \cap \mathfrak{P}_i$*.*

For each $(X, Y) \in \mathcal{M}_A \cup \mathcal{M}_B$ *such that* $\{X, Y\} \cap \bar{\mathcal{V}} = \emptyset$ *we write* $(X, Y)^{\restriction i}$ *for an arbitrary fixed pair* $(X' \subseteq X, Y' \subseteq Y)$ *with the property that* $|X'| = |Y'| = \min\{|X^{\restriction i}|, |Y^{\restriction i}|\}$*. We extend this notion of restriction to an arbitrary semiregular matching* $\mathcal{N} \subseteq \mathcal{M}_A \cup \mathcal{M}_B$ *as follows. We set*

$$\mathcal{N}^{\restriction i} := \left\{ (X, Y)^{\restriction i} \ : \ (X, Y) \in \mathcal{N}, \{X, Y\} \cap \bar{\mathcal{V}} = \emptyset \right\} .$$

The next lemma provides some simple properties of a restriction of a semiregular matching.

Lemma 7.8. *Assume Setting 7.7. Then for each* $i \in \{0, 1, 2\}$*, and for each* $\mathcal{N} \subseteq \mathcal{M}_A \cup \mathcal{M}_B$ *we have that* $\mathcal{N}^{\restriction i}$ *is a* $(\frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi}{200}\mathfrak{c})$*-semiregular matching satisfying*

$$|V(\mathcal{N}^{\restriction i})| \geqslant \mathfrak{p}_i |V(\mathcal{N})| - 2k^{-0.05} n . \tag{7.17}$$

Proof. Let us consider an arbitrary pair $(X, Y) \in \mathcal{N}$. By Definition 7.6(3) we have

$$|X^{\restriction i}| \geqslant \mathfrak{p}_i |X| - k^{0.9} \overset{(7.14)}{\geqslant} \frac{\eta}{200} |X| \quad \text{and} \quad |Y^{\restriction i}| \geqslant \mathfrak{p}_i |Y| - k^{0.9} \overset{(7.14)}{\geqslant} \frac{\eta}{200} |Y| . \tag{7.18}$$

In particular, Fact 2.7 gives that $(X, Y)^{\restriction i}$ is a $400\varepsilon/\eta$-regular pair of density at least $d/2$.

We now turn to (7.17). The total order of pairs $(X, Y) \in \mathcal{N}$ excluded entirely from $\mathcal{N}^{\restriction 1}$ is at most $2\exp(-k^{0.1})n < k^{-0.05} n$ by Definition 7.6(1). Further, for each $(X, Y) \in \mathcal{N}$ whose part is included to $\mathcal{N}^{\restriction 1}$ we have by that $|V((X, Y)^{\restriction i})| \geqslant \mathfrak{p}_i (|X| + |Y|) - 2k^{0.9}$ by (7.18). As $|\mathcal{N}| \leqslant \frac{n}{2k^{0.95}}$, and (7.17) follows. □

The following two Lemmas give us useful bound on some sets defined on page 78.

Lemma 7.9. *Suppose we are in Setting 7.4. Suppose that all but at most βkn edges are captured by ∇. Then,*

$$|L_{\#}| \leqslant \frac{20\beta}{\eta} n \tag{7.19}$$

$$|\mathbb{XA} \setminus \mathbb{YA}| \leqslant \frac{600\beta}{\eta^2} n \text{ , and} \tag{7.20}$$

$$|(\mathbb{XA} \cup \mathbb{XB}) \setminus \mathbb{YB}| \leqslant \frac{600\beta}{\eta^2} n \text{ .} \tag{7.21}$$

Further, if $e_G(\mathbf{\Psi}, \mathbb{XA} \cup \mathbb{XB}) \leqslant \tilde{\beta} kn$ then

$$|V_{\not\leadsto\mathbf{\Psi}}| \leqslant \frac{100\tilde{\beta} n}{\eta} \text{ .} \tag{7.22}$$

Proof. Let $W_1 := \{v \in V(G) : \deg_G(v) - \deg_{G_\nabla}(v) \geqslant \eta k/100\}$. We have $|W_1| \leqslant \frac{200\beta}{\eta} n$.

Observe that $L_{\#}$ sends out at most $(1 + \frac{9}{10}\eta)k|L_{\#}| < \frac{40\beta}{\eta} kn$ edges in G_∇. Let $W_2 := \{v \in V(G) : \deg_{G_\nabla}(v, L_{\#}) \geqslant \eta k/10\}$. We have $|W_2| \leqslant \frac{400\beta}{\eta^2} n$.

Let $W_3 := \{v \in \mathbb{XA} : \deg_{G_\nabla}(v, S^0 \setminus V(\mathcal{M}_A)) \geqslant \sqrt{\gamma} k\}$. By Property 6 we have $|W_3| \leqslant \sqrt{\gamma} n$.

Now, observe that $\mathbb{XA} \setminus \mathbb{YA} \subseteq W_1 \cup W_2 \cup W_3$, and $\mathbb{XB} \setminus \mathbb{YB} \subseteq W_1 \cup W_2$.

The bound (7.22) follows in a straightforward way. □

7.4 Types of configurations

We can now define the following preconfigurations (♣), (♡**1**), (♡**2**), (**exp**), and (**reg**), and the configurations[xix] (⋄**1**)–(⋄**10**). It will follow from results from other sections that at least one of the configurations (⋄**1**)–(⋄**10**) appears in each graph $\mathbf{LKS}(n, k, \eta)$. More precisely, after getting the "rough structure" in Lemma 6.1 we get one of the configurations (⋄**1**)–(⋄**10**) from Lemma 7.31. The latter lemma reduces the situation to one of three cases which are then dealt with in Lemmas 7.32–7.34 separately. Then, in Section 8, we provide with an embedding for a given tree $T_{\triangleright \text{T1.3}} \in \mathbf{trees}(k)$.

We now give a brief overview of these configurations. Configuration (⋄**1**) covers the easy and lucky case when G is contains a subgraph with high minimum degree. A very simple tree-embedding strategy similar to the greedy strategy turns out to work in this case.

The purpose of Preconfiguration (♣) is to utilize vertices of $\mathbf{\Psi}$. On one hand these vertices seem very powerful because of their large degree, on the other hand the edges incident with them are very unstructured. Therefore Preconfiguration (♣) distills some

[xix] The word "configuration" is used for a final structure in a graph which is suitable for embedding purposes while "preconfigurations" are building blocks for configurations.

structure in $\mathbf{\Psi}$. This preconfiguration is then a part of configurations ($\diamond$**2**)–($\diamond$**5**) which deal with the case when $\mathbf{\Psi}$ is substantial. Indeed, Lemma 7.32 asserts that whenever $\mathbf{\Psi}$ is incident with many edges in the setting provided by Lemma 6.1, at least one of configurations ($\diamond$**1**)–($\diamond$**5**) must occur.

The cases when the number of edges incident with $\mathbf{\Psi}$ is negligible are covered by configurations ($\diamond$**6**)–($\diamond$**10**). More precisely, in this setting Lemma 7.31 transforms the output structure of Lemma 6.1 into an input structure for either Lemma 7.33 or Lemma 7.34. These lemmas then assert that indeed one of the Configurations ($\diamond$**6**)–($\diamond$**10**) must occur. The configurations ($\diamond$**6**)–($\diamond$**8**) involve combinations of one of the two preconfigurations ($\heartsuit$**1**) and ($\heartsuit$**2**) and one of the two preconfigurations (**exp**) and (**reg**). The idea here is that the knags are embedded using the structure of (**exp**) or (**reg**) (whichever applicable), the internal shrubs are embedded using the structure which is specific to each of the configurations ($\diamond$**6**)–($\diamond$**8**), and the end shrubs are embedded using the structure of ($\heartsuit$**1**) or ($\heartsuit$**2**). The configuration ($\diamond$**10**) is very similar to the structures obtained in the dense setting in [PS12, HP] (see Section 8.1.5 for a discussion), and ($\diamond$**9**) should be considered as half-way towards it.

The reader may find it helpful to compare the definitions of the configurations with Section 8.1 where an overview is given how these configurations are used to embed the tree $T_{\triangleright\mathrm{T1.3}}$.

Some of the configurations below are accompanied with parameters in the parentheses; note that we do not make explicit those numerical parameters which are inherited from Setting 7.4.

We start off by giving definitions of Configuration ($\diamond$**1**). This is a very easy configuration in which a modification of the greedy tree-embedding strategy works.

Definition 7.10 (Configuration ($\diamond$1)). *We say that a graph G is in* Configuration ($\diamond$**1**) *if there exists a non-empty bipartite graph $H \subseteq G$ with* $\mathrm{deg^{min}}_G(V(H)) \geqslant k$ *and* $\mathrm{deg^{min}}(H) \geqslant k/2$.

We now introduce the configurations ($\diamond$**2**)–($\diamond$**5**) which make use of the set $\mathbf{\Psi}$. These configurations build on Preconfiguration ($\clubsuit$). Figure 8.1 shows common features of the configurations ($\diamond$**2**)–($\diamond$**5**).

Definition 7.11 (Preconfiguration ($\clubsuit$)). *Suppose that we are in Setting 7.4. We say that the graph G is in* Preconfiguration ($\clubsuit$)($\Omega^\star$) *if the following conditions are met. G contains non-empty sets $L'' \subseteq L' \subseteq \mathbb{L}_{\frac{9}{10}\eta,k}(G_\nabla) \setminus \mathbf{\Psi}$, and a non-empty set $\mathbf{\Psi}' \subseteq \mathbf{\Psi}$*

such that

$$\mathrm{deg^{max}}_{G_\nabla}(L', \mathbf{\Psi} \setminus \mathbf{\Psi}') < \frac{\eta k}{100} \;, \tag{7.23}$$

$$\mathrm{deg^{min}}_{G_\nabla}(\mathbf{\Psi}', L'') \geqslant \Omega^\star k \;, \text{ and} \tag{7.24}$$

$$\mathrm{deg^{max}}_{G_\nabla}(L'', \mathbb{L}_{\frac{9}{10}\eta,k}(G_\nabla) \setminus (\mathbf{\Psi} \cup L')) \leqslant \frac{\eta k}{100} \;. \tag{7.25}$$

Definition 7.12 (**Configuration** ($\diamond$**2**))**.** *Suppose that we are in Setting 7.4. We say that the graph G is in* Configuration ($\diamond$**2**)$(\Omega^\star, \tilde{\Omega}, \beta)$ *if the following conditions are met.*

The triple $L'', L', \mathbf{\Psi}'$ witnesses preconfiguration ($\clubsuit$)$(\Omega^\star)$ *in G. There exist a non-empty set $\mathbf{\Psi}'' \subseteq \mathbf{\Psi}'$, a set $V_1 \subseteq V(G_{\mathrm{exp}}) \cap \mathbb{YB} \cap L''$, and a set $V_2 \subseteq V(G_{\mathrm{exp}})$ with the following properties.*

$$\mathrm{deg^{min}}_{G_\nabla}(\mathbf{\Psi}'', V_1) \geqslant \tilde{\Omega} k$$
$$\mathrm{deg^{min}}_{G_\nabla}(V_1, \mathbf{\Psi}'') \geqslant \beta k \;,$$
$$\mathrm{deg^{min}}_{G_{\mathrm{exp}}}(V_1, V_2) \geqslant \beta k \;,$$
$$\mathrm{deg^{min}}_{G_{\mathrm{exp}}}(V_2, V_1) \geqslant \beta k \;.$$

Definition 7.13 (**Configuration** ($\diamond$**3**))**.** *Suppose that we are in Setting 7.4. We say that the graph G is in* Configuration ($\diamond$**3**)$(\Omega^\star, \tilde{\Omega}, \zeta, \delta)$ *if the following conditions are met.*

The triple $L'', L', \mathbf{\Psi}'$ witnesses preconfiguration ($\clubsuit$)$(\Omega^\star)$ *in G. There exist a non-empty set $\mathbf{\Psi}'' \subseteq \mathbf{\Psi}'$, a set $V_1 \subseteq \mathfrak{A} \cap \mathbb{YB} \cap L''$, and a set $V_2 \subseteq V(G) \setminus \mathbf{\Psi}$ such that the following properties are satisfied.*

$$\mathrm{deg^{min}}_{G_\nabla}(\mathbf{\Psi}'', V_1) \geqslant \tilde{\Omega} k \;,$$
$$\mathrm{deg^{min}}_{G_\nabla}(V_1, \mathbf{\Psi}'') \geqslant \delta k \;,$$
$$\mathrm{deg^{max}}_{G_\mathcal{D}}(V_1, V(G) \setminus (V_2 \cup \mathbf{\Psi})) \leqslant \zeta k \;, \tag{7.26}$$
$$\mathrm{deg^{min}}_{G_\mathcal{D}}(V_2, V_1) \geqslant \delta k \;. \tag{7.27}$$

Definition 7.14 (**Configuration** ($\diamond$**4**))**.** *Suppose that we are in Setting 7.4. We say that the graph G is in* Configuration ($\diamond$**4**)$(\Omega^\star, \tilde{\Omega}, \zeta, \delta)$ *if the following conditions are met.*

The triple $L'', L', \mathbf{\Psi}'$ witnesses preconfiguration ($\clubsuit$)$(\Omega^\star)$ *in G. There exists a non-empty set $\mathbf{\Psi}'' \subseteq \mathbf{\Psi}'$, sets $V_1 \subseteq \mathbb{YB} \cap L''$, $\mathfrak{A}' \subseteq \mathfrak{A}$, and $V_2 \subseteq V(G) \setminus \mathbf{\Psi}$ with the following* *properties*

$$\mathrm{deg^{min}}_{G_\nabla}(\mathbf{\Psi}'', V_1) \geqslant \tilde{\Omega}k\ ,$$

$$\mathrm{deg^{min}}_{G_\nabla}(V_1, \mathbf{\Psi}'') \geqslant \delta k\ ,$$

$$\mathrm{deg^{min}}_{G_\nabla \cup G_\mathcal{D}}(V_1, \mathfrak{A}') \geqslant \delta k\ , \tag{7.28}$$

$$\mathrm{deg^{min}}_{G_\nabla \cup G_\mathcal{D}}(\mathfrak{A}', V_1) \geqslant \delta k\ , \tag{7.29}$$

$$\mathrm{deg^{min}}_{G_\nabla \cup G_\mathcal{D}}(V_2, \mathfrak{A}') \geqslant \delta k\ , \tag{7.30}$$

$$\mathrm{deg^{max}}_{G_\nabla \cup G_\mathcal{D}}(\mathfrak{A}', V(G) \setminus (\mathbf{\Psi} \cup V_2)) \leqslant \zeta k\ . \tag{7.31}$$

Definition 7.15 (**Configuration** ($\diamond$**5**))**.** *Suppose that we are in Setting 7.4. We say that the graph G is in* Configuration ($\diamond$**5**)$(\Omega^\star, \tilde{\Omega}, \delta, \zeta, \tilde{\pi})$ *if the following conditions are met.*

The triple $L'', L', \mathbf{\Psi}'$ witnesses preconfiguration ($\clubsuit$)$(\Omega^\star)$ in G. There exists a non-empty set $\mathbf{\Psi}'' \subseteq \mathbf{\Psi}'$, and a set $V_1 \subseteq (\mathbb{YB} \cap L'' \cap \bigcup \mathbf{V}) \setminus V(G_{\mathrm{exp}})$ such that the following conditions are fulfilled.

$$\mathrm{deg^{min}}_{G_\nabla}(\mathbf{\Psi}'', V_1) \geqslant \tilde{\Omega}k\ , \tag{7.32}$$

$$\mathrm{deg^{min}}_{G_\nabla}(V_1, \mathbf{\Psi}'') \geqslant \delta k\ , \tag{7.33}$$

$$\mathrm{deg^{min}}_{G_{\mathrm{reg}}}(V_1) \geqslant \zeta k\ . \tag{7.34}$$

Further, we have

$$C \cap V_1 = \emptyset \text{ or } |C \cap V_1| \geqslant \tilde{\pi}|C| \tag{7.35}$$

for every $C \in \mathbf{V}$.

In remains to introduce configurations ($\diamond$**6**)–($\diamond$**10**) in which the set $\mathbf{\Psi}$ does not appear. A summary picture for Configurations ($\diamond$**6**)–($\diamond$**7**), ($\diamond$**8**), and ($\diamond$**9**) is given in Figures 8.2, 8.3 and 8.4, respectively.

To this end we introduce four preconfigurations ($\heartsuit$**1**), ($\heartsuit$**2**), (**exp**) and (**reg**). The preconfigurations ($\heartsuit$**1**) and ($\heartsuit$**2**) will be used for embedding end shrubs of a fine partition of the tree $T_{\triangleright \mathrm{T1.3}}$, and preconfigurations (**exp**) and (**reg**) will be used for embedding its knags.

An $\mathcal{M}$*-cover* of a semiregular matching $\mathcal{M}$ is a family $\mathcal{F} \subseteq \mathcal{V}(\mathcal{M})$ with the property that at least one of the elements S_1 and S_2 is a member of $\mathcal{F}$, for each $(S_1, S_2) \in \mathcal{M}$.

Definition 7.16 (**Preconfiguration** ($\heartsuit$**1**))**.** *Suppose that we are in Setting 7.4 and Setting 7.7. We say that the graph G is in* Preconfiguration ($\heartsuit$**1**)(γ', h) *of $V(G)$ if there are two non-empty sets $V_0, V_1 \subseteq \mathfrak{P}_0 \setminus \left(\mathbb{F} \cup \mathbf{shadow}_{G_\mathcal{D}}(V_{\not\leadsto \mathbf{\Psi}}, \frac{\eta^2 k}{10^5})\right)$ with the following* *properties.*

$$\mathrm{deg^{min}}_{G_\nabla}\left(V_0, V_{\mathrm{good}}^{\restriction 2}\right) \geqslant h/2 \text{ , and} \tag{7.36}$$

$$\mathrm{deg^{min}}_{G_\nabla}\left(V_1, V_{\mathrm{good}}^{\restriction 2}\right) \geqslant h \text{ .} \tag{7.37}$$

Further, there is an $(\mathcal{M}_A \cup \mathcal{M}_B)$*-cover* $\mathcal{F}$ *such that*

$$\mathrm{deg^{max}}_{G_\nabla}\left(V_1, \bigcup \mathcal{F}\right) \leqslant \gamma' k \text{ .} \tag{7.38}$$

Definition 7.17 (Preconfiguration ($\heartsuit$2)). *Suppose that we are in Setting 7.4 and Setting 7.7. We say that the graph G is in* Preconfiguration $(\heartsuit\mathbf{2})(h)$ *of* $V(G)$ *if there are two non-empty sets* $V_0, V_1 \subseteq \mathfrak{P}_0 \setminus \left(\mathbb{F} \cup \mathbf{shadow}_{G_\mathcal{D}}(V_{\not\leadsto\Psi}, \frac{\eta^2 k}{10^5})\right)$ *with the following properties.*

$$\mathrm{deg^{min}}_{G_\nabla}\left(V_0 \cup V_1, V_{\mathrm{good}}^{\restriction 2}\right) \geqslant h. \tag{7.39}$$

Definition 7.18 (Preconfiguration (exp)). *Suppose that we are in Setting 7.4 and Setting 7.7. We say that the graph G is in* Preconfiguration $(\mathbf{exp})(\beta)$ *if there are two non-empty sets* $V_0, V_1 \subseteq \mathfrak{P}_0$ *with the following properties.*

$$\mathrm{deg^{min}}_{G_{\mathrm{exp}}}(V_0, V_1) \geqslant \beta k \text{ ,} \tag{7.40}$$

$$\mathrm{deg^{min}}_{G_{\mathrm{exp}}}(V_1, V_0) \geqslant \beta k \text{ .} \tag{7.41}$$

Definition 7.19 (Preconfiguration (reg)). *Suppose that we are in Setting 7.4 and Setting 7.7. We say that the graph G is in* Preconfiguration $(\mathbf{reg})(\tilde{\varepsilon}, d', \mu)$ *if there are two non-empty sets* $V_0, V_1 \subseteq \mathfrak{P}_0$ *and a non-empty family of vertex-disjoint* $(\tilde{\varepsilon}, d')$*-super-regular pairs* $\{(Q_0^{(j)}, Q_1^{(j)}\}_{j \in \mathcal{Y}}$ *(with respect to the edge set* $E(G)$*) with* $V_0 := \bigcup Q_0^{(j)}$ *and* $V_1 := \bigcup Q_1^{(j)}$ *such that*

$$\min\left\{|Q_0^{(j)}|, |Q_1^{(j)}|\right\} \geqslant \mu k \text{ .} \tag{7.42}$$

Definition 7.20 (Configuration ($\diamond$6)). *Suppose that we are in Settings 7.4 and 7.7. We say that the graph G is in* Configuration $(\diamond\mathbf{6})(\delta, \tilde{\varepsilon}, d', \mu, \gamma', h_2)$ *if the following conditions are met.*

The vertex sets V_0, V_1 *witness Preconfiguration* $(\mathbf{reg})(\tilde{\varepsilon}, d', \mu)$ *or Preconfiguration* $(\mathbf{exp})(\delta)$ *and either Preconfiguration* $(\heartsuit\mathbf{1})(\gamma', h_2)$ *or Preconfiguration* $(\heartsuit\mathbf{2})(h_2)$*. There exist non-empty sets* $V_2, V_3 \subseteq \mathfrak{P}_1$ *such that*

$$\mathrm{deg^{min}}_{G}(V_1, V_2) \geqslant \delta k \text{ ,} \tag{7.43}$$

$$\mathrm{deg^{min}}_{G}(V_2, V_1) \geqslant \delta k \text{ ,} \tag{7.44}$$

$$\mathrm{deg^{min}}_{G_{\mathrm{exp}}}(V_2, V_3) \geqslant \delta k \text{ , and} \tag{7.45}$$

$$\mathrm{deg^{min}}_{G_{\mathrm{exp}}}(V_3, V_2) \geqslant \delta k \text{ .} \tag{7.46}$$

Definition 7.21 (**Configuration** $(\diamond\mathbf{7})$)**.** *Suppose that we are in Settings 7.4 and 7.7. We say that the graph G is in* Configuration $(\diamond\mathbf{7})(\delta,\rho',\tilde{\varepsilon},d',\mu,\gamma',h_2)$ *if the following conditions are met.*

The vertex sets V_0, V_1 witness Preconfiguration $(\mathbf{reg})(\tilde{\varepsilon},d',\mu)$ and either Preconfiguration $(\heartsuit\mathbf{1})(\gamma',h_2)$ or Preconfiguration $(\heartsuit\mathbf{2})(h_2)$. There exist non-empty sets $V_2 \subseteq \mathfrak{A}^{\restriction 1} \setminus \bar{V}$ and $V_3 \subseteq \mathfrak{P}_1$ such that

$$\mathrm{deg^{min}}_G(V_1,V_2) \geqslant \delta k \ , \tag{7.47}$$

$$\mathrm{deg^{min}}_G(V_2,V_1) \geqslant \delta k \ , \tag{7.48}$$

$$\mathrm{deg^{max}}_{G_{\mathcal{D}}}(V_2,\mathfrak{P}_1 \setminus V_3) < \rho' k \text{ and} \tag{7.49}$$

$$\mathrm{deg^{min}}_{G_{\mathcal{D}}}(V_3,V_2) \geqslant \delta k \ . \tag{7.50}$$

Definition 7.22 (**Configuration** $(\diamond\mathbf{8})$)**.** *Suppose that we are in Settings 7.4 and 7.7. We say that the graph G is in* Configuration $(\diamond\mathbf{8})(\delta,\rho',\varepsilon_1,\varepsilon_2,d_1,d_2,\mu_1,\mu_2,h_1,h_2)$ *if the following conditions are met.*

The vertex sets V_0, V_1 witness Preconfiguration $(\mathbf{reg})(\varepsilon_2,d_2,\mu_2)$ and Preconfiguration $(\heartsuit\mathbf{2})(h_2)$. There exist non-empty sets $V_2 \subseteq \mathfrak{P}_0$, $V_3, V_4 \subseteq \mathfrak{P}_1$, $V_3 \subseteq \mathfrak{A} \setminus \bar{V}$, and an $(\varepsilon_1,d_1,\mu_1 k)$-semiregular matching $\mathcal{N}$ absorbed by $(\mathcal{M}_A \cup \mathcal{M}_B) \setminus \mathcal{N}_{\mathfrak{A}}$, $V(\mathcal{N}) \subseteq \mathfrak{P}_1 \setminus V_3$ such that

$$\mathrm{deg^{min}}_G(V_1,V_2) \geqslant \delta k \ , \tag{7.51}$$

$$\mathrm{deg^{min}}_G(V_2,V_1) \geqslant \delta k \ , \tag{7.52}$$

$$\mathrm{deg^{min}}_{G_\nabla}(V_2,V_3) \geqslant \delta k \ , \tag{7.53}$$

$$\mathrm{deg^{min}}_{G_\nabla}(V_3,V_2) \geqslant \delta k \ , \tag{7.54}$$

$$\mathrm{deg^{max}}_{G_{\mathcal{D}}}(V_3,\mathfrak{P}_1 \setminus V_4) < \rho' k \ , \tag{7.55}$$

$$\mathrm{deg^{min}}_{G_{\mathcal{D}}}(V_4,V_3) \geqslant \delta k \ , \text{ and} \tag{7.56}$$

$$\mathrm{deg}_{G_{\mathcal{D}}}(v,V_3) + \mathrm{deg}_{G_{\mathrm{reg}}}(v,V(\mathcal{N})) \geqslant h_1 \text{ for each } v \in V_2. \tag{7.57}$$

Definition 7.23 (**Configuration** $(\diamond\mathbf{9})$)**.** *Suppose that we are in Settings 7.4, and 7.7. We say that the graph G is in* Configuration $(\diamond\mathbf{9})(\delta,\gamma',h_1,h_2,\varepsilon_1,d_1,\mu_1,\varepsilon_2,d_2,\mu_2)$ *if the following conditions are met.*

The sets V_0, V_1 together with the $(\mathcal{M}_A \cup \mathcal{M}_B)$-cover $\mathcal{F}'$ witness Preconfiguration $(\heartsuit\mathbf{1})(\gamma',h_2)$. There exists an $(\varepsilon_1,d_1,\mu_1 k)$-semiregular matching $\mathcal{N}$ absorbed by $\mathcal{M}_A \cup \mathcal{M}_B$, $V(\mathcal{N}) \subseteq \mathfrak{P}_1$. Further, there is a family $\{(Q_0^{(j)},Q_1^{(j)})\}_{j\in\mathcal{Y}}$ as in Preconfiguration $(\mathbf{reg})(\varepsilon_2,d_2,\mu_2)$. There is a set $V_2 \subseteq V(\mathcal{N}) \setminus \bigcup \mathcal{F}' \subseteq \bigcup \mathbf{V}$ with the following properties:

$$\mathrm{deg^{min}}_{G_{\mathcal{D}}}(V_1,V_2) \geqslant h_1 \ , \tag{7.58}$$

$$\mathrm{deg^{min}}_{G_{\mathcal{D}}}(V_2,V_1) \geqslant \delta k \ . \tag{7.59}$$

Our last configuration, Configuration ($\diamond$**10**), will lead to an embedding very similar to the one in the dense case (treated in [PS12]; see Section 8.1.5). In order to be able to formalize the configuration we need a preliminary definition. We shall generalize the standard concept of a regularity graph (in the context of regular partitions and Szemerédi's Regularity Lemma) to graphs with clusters whose sizes are only bounded from below.

Definition 7.24 (**$(\varepsilon, d, \ell_1\ell_2)$-regularized graph**)**.** *Let G be a graph, and let $\mathcal{V}$ be an ℓ_1-ensemble that partitions $V(G)$. Suppose that $G[X]$ is empty for each $X \in \mathcal{V}$ and suppose $G[X,Y]$ is ε-regular and of density either* 0 *or at least d for all $X, Y \in \mathcal{V}$. Further suppose that for all $X \in \mathcal{V}$ it holds that $|\bigcup \mathrm{N}_G(X)| \leqslant \ell_2$. Then we say that $(G, \mathcal{V})$ is an $(\varepsilon, d, \ell_1, \ell_2)$*-regularized graph.

A semiregular matching $\mathcal{M}$ of G is consistent *with $(G, \mathcal{V})$ if $\mathcal{V}(\mathcal{M}) \subseteq \mathcal{V}$.*

Definition 7.25 (**Configuration** ($\diamond$**10**)$(\tilde{\varepsilon}, d', \ell_1, \ell_2, \eta')$)**.** *Assume Setting 7.4. The graph G contains an $(\tilde{\varepsilon}, d', \ell_1, \ell_2)$-regularized graph $(\tilde{G}, \mathcal{V})$ and there is a $(\tilde{\varepsilon}, d', \ell_1)$-semiregular matching $\mathcal{M}$ consistent with $(\tilde{G}, \mathcal{V})$. There are a family $\mathcal{L}^* \subseteq \mathcal{V}$ and clusters $A, B \in \mathcal{V}$ $(A \neq B)$ with $E(\tilde{G}[A,B]) \neq \emptyset$, and such that we have $\deg_{\tilde{G}}\big(v, V(\mathcal{M}) \cup \bigcup \mathcal{L}^*\big) \geqslant (1+\eta')k$ for all but at most $\tilde{\varepsilon}|A|$ vertices $v \in A$ and for all but at most $\tilde{\varepsilon}|B|$ vertices $v \in B$. For each $X \in \mathcal{L}^*$ we have $\deg_{\tilde{G}}(v) \geqslant (1+\eta')k$ for all but at most $\tilde{\varepsilon}|X|$ vertices $v \in X$.*

7.5 The role of random splitting

The random splitting as introduced in Setting 7.7 is used in Configurations ($\diamond$**6**)–($\diamond$**9**); the set $\mathfrak{P}_0$ will host the cut-vertices $W_A \cup W_B$, the set $\mathfrak{P}_1$ will host the internal shrubs, and the set $\mathfrak{P}_2$ will (essentially) host the end shrubs of a (τk)-fine partition of $T_{\triangleright\text{T1.3}}$.

The need for introducing the random splitting is dictated by Configurations ($\diamond$**6**)–($\diamond$**8**). To see this, let us try to follow the embedding plan from, for example, Section 8.1.2 without the random splitting, i.e., dropping the conditions $\subseteq \mathfrak{P}_0$, $\subseteq \mathfrak{P}_1$, $\subseteq \mathfrak{P}_2$ from Definitions 7.16–7.22. Then the sets V_2 and V_3 in Figure 8.2, which will host the internal shrubs, may interfere with V_0 and V_1 primarily designated for W_A and W_B. In particular, the conditions on degrees between V_0 and V_1 given by (7.40)–(7.41) in Definition 7.18, or given by the super-regularity in Definition 7.19 (in which $\beta_{\triangleright\text{D7.18}} > 0$, or $d'_{\triangleright\text{D7.19}}\mu_{\triangleright\text{D7.19}} > 0$ are tiny) need not be sufficient for embedding greedily all the cut- vertices and all the internal shrubs of $T_{\triangleright\text{T1.3}}$. It should be noted that this problem occurs even in Preconfiguration (**exp**), i.e., the expanding property does not add enough strength to the minimum degree conditions due to the same peculiarity as in Figure 4.2. Restricting V_0 and V_1 to host only the cut-vertices (only $o(k)$ of them in total, cf.

Definition 3.1(c)) resolves the problem.

The above justifies the distinction between the space $\mathfrak{P}_0$ for embedding the cut-vertices and the space $\mathfrak{P}_1 \cup \mathfrak{P}_2$ for embedding the shrubs. There are some other approaches which do not need to further split $\mathfrak{P}_1 \cup \mathfrak{P}_2$ but doing so seems to be the most convenient.

7.6 Cleaning

This section contains five "cleaning lemmas" (Lemma 7.26–Lemma 7.30). The basic setting of all these lemmas is the same. There is a system of vertex sets and some density assumptions on edges between certain sets of this system. The assertion the is that a small number of vertices can be discarded from the sets so that some conditions on the minimum degree are fullfilled. While the cleaning strategy is simply discarding the vertices which violate these minimum degree conditions the analysis of the outcome is non-trivial and employs amortized analysis. The simplest application of such an approach was the proof of Lemma 7.5 above.

Lemmas 7.26–7.30 are used to get the structures required by (pre-)configurations introduced in Section 7.4, based on rough structures found in Lemma 6.1.

The first lemma will be used to obtain preconfiguration (♣) in certain situations.

Lemma 7.26. *Let $\psi \in (0,1)$, and $\Gamma, \Omega \geqslant 1$ be arbitrary. Let P and Q be two disjoint vertex sets in a graph G. Assume that $Y \subseteq V(G)$ is given. We assume that*

$$\deg^{\min}(P,Q) \geqslant \Omega k \,, \tag{7.60}$$

and $\deg^{\max}(Q) \leqslant \Gamma k$. Then there exist sets $P' \subseteq P$, $Q' \subseteq Q \setminus Y$ and $Q'' \subseteq Q'$ such that the following holds.

(a) $\deg^{\min}(P', Q'') \geqslant \frac{\psi^3 \Omega}{4\Gamma^2} k$,

(b) $\deg^{\max}(Q', P \setminus P') < \psi k$,

(c) $\deg^{\max}(Q'', Q \setminus Q') < \psi k$, *and*

(d) $e(P', Q'') \geqslant (1-\psi) e(P,Q) - \frac{2|Y \cap Q|\Gamma^2}{\psi} k$.

Proof. Initially, set $P' := P$, $Q' =: Q \setminus Y$ and $Q'' := Q' \setminus Y$. We shall sequentially discard from the sets P', Q' and Q'' those vertices which violate any of the properties (a)–(c). Further, if a vertex $v \in Q$ is removed from Q' then we remove it from the set Q'' as well. This way, we have $Q'' \subseteq Q'$ in each step. After this sequential cleaning procedure finishes it only remains to establish (d).

First, observe that the way we constructed P' ensures that

$$e(P \setminus P', Q'') \leqslant \frac{\psi^3}{4\Gamma^2} e(P, Q) \ . \tag{7.61}$$

Let $Q^b \subseteq Q \setminus Q'$ be the set of the vertices removed because of condition (b). For a vertex $u \in P \setminus P'$, we write Q''_u for the set Q'' just before the moment when u was removed from P'. Likewise, we define the sets P'_v, Q'_v, Q''_v for each $v \in Q \setminus Q''$. For $u \in P \setminus P'$ let $f(u) := \deg(u, Q''_u)$, for $v \in Q \setminus (Q' \cup Y)$ let $g(v) := \deg(v, P \setminus P'_v)$, and for $w \in Q' \setminus Q''$ let $h(w) := \deg(w, Q \setminus Q'_w)$. Observe that $\sum_{u \in P \setminus P'} f(u) \geqslant \sum_{v \in Q^b} g(v)$. Indeed, at the moment when $v \in Q$ is removed from Q', the $g(v)$ edges that v sends to the set $P \setminus P'_v$ are counted in $\sum_{w \in \mathrm{N}(v) \cap (P \setminus P')} f(w)$. We therefore have

$$\frac{\psi^3}{4\Gamma^2} e(P, Q) \geqslant \frac{\psi^3}{4\Gamma^2} \sum_{u \in P \setminus P'} \deg(u, Q) \geqslant \sum_{u \in P \setminus P'} f(u) \geqslant \sum_{v \in Q^b} g(v) \geqslant |Q^b| \psi k \ ,$$

and consequently,

$$|Q^b| \leqslant \frac{\psi^2}{4\Gamma^2 k} e(P, Q) \ . \tag{7.62}$$

We also have

$$|Q' \setminus Q''| \psi k \leqslant \sum_{w \in Q' \setminus Q''} h(w) \leqslant |Q^b \cup (Y \cap Q)| \Gamma k \overset{(7.62)}{\leqslant} \frac{\psi^2}{4\Gamma} e(P, Q) + |Y \cap Q| \Gamma k \ . \tag{7.63}$$

Finally, we can lower-bound $e(P', Q'')$ as follows.

$$\begin{aligned} e(P', Q'') &\geqslant e(P, Q) - e(P \setminus P', Q'') - |Y \cap Q| \Gamma k - |Q^b| \Gamma k - |Q' \setminus Q''| \Gamma k \\ \text{(by (7.61), (7.62), (7.63))} \quad &\geqslant e(P, Q) \Big(1 - \frac{\psi^3}{4\Gamma^2} - \frac{\psi^2}{4\Gamma} - \frac{\psi}{4} \Big) - |Y \cap Q| \Big(\frac{\Gamma^2 k}{\psi} + \Gamma k \Big) \\ &\geqslant (1 - \psi) e(P, Q) - \frac{2}{\psi} |Y \cap Q| \Gamma^2 k \ . \end{aligned}$$

□

The purpose of the lemmas below (Lemmas 7.27–7.30) is to distill vertex-sets for configurations ($\diamond$**2**)-($\diamond$**10**). They will be applied in Lemmas 7.32, 7.33, 7.34. This is the final "cleaning step" on our way to the proof of Theorem 1.3 — the outputs of these lemmas can by used for a vertex-by-vertex embedding of any tree $T \in \mathbf{trees}(k)$ (although the corresponding embedding procedures given in Section 8 are quite complex).

The first two of these cleaning lemmas (Lemmas 7.27 and 7.28) are suited when the set $\mathbf{\Psi}$ of vertices of huge degrees (cf. Setting 7.4) needs to be considered.

For the following lemma, recall that we defined $[r]$ as the set of the first r natural numbers, *not* including 0.

Lemma 7.27. *For all* $r, \Omega^*, \Omega^{**} \in \mathbb{N}$, *and* $\delta, \gamma, \eta \in (0, 1)$, *with* $\left(\frac{3\Omega^*}{\gamma}\right)^r \delta < \eta/10$, *and* $\Omega^{**} > 1000$ *the following holds. Suppose there are vertex sets* $X_0, X_1, \ldots, X_r$ *and* Y *of an* n*-vertex graph* G *such that*

1. $|Y| < \eta n/(4\Omega^*)$,

2. $e(X_0, X_1) \geqslant \eta kn$,

3. $\deg^{\min}(X_0, X_1) \geqslant \Omega^{**}k$,

4. $\deg^{\min}(X_i, X_{i+1}) \geqslant \gamma k$ *for all* $i \in [r-1]$, *and*

5. $\deg^{\max}\left(Y \cup \bigcup_{i\in[r]} X_i\right) \leqslant \Omega^* k$.

Then there are sets $X'_i \subseteq X_i$ *for* $i = 0, 1, \ldots, r$ *such that*

(a) $X'_1 \cap Y = \emptyset$,

(b) $\deg^{\min}(X'_i, X'_{i-1}) \geqslant \delta k$ *for all* $i \in [r]$,

(c) $\deg^{\max}(X'_i, X_{i+1} \setminus X'_{i+1}) < \gamma k/2$ *for all* $i \in [r-1]$,

(d) $\deg^{\min}(X'_0, X'_1) \geqslant \sqrt{\Omega^{**}}k$, *and*

(e) $e(X'_0, X'_1) \geqslant \eta kn/2$, *in particular* $X'_0 \neq \emptyset$.

Proof. In the formulae below we refer to hypotheses of the lemma as "1."–"5.".

Set $X'_1 := X_1 \setminus Y$. For $i = 0, 2, 3, 4, \ldots, r$, set $X'_i := X_i$. Discard sequentially from X'_i any vertex that violates any of the Properties (b)–(d). Properties (a)–(d) are trivially satisfied when the procedure terminates. To show that Property (e) holds at this point, we bound the number of edges from $e(X_0, X_1)$ that are incident with $X_0 \setminus X'_0$ or with $X_1 \setminus X'_1$ in an amortized way.

For $i \in \{0, \ldots, r\}$ and for $v \in X_i \setminus X'_i$ we write

$$\begin{aligned} f_i(v) &:= \deg\left(v, X_{i+1}(v) \setminus X'_{i+1}(v)\right) , \\ g_i(v) &:= \deg\left(v, X'_{i-1}(v)\right) , \text{ and} \\ h_i(v) &:= \deg\left(v, X'_{i+1}(v)\right) . \end{aligned}$$

where the sets $X'_{i-1}(v), X'_i(v), X'_{i+1}(v)$ above refer to the moment when v is removed from X'_i (we do not define $f_i(v)$ and $h_i(v)$ for $i = r$ and $g_i(v)$ for $i = 0$).

For $i \in [r]$ let X^b_i denote the vertices in $X_i \setminus X'_i$ that were removed from X'_i because of violating Property (b). Then for a given $i \in [r]$ we have that

$$\sum_{v \in X^b_i} g_i(v) < \delta kn. \tag{7.64}$$

For $i = 1, \ldots, r-1$ let X^c_i denote the vertices in $X_i \setminus X'_i$ that violated Property (c). Set $X^c_r := \emptyset$. For a given $i \in [r-1]$ we have

$$|X^c_i| \cdot \gamma k/2 \leqslant \sum_{v \in X^c_i} f_i(v) \leqslant \sum_{v \in X_{i+1} \setminus X'_{i+1}} g_{i+1}(v) \overset{5.,(7.64)}{<} \delta kn + |X^c_{i+1}| \cdot \Omega^* k , \tag{7.65}$$

as $X_i \setminus X'_i = X_i^b \cup X_i^c$, for $i = 2, \ldots, r$. Using (7.65) for $j = 0, \ldots, r-1$, we inductively deduce that

$$|X_{r-j}^c|\frac{\gamma}{2} \leqslant \sum_{i=0}^{j-1} \left(\frac{2\Omega^*}{\gamma}\right)^i \delta n \ . \tag{7.66}$$

(The left-hand side is zero for $j = 0$.) The bound (7.66) for $j = r-1$ gives

$$|X_1^c| \leqslant \frac{2}{\gamma} \cdot \sum_{i=0}^{r-2} \left(\frac{2\Omega^*}{\gamma}\right)^i \delta n \leqslant \frac{2(2\Omega^*)^{r-1}}{\gamma^r} \delta n \ . \tag{7.67}$$

Therefore,

$$e(X_0, Y \cup X_1^c) \leqslant |Y \cup X_1^c| \cdot \Omega^* k \overset{(7.67),1.}{\leqslant} \frac{\eta k n}{4} + \left(\frac{2\Omega^*}{\gamma}\right)^r \delta k n \ . \tag{7.68}$$

For any vertex $v \in X_0 \setminus X'_0$ we have $h_0(v) < \sqrt{\Omega^{**}}k$, and at the same time by Hypothesis 3. we have $\deg(v, X_1) \geqslant \Omega^{**}k$. So,

$$\sum_{v \in X_0 \setminus X'_0} h_0(v) \leqslant \frac{e(X_0, X_1)}{\sqrt{\Omega^{**}}} \ . \tag{7.69}$$

We have

$$e(X'_0, X'_1) \geqslant e(X_0, X_1) - e(X_0, Y \cup X_1^c) - \sum_{v \in X_0 \setminus X'_0} h_0(v) - \sum_{v \in X_1^b} g_1(v) \ .$$

(It requires a minute of meditation to see that edges between $X_0 \setminus X'_0$ and X_1^b are indeed not counted on the right-hand side.) Therefore,

$$\begin{aligned} e(X'_0, X'_1) &\geqslant e(X_0, X_1) - e(X_0, Y \cup X_1^c) - \sum_{v \in X_0 \setminus X'_0} h_0(v) - \sum_{v \in X_1^b} g_1(v) \\ \text{(by (7.64), (7.68), (7.69))} \quad &\geqslant e(X_0, X_1) - \frac{\eta k n}{4} - \left(\frac{2\Omega^*}{\gamma}\right)^r \delta k n - \frac{e(X_0, X_1)}{\sqrt{\Omega^{**}}} - \delta k n \\ \text{(by 2.)} \quad &\geqslant \eta k/2 \ , \end{aligned}$$

proving Property (e). □

Lemma 7.28. *Let $\delta, \eta, \Omega^*, \Omega^{**}, h > 0$, let G be an n-vertex graph, let $X_0, X_1, Y \subseteq V(G)$, and let $\mathcal{C}$ be a system of subsets of $V(G)$ such that*

1. $20(\delta + \frac{2}{\sqrt{\Omega^{**}}}) < \eta$,
2. $2kn \geqslant e(X_0, X_1) \geqslant \eta k n$,
3. $\deg^{\min}(X_0, X_1) \geqslant \Omega^{**}k$,
4. $\deg^{\max}(X_1) \leqslant \Omega^* k$,
5. $|Y| < \eta n/(4\Omega^*)$, *and*

6. $10h|\mathcal{C}|\Omega^* < \eta n$.

Then there are sets $X_0' \subseteq X_0$ and $X_1' \subseteq X_1 \setminus Y$ such that

a) $\deg^{\min}(X_0', X_1') \geqslant \sqrt{\Omega^{**}}k$,

b) $\deg^{\min}(X_1', X_0') \geqslant \delta k$,

c) for all $C \in \mathcal{C}$, either $X_1' \cap C = \emptyset$, or $|X_1' \cap C| \geqslant h$, and

d) $e(X_0', X_1') \geqslant \eta kn/2$.

Proof. Set $X_0' := X_0$ and $X_1' := X_1 \setminus Y$ and discard sequentially from X_0', any vertex violating Property a). Further, we discard from X_1' any vertex violating Property b), or any $C \in \mathcal{C}$ violating c). When the process ends, we verify Property d) by bounding the number of edges in $e(X_0, X_1)$ incident with $X_0 \setminus X_0'$ or with $X_1 \setminus X_1'$. Given Assumption 2, and since by Assumption 5 there are at most $\frac{1}{4}\eta kn$ edges incident with $Y \cap X_1$ it suffices to prove that

$$e(X_0, X_1) - e(X_0', X_1') - e(Y \cap X_1, X_0) < \frac{\eta kn}{4}\ . \tag{7.70}$$

Denote by X_1^b the set of vertices in $X_1 \setminus (Y \cup X_1')$ that violated Property b), and by X_1^c the set of vertices in $X_1 \setminus (Y \cup X_1')$ that violated Property c). For a vertex $v \in X_1 \setminus (Y \cup X_1')$, let $g(v)$ denote the number $\deg(v, X_0')$ at the very time when v is removed from X_1'. Analogously we define $f(v)$, for $v \in X_0 \setminus X_0'$, as $\deg(v, X_1')$ where the set X_1' is considered at the point of removal of v. We have $\sum_{v \in X_1^b} g(v) < \delta kn$, $\sum_{v \in X_1^c} g(v) \leqslant |X_1^c|\Omega^* k < h|\mathcal{C}| \cdot \Omega^* k$, and

$$\sum_{v \in X_0 \setminus X_0'} f(v) \leqslant \frac{e(X_0, X_1)}{\sqrt{\Omega^{**}}} \overset{2.}{\leqslant} \frac{2}{\sqrt{\Omega^{**}}} kn\ .$$

Thus,

$$\begin{aligned} & e(X_0, X_1) - e(X_0', X_1') - e(Y \cap X_1, X_0) \\ & \qquad = \sum_{v \in X_1^b} g(v) + \sum_{v \in X_1^c} g(v) + \sum_{v \in X_0 \setminus X_0'} f(v) \\ & \qquad < \left(\delta + \frac{2}{\sqrt{\Omega^{**}}}\right) kn + h|\mathcal{C}|\Omega^* k \\ & \text{(by 1. and 6.)} \qquad < \frac{\eta kn}{4}\ . \end{aligned}$$

establishing (7.70). □

The next two lemmas (Lemmas 7.29 and 7.30) deal with cleaning outside the set of huge degree vertices $\mathbf{\Psi}$.

Lemma 7.29. *For all $r, \Omega \in \mathbb{N}$, $r \geqslant 2$ and all $\gamma, \delta, \eta > 0$ such that*

$$\left(\frac{8\Omega}{\gamma}\right)^r \delta \leqslant \frac{\eta}{10} \tag{7.71}$$

the following holds. Suppose there are vertex sets $Y, X_0, X_1, \ldots, X_r \subseteq V$, where V is a set of n vertices. Suppose that edge sets $E_1, \ldots, E_r$ are given on V. The expressions $\deg_i$, ${\deg^{\max}}_i$, ${\deg^{\min}}_i$, and e_i below refer to the edge set E_i. Suppose that the following properties are fulfilled

1. $|Y| < \delta n$,
2. $e_1(X_0, X_1) \geqslant \eta k n$,
3. *for all $i \in [r-1]$ we have* ${\deg^{\min}}_{i+1}(X_i \setminus Y, X_{i+1}) \geqslant \gamma k$,
4. *for all $i \in \{0, \ldots, r-1\}$, we have* ${\deg^{\max}}_{i+1}(X_i) \leqslant \Omega k$, *and* ${\deg^{\max}}_{i+1}(X_{i+1}) \leqslant \Omega k$.

Then there are sets $X'_i \subseteq X_i \setminus Y$ $(i = 0, \ldots, r)$ satisfying the following.

a) For all $i \in [r]$ and we have ${\deg^{\min}}_i(X'_i, X'_{i-1}) \geqslant \delta k$,

b) for all $i \in [r-1]$ we have ${\deg^{\max}}_{i+1}(X'_i, X_{i+1} \setminus X'_{i+1}) < \gamma k/2$,

c) ${\deg^{\min}}_1(X'_0, X'_1) \geqslant \delta k$, *and*

d) $e_1(X'_0, X'_1) \geqslant \eta k n/2$

Proof. We proceed similarly as in the proof of Lemma 7.27. Set $X'_i := X_i \setminus Y$ for each $i = 0, \ldots, r$. Discard sequentially from X'_i any vertex that violates Property a) or b), or c). When the procedure terminates, we certainly have that a)–c) hold. We then show that Property d) holds by bounding the number of edges from $e_1(X_0, X_1)$ that are incident with $X_0 \setminus X'_0$ or with $X_1 \setminus X'_1$. For $i \in \{0, \ldots, r\}$ and for $v \in X_i \setminus X'_i$ we write

$$\begin{aligned} f_{i+1}(v) &:= \deg_{i+1}(v, X_{i+1} \setminus X'_{i+1}) \,, \\ g_i(v) &:= \deg_i(v, X'_{i-1}) \,, \text{ and} \\ h(v) &:= \deg_1(v, X'_1) \,, \end{aligned}$$

where the sets X'_1, X'_{i-1} and X'_{i+1} above refer to the moment[xx] when v is removed from X'_i or from X'_1 (we do not define $f_{i+1}(v)$ for $i = r$ and $g_i(v)$ for $i = 0$).

[xx] if $v \in Y$ then this moment is the zero-th step

Let $X_i^a \subseteq X_i$, $X_i^b \subseteq X_i$ for $i \in [r-1]$ be the sets of vertices removed from X_i' because of Property a) and b), respectively. Set $X_r^a := X_r \setminus X_r'$ and $X_0^c := X_0 \setminus X_0'$. We have for each $i \in [r]$,

$$\sum_{v \in X_i^a} g_i(v) < \delta kn \; . \tag{7.72}$$

Also, note that we have

$$\sum_{v \in X_0^c} h(v) \leqslant \delta kn \; . \tag{7.73}$$

We set $X_r^b := \emptyset$. For a given $i \in [r-1]$ we have

$$\begin{aligned} |X_i^b| \cdot \frac{\gamma k}{2} &\leqslant \sum_{v \in X_i^b} f_{i+1}(v) \\ &\leqslant \sum_{v \in X_{i+1} \setminus X_{i+1}'} g_{i+1}(v) \\ \text{(by 4., (7.72))} \qquad &\leqslant \delta kn + |X_{i+1}^b| \Omega k \; , \end{aligned} \tag{7.74}$$

as $X_i \setminus X_i' \subseteq X_i^a \cup X_i^b \cup Y$, for $i = 2, \dots, r$. Using (7.74), we deduce inductively that

$$|X_{r-j}^b| \leqslant \left(\frac{8\Omega}{\gamma} \right)^j \delta n \; , \tag{7.75}$$

for $j = 0, \dots, r-1$. (The left-hand side is zero for $j = 0$.) Therefore,

$$\begin{aligned} e_1(X_0', X_1') &\geqslant e_1(X_0, X_1) - (|Y| + |X_1^b|)\Omega k - \sum_{v \in X_1^a} g_1(v) - \sum_{v \in X_0^c} h(v) \\ \text{(by 2, (7.75), (7.72), (7.73))} \qquad &\geqslant \eta kn - \left(\frac{8\Omega}{\gamma} \right)^r \delta kn - 2\delta kn \\ &\geqslant \frac{\eta}{2} kn \; , \end{aligned}$$

establishing Property d). □

Lemma 7.30. *For all $r, \Omega \in \mathbb{N}$, $r \geqslant 2$ and all $\gamma, \delta, \varepsilon, \eta, d > 0$ with*

$$20\varepsilon < d \quad \text{and} \quad \left(\frac{8\Omega}{\gamma} \right)^r \delta \leqslant \frac{\eta}{30} \tag{7.76}$$

the following holds. Suppose there are vertex sets $Y, X_0, X_1, \dots, X_r \subseteq V$, where V is a set of n vertices. Let $P_i^{(1)}, \dots, P_i^{(p)}$ partition X_i, for $i = 0, 1$. Suppose that edge sets $E_1, E_2, E_3, \dots, E_r$ are given on V. The expressions deg_i, $\mathrm{deg^{max}}_i$, and $\mathrm{deg^{min}}_i$ below refer to the edge set E_i. Suppose that

1. *$|Y| < \delta n$,*
2. *$|X_1| \geqslant \eta n$,*
3. *for all $i \in [r-1]$ we have $\mathrm{deg^{min}}_{i+1}(X_i \setminus Y, X_{i+1}) \geqslant \gamma k$,*

4. *the family* $\left\{(P_0^{(j)}, P_1^{(j)})\right\}_{j\in[p]}$ *is an* $(\varepsilon, d, \mu k)$*-semiregular matching with respect to the edge set* E_1*, and*

5. *for all* $i \in \{0, \dots, r-1\}$, $\deg^{\max}{}_{i+1}(X_{i+1}) \leqslant \Omega k$*, and (when* $i \neq r$*)* $\deg^{\max}{}_{i+1}(X_i) \leqslant \Omega k$*.*

Then

a) there is a non-empty family $\{(Q_0^{(j)}, Q_1^{(j)})\}_{j\in\mathcal{Y}}$ *of vertex-disjoint* $(4\varepsilon, \frac{d}{4})$*-super-regular pairs with respect to* E_1*, with* $|Q_0^{(j)}|, |Q_1^{(j)}| \geqslant \frac{\mu k}{2}$*,*

and further sets $X_0' := \bigcup Q_0^{(j)} \subseteq X_0 \setminus Y$*,* $X_1' := \bigcup Q_1^{(j)} \subseteq X_1 \setminus Y$*,* $X_i' \subseteq X_i \setminus Y$ *(*$i = 2, \dots, r$*) satisfying the following,*

b) for all $i \in [r-1]$ *we have* $\deg^{\min}{}_{i+1}(X_{i+1}', X_i') \geqslant \delta k$*,*

c) for all $i \in [r-1]$*, we have* $\deg^{\max}{}_{i+1}(X_i', X_{i+1} \setminus X_{i+1}') < \gamma k/2$*.*

Proof. Initially, set $\mathcal{J} := \emptyset$ and $X_i' := X_i \setminus Y$ for each $i = 0, \dots, r$. Discard sequentially from X_i' any vertex that violates any of the Properties b) or c). We would like to keep track of these vertices and therefore we call $X_i^b, X_i^c \subseteq X_i$ the sets of vertices removed from X_i' because of Property b), and c), respectively. Further, for $i = 0, 1$ and for $j \in [p]$ remove any vertex $v \in X_i' \cap P_i^{(j)}$ from X_i' if

$$\deg_1(v, X_{1-i}' \cap P_{1-i}^{(j)}) \leqslant \frac{d|P_{1-i}^{(j)}|}{4} . \tag{7.77}$$

For $i = 0, 1$, let X_i^a be the set of those vertices of X_i that were removed because of (7.77).

Last, if for some $j \in [p]$ we have $|P_0^{(j)} \cap Y| > \frac{|P_0^{(j)}|}{4}$ or $|P_1^{(j)} \cap (Y \cup X_1^c)| > \frac{|P_1^{(j)}|}{4}$ we remove simultaneously the sets $P_0^{(j)}$ and $P_1^{(j)}$ entirely from X_0' and X_1', i.e., we set $X_0' := X_0' \setminus P_0^{(j)}$ and $X_1' := X_1' \setminus P_1^{(j)}$. We also add the index j to the set $\mathcal{J}$ in this case.

When the procedure terminates define $\mathcal{Y} := [p] \setminus \mathcal{J}$, and for $j \in \mathcal{Y}$ set $(Q_0^{(j)}, Q_1^{(j)}) := (P_0^{(j)} \cap X_0', P_1^{(j)} \cap X_1')$. The sets X_i' obviously satisfy Properties b)–c). We now turn to verifying Property a). This relies on the following claim.

Claim 7.30.1. If $j \in [p] \setminus \mathcal{J}$ then $|P_0^{(j)} \cap X_0^a| \leqslant \frac{|P_0^{(j)}|}{4}$ and $|P_1^{(j)} \cap X_1^a| \leqslant \frac{|P_1^{(j)}|}{4}$.

Proof of Claim 7.30.1. Recall that E_1 is the relevant underlying edge set when working with the pairs $(P_0^{(j)}, P_1^{(j)})$. Also, recall that only vertices from $Y \cup X_0^a$ were removed from $P_0^{(j)}$ and only vertices from $Y \cup X_1^a \cup X_1^c$ were removed from $P_1^{(j)}$.

Since $j \notin \mathcal{J}$, the pair $(P_0^{(j)} \setminus Y, P_1^{(j)} \setminus (Y \cup X_1^c))$ is 2ε-regular of density at least $0.9d$ by Fact 2.7. Let

$$K_0 := \left\{v \in P_0^{(j)} \setminus Y \;:\; \deg_1(v, P_1^{(j)} \setminus (Y \cup X_1^c)) < 0.8d|P_1^{(j)} \setminus (Y \cup X_1^b)|\right\} \text{, and}$$
$$K_1 := \left\{v \in P_1^{(j)} \setminus (Y \cup X_1^c) \;:\; \deg_1(v, P_0^{(j)} \setminus Y) < 0.8d|P_0^{(j)} \setminus Y|\right\} .$$

By Fact 2.8, we have $|K_0| \leqslant 2\varepsilon|P_0^{(j)} \setminus Y| \leqslant 0.1d|P_0^{(j)}|$ and $|K_1| \leqslant 0.1d|P_1^{(j)}|$. In particular, we have

$$\begin{aligned}
\deg^{\min}{}_1(P_0^{(j)} \setminus (Y \cup K_0), P_1^{(j)} \setminus (Y \cup X_1^c \cup K_1)) &\geqslant 0.8d|P_1^{(j)} \setminus (Y \cup X_1^c)| - |K_1| \\
&\geqslant 0.8d \cdot 0.75|P_1^{(j)}| - 0.1d|P_1^{(j)}| \quad (7.78) \\
&> 0.25d|P_1^{(j)}| \text{ , and} \\
\deg^{\min}{}_1(P_1^{(j)} \setminus (Y \cup X_1^c \cup K_1), P_0^{(j)} \setminus (Y \cup K_0)) &\geqslant 0.8d|P_0^{(j)} \setminus Y| - |K_0| \\
&\geqslant 0.8d \cdot 0.75|P_0^{(j)}| - 0.1d|P_0^{(j)}| \quad (7.79) \\
&> 0.25d|P_0^{(j)}| \ .
\end{aligned}$$

Then (7.78) and (7.79) allow us to prove that $P_i^{(j)} \cap X_i^a \subseteq K_i$ for $i = 0, 1$. Indeed, assume inductively that $P_i^{(j)} \cap X_i^a \subseteq K_i$ for $i = 0, 1$ throughout the cleaning process until a certain step. Then (7.78) and (7.79) assert that no vertex outside of $P_0^{(j)} \setminus (Y \cup K_0)$ or of $P_1^{(j)} \setminus (Y \cup X_1^c \cup K_1)$ can be removed because of (7.77), proving the induction step. The claim follows. □

Putting together the definition of $\mathcal{J}$ (through which one controls the size of $P_i^{(j)} \cap (Y \cup X_i^c)$) and Claim 7.30.1 (which controls the size of $P_i^{(j)} \cap X_i^a$) we get for each $j \in \mathcal{Y}$ and $i = 0, 1$,

$$|Q_i^{(j)}| \geqslant \frac{|P_i^{(j)}|}{2} \geqslant \frac{\mu k}{2} \ .$$

Therefore, these pairs are 4ε-regular (cf. Fact 2.7). Last, we get the property of $(4\varepsilon, \frac{d}{4})$-super-regularity from the definition of X_i^c (cf. (7.77)). Thus, the pairs $(Q_0^{(j)}, Q_1^{(j)})$ are as required for Property a).

The only thing we have to prove is that the set X_1' is nonempty. By the definition, for each $j \in \mathcal{J}$, we either have $|P_1^{(j)}| \leqslant 4(|(Y \cup X_1^c) \cap P_1^{(j)}|)$ or $|P_0^{(j)}| \leqslant 4|Y \cap P_0^{(j)}|$. We use that that $|P_0^{(j)}| = |P_1^{(j)}|$ to see that

$$\left| \bigcup_{\mathcal{J}} P_1^{(j)} \right| \leqslant 4(|Y| + |X_1^c|) \ . \tag{7.80}$$

For $i \in \{1, \ldots, r\}$ and for $v \in X_i \setminus X_i'$ write

$$\begin{aligned}
f_{i+1}(v) &:= \deg_{i+1}(v, X_{i+1} \setminus X_{i+1}') \text{ , and} \\
g_i(v) &:= \deg_i(v, X_{i-1}') \ .
\end{aligned}$$

where the sets X_1', X_{i-1}' and X_{i+1}' above refer to the moment[xxi] when v is removed from X_i' (we do not define $f_{i+1}(v)$ for $i = r$).

[xxi] if $v \in Y$ then this moment is the zero-th step

Observe that for each $i \in \{2, \dots, r\}$, we have

$$\sum_{v \in X_i^b} g_i(v) < \delta kn . \tag{7.81}$$

We set $X_r^c := \emptyset$. For a given $i \in [r-1]$ we have

$$\begin{aligned}
|X_i^c| \cdot \frac{\gamma k}{2} &\leqslant \sum_{v \in X_i^c} f_{i+1}(v) \\
&\leqslant \sum_{v \in X_{i+1} \setminus X'_{i+1}} g_{i+1}(v) && (7.82) \\
\text{(by 1. ,5. , (7.81))} \quad &< \delta kn + |X_{i+1}^c| \Omega k, && (7.83)
\end{aligned}$$

as $X_i \setminus X'_i \subseteq X_i^b \cup X_i^c \cup Y$, for $i = 2, \dots, r$. Using (7.83), we deduce inductively that $|X_{r-j}^c| \leqslant \left(\frac{8\Omega}{\gamma}\right)^j \delta n$ for $j = 1, 2, \dots, r-1$, and in particular that

$$|X_1^c| \leqslant \left(\frac{8\Omega}{\gamma}\right)^{r-1} \delta n . \tag{7.84}$$

As $X_1^a = \emptyset$, we obtain that

$$\begin{aligned}
|X'_1| &= \left| X_1 \setminus \left(\bigcup_{j \in \mathcal{J}} P_1^{(j)} \cup \bigcup_{j \in \mathcal{Y}} \left(P_1^{(j)} \cap (Y \cup X_1^a \cup X_1^c)\right) \right) \right| \\
\text{(by (7.80))} \quad &\geqslant |X_1| - 4(|Y| + |X_1^c|) - \left| \bigcup_{j \in \mathcal{Y}} \left(P_1^{(j)} \cap X_1^a \right) \right| \\
\text{(by 1., (7.76), (7.84))} \quad &\geqslant |X_1| - \frac{\eta n}{2} - \left| \bigcup_{j \in \mathcal{Y}} (P_1^{(j)} \cap X_1^a) \right| \\
\text{(by Cl 7.30.1)} \quad &\geqslant |X_1| - \frac{\eta n}{2} - \frac{|X_1|}{4} \\
\text{(by 2.)} \quad &> 0,
\end{aligned}$$

as desired. □

7.7 Obtaining a configuration

In this section we prove that the structure in the graph $G \in \mathbf{LKS}(n, k, \eta)$ guaranteed by Lemma 6.1 always leads to one of the configurations ($\diamond$**1**)–($\diamond$**10**). We distinguish two cases. When the set $\mathbf{\Psi}$ of vertices of huge degree (coming from a sparse decomposition of G) sees many edges, then one of the configurations ($\diamond$**1**)–($\diamond$**5**) must occur (cf. Lemma 7.32). Otherwise, when the edges incident with $\mathbf{\Psi}$ can be neglected, we obtain one of the configurations ($\diamond$**6**)–($\diamond$**10**) (cf. Lemmas 7.33 and 7.34). How these configurations help in embedding the tree $T_{\triangleright \mathrm{T}1.3} \in \mathbf{trees}(k)$ will be shown in Section 8.

Lemmas 7.32, 7.33, and 7.34 are stated in the next section, and their proofs occupy Sections 7.7.3, 7.7.4, and 7.7.5, respectively. These results are put together in Lemma 7.31 of Section 7.7.1.

7.7.1 Statements of the results

We first state the main result of this section, Lemma 7.31. Its proof is given in Section 7.7.2.

Lemma 7.31. *Suppose Settings 7.4 and 7.7. Further suppose that at least on of the cases* **(K1)** *or* **(K2)** *from Lemma 6.1 occurs in G. Then one of the configurations*

- $(\diamond\mathbf{1})$,
- $(\diamond\mathbf{2})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\eta^9\rho^2}{128\cdot 10^{22}\cdot(\Omega^*)^5}\right)$,
- $(\diamond\mathbf{3})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\gamma}{2}, \frac{\eta^9\gamma^2}{128\cdot 10^{22}\cdot(\Omega^*)^5}\right)$,
- $(\diamond\mathbf{4})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\gamma}{2}, \frac{\eta^9\gamma^3}{384\cdot 10^{22}(\Omega^*)^5}\right)$,
- $(\diamond\mathbf{5})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\eta^9}{128\cdot 10^{22}\cdot(\Omega^*)^3}, \frac{\eta}{2}, \frac{\eta^9}{128\cdot 10^{22}\cdot(\Omega^*)^4}\right)$,
- $(\diamond\mathbf{6})\big(\frac{\eta^3\rho^4}{10^{14}(\Omega^*)^4}, 4\varepsilon_\odot, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2000}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{7})\big(\frac{\eta^3\gamma^3\rho}{10^{12}(\Omega^*)^4}, \frac{\eta\gamma}{400}, 4\varepsilon_\odot, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{8})\big(\frac{\eta^4\gamma^4\rho}{10^{15}(\Omega^*)^5}, \frac{\eta\gamma}{400}, \frac{400\varepsilon}{\eta}, 4\varepsilon_\odot, \frac{d}{2}, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta\pi\mathfrak{c}}{200k}, \frac{\eta^2\nu}{2\cdot 10^4}, \mathfrak{p}_1(1+\frac{\eta}{20})k, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{9})\big(\frac{\rho\eta^8}{10^{27}(\Omega^*)^3}, \frac{2\eta^3}{10^3}, \mathfrak{p}_1(1+\frac{\eta}{40})k, \mathfrak{p}_2(1+\frac{\eta}{20})k, \frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi\mathfrak{c}}{200k}, 4\varepsilon_\odot, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}\big)$,
- $(\diamond\mathbf{10})\big(\varepsilon, \frac{\gamma^2 d}{2}, \pi\sqrt{\varepsilon'}\nu k, \frac{2(\Omega^*)^2 k}{\gamma^2}, \frac{\eta}{40}\big)$

occurs in G.

Lemma 7.31 will be proved in Section 7.7.2. The proof relies on Lemmas 7.32, 7.33 and 7.34 below. For an input graph $G_{\rhd \mathrm{L}7.31}$ one of these lemmas is applied depending on the majority type of "good" edges in $G_{\rhd \mathrm{L}7.31}$. Observe that **(K1)** of Lemma 6.1 guarantees edges between $\mathbf{\Psi}$ and $\mathbb{XA} \cup \mathbb{XB}$, or between $\mathbb{XA}$ and $\mathbb{XA} \cup \mathbb{XB}$ either in $E(G_{\mathrm{exp}})$ or in $E(G_{\mathcal{D}})$. Lemma 7.32 is used if we find edges between $\mathbf{\Psi}$ and $\mathbb{XA} \cup \mathbb{XB}$. Lemma 7.33 is used if we find edges of $E(G_{\mathrm{exp}})$ between $\mathbb{XA}$ and $\mathbb{XA} \cup \mathbb{XB}$. The remaining case can be reduced to the setting of Lemma 7.34. Lemma 7.34 is also used to obtain a configuration if we are in case **(K2)** of Lemma 6.1.

Lemma 7.32. *Suppose we are in Setting 7.4. Assume that*

$$e_{G_\nabla}(\mathbf{\Psi}, \mathbb{XA} \cup \mathbb{XB}) \geqslant \frac{\eta^{13} kn}{10^{28}(\Omega^*)^3}. \tag{7.85}$$

Then G contains at least one of the configurations

- $(\diamond\mathbf{1})$,

- $(\diamond\mathbf{2})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\eta^9\rho^2}{128\cdot 10^{22}\cdot(\Omega^*)^5}\right)$,
- $(\diamond\mathbf{3})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\gamma}{2}, \frac{\eta^9\gamma^2}{128\cdot 10^{22}\cdot(\Omega^*)^5}\right)$,
- $(\diamond\mathbf{4})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\gamma}{2}, \frac{\eta^9\gamma^3}{384\cdot 10^{22}(\Omega^*)^5}\right)$, *or*
- $(\diamond\mathbf{5})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \frac{\sqrt[4]{\Omega^{**}}}{2}, \frac{\eta^9}{128\cdot 10^{22}\cdot(\Omega^*)^3}, \frac{\eta}{2}, \frac{\eta^9}{128\cdot 10^{22}\cdot(\Omega^*)^4}\right)$.

Lemma 7.33. *Suppose that we are in Setting 7.4 and Setting 7.7. If there exist two disjoint sets* $\mathbb{YA}_1, \mathbb{YA}_2 \subseteq V(G)$ *such that*

$$e_{G_{\mathrm{exp}}}(\mathbb{YA}_1, \mathbb{YA}_2) \geqslant 2\rho k n\ , \tag{7.86}$$

and either

$$\mathbb{YA}_1 \cup \mathbb{YA}_2 \subseteq \mathbb{XA}^{\restriction 0} \setminus (\mathtt{P} \cup \bar{V} \cup \mathbb{F}),\ or \tag{7.87}$$

$$\mathbb{YA}_1 \subseteq \mathbb{XA}^{\restriction 0} \setminus (\mathtt{P} \cup \bar{V} \cup \mathbb{F} \cup \mathtt{P}_2 \cup \mathtt{P}_3),\ and\ \mathbb{YA}_2 \subseteq \mathbb{XB}^{\restriction 0} \setminus (\mathtt{P} \cup \bar{V} \cup \mathbb{F}) \tag{7.88}$$

then G *has configuration* $(\diamond\mathbf{6})(\frac{\eta^3\rho^4}{10^{14}(\Omega^*)^3}, 0, 1, 1, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k)$.

Lemma 7.34. *Suppose that we are in Setting 7.4 and Setting 7.7. Let* $\mathcal{D}_\nabla$ *be as in Lemma 7.5. Suppose that there exists an* $(\bar{\varepsilon}, \bar{d}, \beta k)$*-semiregular matching* $\mathcal{M}$*,* $V(\mathcal{M}) \subseteq \mathfrak{P}_0$*,* $|V(\mathcal{M})| \geqslant \frac{\rho n}{\Omega^*}$*, with one of the following two sets of properties.*

(M1) $\mathcal{M}$ *is absorbed by* $\mathcal{M}_{\mathrm{good}}$*,* $\bar{\varepsilon} := \frac{10^5\varepsilon'}{\eta^2}$*,* $\bar{d} := \frac{\gamma^2}{4}$*, and* $\beta := \frac{\eta^2\mathfrak{c}}{8\cdot 10^3 k}$.

(M2) $E(\mathcal{M}) \subseteq E(\mathcal{D}_\nabla)$*,* $\mathcal{M}$ *is absorbed by* $\mathcal{D}_\nabla$*,* $\bar{\varepsilon} := \varepsilon_\odot$*,* $\bar{d} := \frac{\gamma^3\rho}{32\Omega^*}$*, and* $\beta := \frac{\alpha_\odot\rho}{\Omega^*}$.

Suppose further that one of the following occurs.

(cA) $V(\mathcal{M}) \subseteq \mathbb{XA}^{\restriction 0} \setminus (\mathtt{P} \cup \bar{V} \cup \mathbb{F})$*, and we have for the set*

$$R := \mathbf{shadow}_{G_\nabla}\left((V_{\rightsquigarrow\mathfrak{A}} \cap \mathbb{L}_{\eta,k}(G)) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), \frac{2\eta^2 k}{10^5}\right)$$

one of the following

(t1) $V_1(\mathcal{M}) \subseteq \mathbf{shadow}_{G_\nabla}(V(G_{\mathrm{exp}}), \rho k)$,

(t2) $V_1(\mathcal{M}) \subseteq V_{\rightsquigarrow\mathfrak{A}}$,

(t3) $V_1(\mathcal{M}) \subseteq R \setminus (\mathbf{shadow}_{G_\nabla}(V(G_{\mathrm{exp}}), \rho k) \cup V_{\rightsquigarrow\mathfrak{A}})$*, or*

(t5) $V(\mathcal{M}) \subseteq V(G_{\mathrm{reg}}) \setminus (\mathbf{shadow}_{G_\nabla}(V(G_{\mathrm{exp}}), \rho k) \cup V_{\rightsquigarrow\mathfrak{A}} \cup R)$.

(cB) $V_1(\mathcal{M}) \subseteq \mathbb{XA}^{\restriction 0} \setminus (\mathtt{P} \cup \mathtt{P}_2 \cup \mathtt{P}_3 \cup \bar{V} \cup \mathbb{F})$ *and* $V_2(\mathcal{M}) \subseteq \mathbb{XB}^{\restriction 0} \setminus (\mathtt{P} \cup \bar{V} \cup \mathbb{F})$*, and we have*

(t1) $V_1(\mathcal{M}) \subseteq \mathbf{shadow}_{G_\nabla}(V(G_{\mathrm{exp}}), \rho k)$,

(t2) $V_1(\mathcal{M}) \subseteq V_{\rightsquigarrow\mathfrak{A}}$, *or*

(t3–5) $V_1(\mathcal{M}) \cap (\mathbf{shadow}_{G_\nabla}(V(G_{\mathrm{exp}}), \rho k) \cup V_{\rightsquigarrow\mathfrak{A}}) = \emptyset$.

then at least one of the following configurations occurs:

- $(\diamond\mathbf{6})\big(\frac{\eta^3\rho^4}{10^{12}(\Omega^*)^4}, 4\varepsilon_\odot, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2000}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{7})\big(\frac{\eta^3\gamma^3\rho}{10^{12}(\Omega^*)^4}, \frac{\eta\gamma}{400}, 4\varepsilon_\odot, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2000}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{8})\big(\frac{\eta^4\gamma^4\rho}{10^{15}(\Omega^*)^5}, \frac{\eta\gamma}{400}, \frac{400\varepsilon}{\eta}, 4\varepsilon_\odot, \frac{d}{2}, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta\pi\mathfrak{c}}{200k}, \frac{\eta^2\nu}{2\cdot 10^4}, \mathfrak{p}_1(1+\frac{\eta}{20})k, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{9})\big(\frac{\rho\eta^8}{10^{27}(\Omega^*)^3}, \frac{2\eta^3}{10^3}, \mathfrak{p}_1(1+\frac{\eta}{40})k, \mathfrak{p}_2(1+\frac{\eta}{20})k, \frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi\mathfrak{c}}{200k}, 4\varepsilon_\odot, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}\big)$,
- $(\diamond\mathbf{10})\big(\varepsilon, \frac{\gamma^2 d}{2}, \pi\sqrt{\varepsilon'}\nu k, \frac{2(\Omega^*)^2 k}{\gamma^2}, \frac{\eta}{40}\big)$.

We finish this section with an auxiliary result which will be used in the proofs of Lemmas 7.33 and 7.34.

Lemma 7.35. *Assume Settings 7.4 and 7.7. We have that*

$$\mathbb{XA}^{\restriction 0} \setminus (\mathtt{P} \cup \mathbb{F}) \subseteq \mathfrak{P}_0 \setminus \left(\mathbb{F} \cup \mathbf{shadow}_{G_\mathcal{D}}(V_{\not\rightsquigarrow\mathbf{\Psi}}, \frac{\eta^2 k}{10^5})\right) , \tag{7.89}$$

$$\mathrm{deg^{min}}_{G_\nabla}\left(\mathbb{XA} \setminus (\mathtt{P} \cup \bar{V}), V_{\mathrm{good}}^{\restriction 2}\right) \geqslant \mathfrak{p}_2(1+\frac{\eta}{20})k , \tag{7.90}$$

$$\mathrm{deg^{min}}_{G_\nabla}\left(\mathbb{XB} \setminus (\mathtt{P} \cup \bar{V}), V_{\mathrm{good}}^{\restriction 2}\right) \geqslant \mathfrak{p}_2(1+\frac{\eta}{20})\frac{k}{2} , \text{ and} \tag{7.91}$$

$$\mathrm{deg^{max}}_{G_\nabla}\left(\mathbb{XA} \setminus (\mathtt{P}_2 \cup \mathtt{P}_3), \bigcup \mathcal{F}\right) \leqslant \frac{3\eta^3}{2\cdot 10^3}k . \tag{7.92}$$

Moreover, $\mathcal{F}$ defined in (7.12) *is an $(\mathcal{M}_A \cup \mathcal{M}_B)$-cover.*

Proof. The definition of $\mathtt{P}$ gives (7.89).

Observe that

$$\begin{aligned}
&\mathrm{deg^{min}}_{G_\nabla}\left(\mathbb{YA} \setminus (V_{\not\rightsquigarrow\mathbf{\Psi}} \cup \bar{V}), V_{\mathrm{good}}^{\restriction 2}\right) \\
\text{(by Def 7.6(6))} \quad &\geqslant \mathfrak{p}_2 \cdot \mathrm{deg^{min}}_{G_\nabla}(\mathbb{YA} \setminus V_{\not\rightsquigarrow\mathbf{\Psi}}, V_{\mathrm{good}}) - k^{0.9} \\
\text{(by (7.8))} \quad &\geqslant \mathfrak{p}_2 \cdot \left(\mathrm{deg^{min}}_{G_\nabla}(\mathbb{YA}, V_+ \setminus L_\#) - \mathrm{deg^{max}}_{G_\nabla}(\mathbb{YA} \setminus V_{\not\rightsquigarrow\mathbf{\Psi}}, \mathbf{\Psi})\right) - k^{0.9} \\
\text{(by (7.9), (7.11))} \quad &\geqslant \mathfrak{p}_2 \cdot \left((1+\frac{\eta}{10})k - \frac{\eta k}{100}\right) - k^{0.9} \\
\text{(by (7.3), (7.14))} \quad &\geqslant \mathfrak{p}_2 \cdot (1+\frac{\eta}{20})k ,
\end{aligned}$$

which proves (7.90), as $\mathbb{XA} \setminus (\mathtt{P} \cup \bar{V}) \subseteq \mathbb{YA} \setminus (V_{\not\rightsquigarrow\mathbf{\Psi}} \cup \bar{V})$. Similarly, we obtain that

$$\mathrm{deg^{min}}_{G_\nabla}\left(\mathbb{YB} \setminus (V_{\not\rightsquigarrow\mathbf{\Psi}} \cup \bar{V}), V_{\mathrm{good}}^{\restriction 2}\right) \geqslant \mathfrak{p}_2(1+\frac{\eta}{20})\frac{k}{2} ,$$

which proves (7.91).

We have $\deg^{\max}{}_{G_\nabla}(\mathbb{XA}\setminus \mathtt{P}_3, \mathbb{XA}) < \frac{\eta^3}{10^3}k$, and $\deg^{\max}{}_{G_\nabla}(\mathbb{XA}\setminus \mathtt{P}_2, S^0\setminus V(\mathcal{M}_A)) < \sqrt{\gamma}k$. Thus (7.92) follows from Setting 7.4(2) and by (7.3).

For the "moreover" part, it suffices to prove that $\{C\in\mathcal{V}(\mathcal{M}_A) : C\subseteq \mathbb{XA}\} = \mathcal{F}\setminus \mathcal{V}_1(\mathcal{M}_B)$ is an $\mathcal{M}_A$-cover. Let $(T_1,T_2)\subseteq \mathcal{M}_A$. As $G\in \mathbf{LKSsmall}(n,k,\eta)$, we have by Setting 7.4(3) that for some $i\in\{1,2\}$, T_i is contained in $\mathbb{L}_{\eta,k}(G)$. Then by Setting 7.4(1), $T_i\subseteq \mathbb{XA}$, as desired.

□

7.7.2 Proof of Lemma 7.31

In the proof, we distinguish different types of edges captured in cases **(K1)** and **(K2)**. If in case **(K1)** many of the captured edges from $\mathbb{XA}$ to $\mathbb{XA}\cup\mathbb{XB}$ are incident with $\mathbf{\Psi}$, we will get one of the configurations ($\diamond$**1**)–($\diamond$**5**) by employing Lemma 7.32. Otherwise, there must be many edges from $\mathbb{XA}$ to $\mathbb{XA}\cup\mathbb{XB}$ in the graph G_{exp}, or in $G_{\mathcal{D}}$. Lemma 7.33 shows that the former case leads to configuration ($\diamond$**6**). We will reduce the latter case to the situation in Lemma 7.34 which gives one of the configurations ($\diamond$**6**)–($\diamond$**10**).

We use Lemma 7.34 to give one of the configurations ($\diamond$**6**)–($\diamond$**10**) also in case **(K2)**. [xxii]

Let us now turn to the details of the proof. If $e_G(\mathbf{\Psi},\mathbb{XA}\cup\mathbb{XB}) \geqslant \frac{\eta^{13}kn}{10^{28}(\Omega^*)^3}$ then we use Lemma 7.32 to obtain one of the configurations ($\diamond$**1**)–($\diamond$**5**), with the parameters as in the statement of Lemma 7.31.

Thus, in the remainder of the proof we assume that

$$e_G(\mathbf{\Psi},\mathbb{XA}\cup\mathbb{XB}) < \frac{\eta^{13}kn}{10^{28}(\Omega^*)^3}\ . \tag{7.93}$$

We now bound the size of the set $\mathtt{P}$. By Setting 7.4(9) we have at most $2\rho kn$ uncaptured edges. Plugging this into Lemma 7.9 we get $|L_\#|\leqslant \frac{40\rho n}{\eta}$, $|\mathbb{XA}\setminus\mathbb{YA}|\leqslant \frac{1200\rho n}{\eta^2}$, and $|(\mathbb{XA}\cup\mathbb{XB})\setminus\mathbb{YB}|\leqslant\frac{1200\rho n}{\eta^2}$. Further, using (7.93), Lemma 7.9 also gives that $|V_{\not\leadsto\mathbf{\Psi}}|\leqslant \frac{\eta^{12}n}{10^{26}(\Omega^*)^3}$. It follows from Setting 7.4(8) that $|\mathtt{P}_{\mathfrak{A}}|\leqslant\gamma n$. Last, by Setting 7.4(7) we have $|\mathtt{P}_1|\leqslant 2\gamma n$. Thus,

$$\begin{aligned}|\mathtt{P}| &\leqslant |\mathbb{XA}\setminus\mathbb{YA}| + |(\mathbb{XA}\cup\mathbb{XB})\setminus\mathbb{YB}| + |V_{\not\leadsto\mathbf{\Psi}}| + |L_\#| \\ &\quad + \left|\mathbf{shadow}_{G_{\mathcal{D}}\cup G_\nabla}(V_{\not\leadsto\mathbf{\Psi}}\cup L_\#\cup \mathtt{P}_{\mathfrak{A}}\cup\mathtt{P}_1, \frac{\eta^2 k}{10^5})\right| \\ &\overset{\text{by (7.3)}}{\leqslant} \frac{2\eta^{10}n}{10^{21}(\Omega^*)^2}\ ,\end{aligned} \tag{7.94}$$

[xxii] Actually, our proof of Lemma 7.34 implies that one does not get configuration ($\diamond$**9**) in case **(K2)**; but this fact is not needed for the proof to work.

where we used Fact 7.1 to bound the size of the shadows.

Let us first turn our attention to case **(K1)**. By Definition 7.6 we have $\mathbf{\Psi}\cap\mathfrak{P}_0=\emptyset$. Therefore,

$$\begin{aligned}
e_{G_\nabla}\big(\mathbb{XA}^{\restriction 0}\setminus \mathtt{P},(\mathbb{XA}\cup\mathbb{XB})^{\restriction 0}\setminus \mathtt{P}\big) &= e_{G_\nabla}\big((\mathbb{XA}\setminus(\mathbf{\Psi}\cup\mathtt{P}))^{\restriction 0},(\mathbb{XA}\setminus(\mathbf{\Psi}\cup\mathtt{P}))^{\restriction 0}\cup(\mathbb{XB}\setminus\mathtt{P})^{\restriction 0}\big)\\
\text{(by Def 7.6 (7))}\quad &\geqslant \mathfrak{p}_0^2\cdot e_{G_\nabla}\big(\mathbb{XA}\setminus(\mathbf{\Psi}\cup\mathtt{P}),(\mathbb{XA}\cup\mathbb{XB})\setminus(\mathbf{\Psi}\cup\mathtt{P})\big)-k^{0.6}n^{0.6}\\
\text{(by (7.14))}\quad &\geqslant \frac{\eta^2}{10^4}\big(e_{G_\nabla}(\mathbb{XA},\mathbb{XA}\cup\mathbb{XB})-2e_{G_\nabla}(\mathbf{\Psi},\mathbb{XA}\cup\mathbb{XB})-2|\mathtt{P}|\Omega^*k\big)-k^{0.6}n^{0.6}\\
\text{(by (K1), (7.93), (7.94))}\quad &\geqslant \frac{\eta^2}{10^4}\Big(\frac{\eta kn}{4}-\frac{2\eta^{13}kn}{10^{28}(\Omega^*)^3}-\frac{4\eta^{10}kn}{10^{21}\Omega^*}\Big)-k^{0.6}n^{0.6}\\
&> \frac{\eta^3kn}{10^5}\;. && (7.95)
\end{aligned}$$

We consider the following two complementary cases:

(cA) $e_{G_\nabla}((\mathbb{XA}\setminus\mathtt{P})^{\restriction 0})\geqslant 40\rho kn.$

(cB) $e_{G_\nabla}((\mathbb{XA}\setminus\mathtt{P})^{\restriction 0})< 40\rho kn.$

Note that $\mathbb{XA}\setminus\mathtt{P}\subseteq\mathbb{YA}$, and $(\mathbb{XA}\cup\mathbb{XB})\setminus\mathtt{P}\subseteq\mathbb{YB}$.

In Case (**cA**) there are disjoint sets $\mathbb{YA}_1,\mathbb{YA}_2\subseteq(\mathbb{XA}\setminus(\mathtt{P}\cup\bar{V}\cup\mathbb{F}))^{\restriction 0}\subseteq\mathbb{YA}$ with

$$\begin{aligned}
e_{G_\nabla}(\mathbb{YA}_1,\mathbb{YA}_2) &\geqslant\frac{1}{2}(e_{G_\nabla}(\mathbb{XA}\setminus\mathtt{P})^{\restriction 0}-|\bar{V}\cup\mathbb{F}|\cdot\Omega^*k)\\
\text{(by Def 7.6(1) and by (7.16))}\quad &\geqslant\frac{1}{2}(40\rho kn-2\varepsilon\Omega^*kn)>19\rho kn\;. && (7.96)
\end{aligned}$$

Let us now introduce some setting specific to Case (**cB**). Property 6 of Setting 7.4 implies that

$$|\mathtt{P}_2|\leqslant\sqrt{\gamma}n\;. \tag{7.97}$$

Also, by Definition 7.6(7) we have

$$\begin{aligned}
e_{G_\nabla}(\mathbb{XA}) &\leqslant\frac{1}{\mathfrak{p}_0^2}\big(e_{G_\nabla}((\mathbb{XA}\setminus\mathtt{P})^{\restriction 0})+k^{0.6}n^{0.6}\big)+e_{G_\nabla}(\mathbf{\Psi},\mathbb{XA})+|\mathtt{P}|\Omega^*k\\
\text{(by (7.14), (cB), (7.93), and (7.94))}\quad &\leqslant\frac{10^4}{\eta^2}\cdot\big(40\rho kn+k^{0.6}n^{0.6}\big)+\frac{\eta^{13}}{10^{28}(\Omega^*)^3}kn+\frac{\eta^{10}}{10^{20}\Omega^*}kn\\
\text{(by (7.3))}\quad &<\frac{\eta^8}{10^{15}\Omega^*}kn\;.
\end{aligned}$$

Consequently,

$$|\mathtt{P}_3|\cdot\frac{\eta^3k}{10^3}\leqslant e_{G_\nabla}(\mathtt{P}_3,\mathbb{XA})\leqslant 2\cdot\frac{\eta^8}{10^{15}\Omega^*}kn,$$

and thus,

$$|\mathtt{P}_3|\leqslant 2\cdot\frac{\eta^5}{10^{12}\Omega^*}n\;. \tag{7.98}$$

Set $\mathbb{YA}_1 := (\mathbb{XA} \setminus (\mathtt{P} \cup \mathtt{P}_2 \cup \mathtt{P}_3 \cup \bar{V} \cup \mathbb{F}))^{\restriction 0} \subseteq \mathbb{YA}$ and $\mathbb{YA}_2 := (\mathbb{XB} \setminus (\mathtt{P} \cup \bar{V} \cup \mathbb{F}))^{\restriction 0} \subseteq \mathbb{YB}$. Then the sets $\mathbb{YA}_1$ and $\mathbb{YA}_2$ are disjoint and we have

$$\begin{aligned} e_{G_\nabla}(\mathbb{YA}_1, \mathbb{YA}_2) &\geqslant e_{G_\nabla}\left((\mathbb{XA} \setminus \mathtt{P})^{\restriction 0}, ((\mathbb{XA} \cup \mathbb{XB}) \setminus \mathtt{P})^{\restriction 0}\right) - 2e_{G_\nabla}((\mathbb{XA} \setminus \mathtt{P})^{\restriction 0}) \\ &\quad - (|\mathtt{P}_2| + |\mathtt{P}_3| + 2|\bar{V}| + 2|\mathbb{F}|) \cdot \Omega^* k \\ \text{(by (7.95), (\textbf{cB}), (7.97), (7.98), D7.6(1), (7.16))} \quad &\geqslant \frac{\eta^3 kn}{10^5} - 80\rho kn - \sqrt{\gamma}\Omega^* kn - \frac{2\eta^5}{10^{12}} kn - 4\varepsilon\Omega^* kn \\ &\overset{(7.3)}{\geqslant} 19\rho kn \ . \end{aligned} \tag{7.99}$$

This finishes the setting specific to Case (**cB**). We resume considering Cases (**cA**) and (**cB**) simultaneously.

By (7.96) and (7.99) we have at least one of the following cases.

(L1) $e_{G_{\mathrm{exp}}}(\mathbb{YA}_1, \mathbb{YA}_2) \geqslant 2\rho kn$, or

(L2) $e_{G_{\mathcal{D}}}(\mathbb{YA}_1, \mathbb{YA}_2) \geqslant 17\rho kn$.

In the case **(L1)**, Lemma 7.33 outputs Configuration $(\diamond\mathbf{6})(\frac{\eta^3\rho^4}{10^{14}(\Omega^*)^3}, 0, 1, 1, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k)$.

Let us now consider the case **(L2)**. We fix a family $\mathcal{D}_\nabla$ as in Lemma 7.5. We have $e_{\mathcal{D}_\nabla}(\mathbb{YA}_1, \mathbb{YA}_2) \geqslant 16\rho kn$.

Let $R := \mathbf{shadow}_{G_\nabla}\left((V_{\rightsquigarrow\mathfrak{A}} \cap \mathbb{L}_{\eta,k}(G)) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), \frac{2\eta^2 k}{10^5}\right)$. For $i = 1, 2$ define

$$\begin{aligned} \mathbb{Y}_i^{(1)} &:= \mathbf{shadow}_G(V(G_{\mathrm{exp}}), \rho k) \cap \mathbb{YA}_i \ , \\ \mathbb{Y}_i^{(2)} &:= (V_{\rightsquigarrow\mathfrak{A}} \cap \mathbb{YA}_i) \setminus \mathbb{Y}_i^{(1)} \ , \\ \mathbb{Y}_i^{(3)} &:= (R \cap \mathbb{YA}_i) \setminus (\mathbb{Y}_i^{(1)} \cup \mathbb{Y}_i^{(2)}) \ , \\ \mathbb{Y}_i^{(4)} &:= (\mathfrak{A} \cap \mathbb{YA}_i) \setminus (\mathbb{Y}_i^{(1)} \cup \mathbb{Y}_i^{(2)} \cup \mathbb{Y}_i^{(3)}) \ , \\ \mathbb{Y}_i^{(5)} &:= \mathbb{YA}_i \setminus (\mathbb{Y}_i^{(1)} \cup \ldots \cup \mathbb{Y}_i^{(4)}) \ . \end{aligned} \tag{7.100}$$

We consider subcase **(cA)**. We shall distinguish four cases based on the majority type of edges from $\mathbb{YA}_1$ to $\mathbb{YA}_2$.

Lemma 7.36. *We have one of the following.*

(t1) $e_{\mathcal{D}_\nabla}(\mathbb{Y}_1^{(1)}, \mathbb{YA}_2) + e_{\mathcal{D}_\nabla}(\mathbb{YA}_1, \mathbb{Y}_2^{(1)}) \geqslant 4\rho kn$,

(t2) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(2)}, \mathbb{YA}_2 \setminus \mathbb{Y}_2^{(1)}\right) + e_{\mathcal{D}_\nabla}\left(\mathbb{YA}_1 \setminus \mathbb{Y}_1^{(1)}, \mathbb{Y}_2^{(2)}\right) \geqslant 4\rho kn$,

(t3) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(3)}, \mathbb{YA}_2 \setminus (\mathbb{Y}_2^{(1)} \cup \mathbb{Y}_2^{(2)})\right) + e_{\mathcal{D}_\nabla}\left(\mathbb{YA}_1 \setminus (\mathbb{Y}_1^{(1)} \cup \mathbb{Y}_1^{(2)}), \mathbb{Y}_2^{(3)}\right) \geqslant 4\rho kn$, *or*

(t5) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(5)}, \mathbb{Y}_2^{(5)}\right) \geqslant 2\rho kn$.

Proof. We only need to establish that

$$e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(4)}, \mathbb{YA}_2 \setminus (\mathbb{Y}_2^{(1)} \cup \mathbb{Y}_2^{(2)} \cup \mathbb{Y}_2^{(3)})\right) + e_{\mathcal{D}_\nabla}\left(\mathbb{YA}_1 \setminus (\mathbb{Y}_1^{(1)} \cup \mathbb{Y}_1^{(2)} \cup \mathbb{Y}_1^{(3)}), \mathbb{Y}_2^{(4)}\right) < \rho kn .$$

As $\mathbb{Y}_1^{(4)} \subseteq \mathfrak{A}$ and $\mathbb{YA}_2 \setminus (\mathbb{Y}_2^{(1)} \cup \mathbb{Y}_2^{(2)} \cup \mathbb{Y}_2^{(3)})$ is disjoint from $V_{\leadsto\mathfrak{A}}$ we have

$$e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(4)}, \mathbb{YA}_2 \setminus (\mathbb{Y}_2^{(1)} \cup \mathbb{Y}_2^{(2)} \cup \mathbb{Y}_2^{(3)})\right) < \frac{\rho kn}{100\Omega^*} .$$

The other summand can be bound by a symmetric argument. □

In subcase **(cB)** we clearly have one of the following five possibilities.

(t1) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(1)}, \mathbb{YA}_2\right) \geqslant 2\rho kn$,

(t2) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(2)}, \mathbb{YA}_2\right) \geqslant 2\rho kn$,

(t3) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(3)}, \mathbb{YA}_2\right) \geqslant 2\rho kn$,

(t4) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(4)}, \mathbb{YA}_2\right) \geqslant 2\rho kn$, or

(t5) $e_{\mathcal{D}_\nabla}\left(\mathbb{Y}_1^{(5)}, \mathbb{YA}_2\right) \geqslant 2\rho kn$.

In both subcase **(cA)** and subcase **(cB)** we claim the following.

Lemma 7.37. *Let G^* be the spanning subgraph of $G_\mathcal{D}$ formed by the edges of $\mathcal{D}_\nabla$. If there are two disjoint sets Z_1 and Z_2 with $e_{G^*}(Z_1, Z_2) \geqslant 2\rho kn$ then there exists an $(\varepsilon_\odot, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\alpha_\odot \rho k}{\Omega^*})$-semiregular matching $\mathcal{N}$ in G^* with $V_i(\mathcal{N}) \subseteq Z_i$ $(i = 1, 2)$, and $|V(\mathcal{N})| \geqslant \frac{\rho n}{\Omega^*}$.*

Proof. As the maximum degree G^* is bounded by $\Omega^* k$, we have $|Z_1| \geqslant \frac{2\rho n}{\Omega^*} \geqslant \frac{2\rho k}{\Omega^*}$. Thus,

$$(G^*, \mathcal{D}_\nabla, G^*[Z_1, Z_2], \{Z_1\}) \in \mathcal{G}\left(v(G_\mathcal{D}), k, \Omega^*, \frac{\gamma^3}{4}, \frac{\rho}{\Omega^*}, 2\rho\right) .$$

Lemma 5.6 immediately gives the desired output. □

We use Lemma 7.37 with Z_1, Z_2 being the pair of sets containing many edges as in the cases **(t1)**–**(t3)** and **(t5)** (in subcase **(cA)**)[xxiii] and **(t1)**–**(t5)** (in subcase **(cB)**). The lemma outputs a semiregular matching $\mathcal{M}_{\triangleright L7.34} := \mathcal{N}_{\triangleright L7.37}$. This matching is a basis of the input for Lemma 7.34**(M2)** (subcase **(t1)**–**(t3)**, **(t5)**, or **(t3–5)**). Thus, we get one of the configurations ($\diamond$**6**)–($\diamond$**10**) as in the statement of the lemma.

[xxiii] The quantities in Lemma 7.36 have two summands. We take the sets Z_1,Z_2 as those appearing in the majority summand.

Now we turn our attention to case **(K2)**. For every pair $(X,Y)\in\mathcal{M}_{\mathrm{good}}$, let $X'\subseteq X^{\restriction 0}\setminus(\mathtt{P}\cup\bar{V}\cup\mathbb{F})$ and $Y'\subseteq Y^{\restriction 0}\setminus(\mathtt{P}\cup\bar{V}\cup\mathbb{F})$ maximal with $|X'|=|Y'|$. Define $\mathcal{N}:=\{(X',Y')\,:\,(X,Y)\in\mathcal{M}_{\mathrm{good}}\ ,\ |X'|\geqslant\frac{\eta^2\mathfrak{c}}{2\cdot 10^3}\}$. We have

$$\begin{aligned}|V(\mathcal{N})| &\geqslant |V(\mathcal{M}_{\mathrm{good}}^{\restriction 0})|-2|\mathtt{P}\cup\bar{V}\cup\mathbb{F}|-2\frac{\eta^2 n}{2\cdot 10^3}\\ \text{(by (K2), (7.17),(7.94), Def7.6(1), (7.16))}\quad &\geqslant \frac{\eta^2 n}{400}-\frac{4\cdot\eta^{10}n}{10^{21}(\Omega^*)^2}-4\varepsilon n-\frac{\eta^2 n}{10^3}\\ &> \frac{\eta^2 n}{1000}\ .\end{aligned}$$

By Fact 2.7, $\mathcal{N}$ is a $(\frac{4\cdot 10^3\varepsilon'}{\eta^2},\frac{\gamma^2}{2},\frac{\eta^2\mathfrak{c}}{2\cdot 10^3})$-semiregular matching.

We use definitions of the sets $\mathbb{Y}_i^{(1)},\ldots,\mathbb{Y}_i^{(5)}$ as given in (7.100) with $\mathbb{YA}_i:=V_i(\mathcal{N})$ $(i=1,2)$. As $V(\mathcal{N})\subseteq V(G_{\mathrm{reg}})$, we have that $\mathbb{Y}_i^{(4)}=\emptyset$ $(i=1,2)$. A set $X\in\mathcal{V}_i(\mathcal{N})$ is said to be of *Type 1* if $\left|X\cap\mathbb{Y}_i^{(1)}\right|\geqslant\frac{1}{4}|X|$. Analogously, we define elements of $\mathcal{V}(\mathcal{N})$ of *Type 2*, *Type 3*, and *Type 5*.

Recall that we are in subcase (**cA**) as $V(\mathcal{M}_{\mathrm{good}})\subseteq\mathbb{XA}$. For each $(X_1,X_2)\in\mathcal{N}$ with at least one $X_i\in\{X_1,X_2\}$ being of Type 1, set $X_i':=X_i\cap\mathbb{Y}_i^{(1)}$ and take an arbitrary set $X_{3-i}'\subseteq X_{3-i}$ of size $|X_i'|$. Note that by Fact 2.7 (X_i',X_{3-i}') forms a $\frac{10^5\varepsilon'}{\eta^2}$-regular pair of density at least $\gamma^2/4$. We let $\mathcal{N}_1$ be the semiregular matching consisting of all pairs (X_i',X_{3-i}') obtained in this way.[xxiv]

Likewise, we construct $\mathcal{N}_2,\mathcal{N}_3$ and $\mathcal{N}_5$ using the features of Type 2, 3, and 5.

Since we included at least one quarter of each $\mathcal{N}$-edge into one of $\mathcal{N}_1,\mathcal{N}_2,\mathcal{N}_3$ and $\mathcal{N}_5$, we obtain one of the inputs of Lemma 7.34**(M1)**. Thus, we get one of the configurations ($\diamond$**6**)–($\diamond$**10**) as in the statement of the lemma.

7.7.3 Proof of Lemma 7.32

Set $\tilde{\eta}:=\frac{\eta^{13}}{10^{28}(\Omega^*)^3}$.

Define $\mathrm{N}^{\uparrow}:=\{v\in V(G)\,:\,\deg_{G_\nabla}(v,\mathbf{\Psi})\geqslant k\}$, and $\mathrm{N}^{\downarrow}:=\mathrm{N}_{G_\nabla}(\mathbf{\Psi})\setminus\mathrm{N}^{\uparrow}$. Recall that by the definition of the class **LKSsmall**(n,k,η), the set $\mathbf{\Psi}$ is independent, and thus the sets $\mathrm{N}^{\uparrow}$ and $\mathrm{N}^{\downarrow}$ are disjoint from $\mathbf{\Psi}$. Also, using the same definition, we have

$$\mathrm{N}_{G_\nabla}(\mathbf{\Psi})\subseteq\mathbb{L}_{\eta,k}(G)\setminus\mathbf{\Psi}\quad\text{, and thus}\tag{7.101}$$

$$e_{G_\nabla}(\mathbf{\Psi},B)=e_{G_\nabla}(\mathbf{\Psi},B\cap\mathbb{L}_{\eta,k}(G))\text{ for any }B\subseteq V(G).\tag{7.102}$$

We shall distinguish two cases.

Case A: $e_{G_\nabla}(\mathbf{\Psi},\mathrm{N}^{\uparrow})\geqslant e_{G_\nabla}(\mathbf{\Psi},\mathbb{XA}\cup\mathbb{XB})/8$.

Let us focus on the bipartite subgraph H' of G_∇ induced by the sets $\mathbf{\Psi}$ and $\mathrm{N}^{\uparrow}$. Obviously, the average degree of the vertices of $\mathrm{N}^{\uparrow}$ in H' is at least k.

[xxiv] Note that we are thus changing the orientation of some subpairs.

First, suppose that $|\mathbf{\Psi}| \leqslant |\mathrm{N}^{\uparrow}|$. Then, the average degree of $\mathbf{\Psi}$ in H' is at least k, and hence, the average degree of H' is at least k. Thus, there exists a bipartite subgraph $H \subseteq H'$ with $\deg^{\min}(H) \geqslant k/2$. Furthermore, $\deg^{\min}{}_{G_\nabla}(V(H)) \geqslant k$. We conclude that we are in Configuration $(\diamond\mathbf{1})$.

Now, suppose $|\mathbf{\Psi}| > |\mathrm{N}^{\uparrow}|$. Using the bounds given by Case A, and using (7.85), we get

$$|\mathrm{N}^{\uparrow}| \geqslant \frac{e_{G_\nabla}(\mathbf{\Psi}, \mathrm{N}^{\uparrow})}{\Omega^* k} \geqslant \frac{\tilde{\eta} kn}{8\Omega^* k} = \frac{\tilde{\eta} n}{8\Omega^*} \; .$$

Therefore, we have

$$e(G) \geqslant \sum_{v \in \mathbf{\Psi}} \deg_{G_\nabla}(v) \geqslant |\mathbf{\Psi}|\Omega^{**} k > |\mathrm{N}^{\uparrow}|\Omega^{**} k \geqslant \frac{\tilde{\eta} n}{8\Omega^*}\Omega^{**} k \overset{(7.3)}{\geqslant} kn \; ,$$

a contradiction to Property 3 of Definition 2.6.

Case B: $e_{G_\nabla}(\mathbf{\Psi}, \mathrm{N}^{\uparrow}) < e_{G_\nabla}(\mathbf{\Psi}, \mathbb{XA} \cup \mathbb{XB})/8$.
Consequently, we get

$$e_{G_\nabla}(\mathbf{\Psi}, (\mathbb{XA} \cup \mathbb{XB}) \setminus \mathrm{N}^{\uparrow}) \geqslant \frac{7}{8} e_{G_\nabla}(\mathbf{\Psi}, \mathbb{XA} \cup \mathbb{XB}) \overset{(7.85)}{\geqslant} \frac{7}{8}\tilde{\eta} kn \; . \tag{7.103}$$

We now apply Lemma 7.26 to G_∇ with input sets $P_{\triangleright \mathrm{L7.26}} := \mathbf{\Psi}$, $Q_{\triangleright \mathrm{L7.26}} := \mathbb{L}_{\eta,k}(G) \setminus \mathbf{\Psi}$, $Y_{\triangleright \mathrm{L7.26}} := \mathbb{L}_{\eta,k}(G) \setminus \mathbb{L}_{\frac{9}{10}\eta,k}(G_\nabla)$, and parameters $\psi_{\triangleright \mathrm{L7.26}} := \tilde{\eta}/100$, $\Gamma_{\triangleright \mathrm{L7.26}} := \Omega^*$, and $\Omega_{\triangleright \mathrm{L7.26}} := \Omega^{**}$. Assumption (7.60) of the lemma follows from (7.101). The lemma yields three sets $L'' := Q''_{\triangleright \mathrm{L7.26}}$, $L' := Q'_{\triangleright \mathrm{L7.26}}$, $\mathbf{\Psi}' := P'_{\triangleright \mathrm{L7.26}}$, and it is easy to check that these witness Preconfiguration $(\clubsuit)(\frac{\tilde{\eta}^3 \Omega^{**}}{4 \cdot 10^6 (\Omega^*)^2})$.

Recall that $e(G) \leqslant kn$. Since by the definition of $Y_{\triangleright \mathrm{L7.26}}$, we have $|Y_{\triangleright \mathrm{L7.26}}| \leqslant \frac{40\rho}{\eta} n$, we obtain from Lemma 7.26(d) that

$$\begin{aligned}
e_{G_\nabla}(\mathbf{\Psi}, \mathbb{L}_{\eta,k}(G)) - e_{G_\nabla}(\mathbf{\Psi}', L'') &\leqslant \frac{\tilde{\eta}}{100} e_{G_\nabla}(\mathbf{\Psi}, \mathbb{L}_{\eta,k}(G)) + \frac{|Y_{\triangleright \mathrm{L7.26}}| 200(\Omega^*)^2 k}{\tilde{\eta}} \\
&\leqslant \frac{\tilde{\eta}}{100} kn + \frac{40\rho n}{\eta} \cdot \frac{200(\Omega^*)^2 k}{\tilde{\eta}} \\
&\overset{(7.3)}{\leqslant} \frac{\tilde{\eta}}{2} kn.
\end{aligned} \tag{7.104}$$

So,

$$\begin{aligned}
e_{G_\nabla}\big(\mathbf{\Psi}', (L'' \cap (\mathbb{XA} \cup \mathbb{XB})) \setminus \mathrm{N}^{\uparrow}\big) &\geqslant e_{G_\nabla}\big(\mathbf{\Psi}, (\mathbb{L}_{\eta,k}(G) \cap (\mathbb{XA} \cup \mathbb{XB})) \setminus \mathrm{N}^{\uparrow}\big) \\
&\quad - \big(e_{G_\nabla}(\mathbf{\Psi}, \mathbb{L}_{\eta,k}(G)) - e_{G_\nabla}(\mathbf{\Psi}', L'')\big) \\
&= e_{G_\nabla}(\mathbf{\Psi}, (\mathbb{XA} \cup \mathbb{XB}) \setminus \mathrm{N}^{\uparrow}) \\
&\quad - \big(e_{G_\nabla}(\mathbf{\Psi}, \mathbb{L}_{\eta,k}(G)) - e_{G_\nabla}(\mathbf{\Psi}', L'')\big) \\
&\overset{(7.104)}{\geqslant} e_{G_\nabla}(\mathbf{\Psi}, (\mathbb{XA} \cup \mathbb{XB}) \setminus \mathrm{N}^{\uparrow}) - \frac{\tilde{\eta}}{2} kn \\
&\overset{(7.103)}{\geqslant} \frac{3}{8}\tilde{\eta} kn \; .
\end{aligned} \tag{7.105}$$

We define

$$\mathbf{\Psi}^* := \left\{ v \in \mathbf{\Psi'} \ : \ \deg_{G_\nabla}(v, L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow}) \geqslant \sqrt{\Omega^{**}} k \right\} \ .$$

Using that $e(G) \leqslant kn$, we shall show the following.

Lemma 7.38. *We have* $e_{G_\nabla}(\mathbf{\Psi}^*, L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow}) \geqslant \frac{1}{8}\tilde{\eta}kn$.

Proof. Suppose otherwise. Then by (7.105), we obtain that

$$e_{G_\nabla}(\mathbf{\Psi'} \setminus \mathbf{\Psi}^*, L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow}) \geqslant \frac{1}{4}\tilde{\eta}kn \ .$$

On the other hand, by the definition of $\mathbf{\Psi}^*$,

$$|\mathbf{\Psi'} \setminus \mathbf{\Psi}^*| \sqrt{\Omega^{**}} k \geqslant e_{G_\nabla}(\mathbf{\Psi'} \setminus \mathbf{\Psi}^*, L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow}) \ .$$

Consequently, we have

$$|\mathbf{\Psi'} \setminus \mathbf{\Psi}^*| \geqslant \frac{\tilde{\eta}kn}{4\sqrt{\Omega^{**}}k} = \frac{\tilde{\eta}n}{4\sqrt{\Omega^{**}}} \ .$$

Thus, as $\mathbf{\Psi}$ is independent,

$$e(G) \geqslant \sum_{v \in \mathbf{\Psi}} \deg_{G_\nabla}(v) \geqslant |\mathbf{\Psi}|\Omega^{**}k \geqslant |\mathbf{\Psi'} \setminus \mathbf{\Psi}^*|\Omega^{**}k \geqslant \frac{\tilde{\eta}}{4}\sqrt{\Omega^{**}}kn \overset{(7.3)}{>} kn \ ,$$

a contradiction. □

Let us define $O := \mathbf{shadow}_{G_\nabla}(\mathfrak{A}, \gamma k)$. Next, we define

$$\begin{aligned} N_1 &:= V(G_{\mathrm{exp}}) \cap L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow} \ , \\ N_2 &:= \mathfrak{A} \cap L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow} \ , \\ N_3 &:= O \cap L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow} \ , \text{ and} \\ N_4 &:= (L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow}) \setminus (N_1 \cup N_2 \cup N_3) \ . \end{aligned}$$

Observe that

$$O \cap N_4 = \emptyset \ . \tag{7.106}$$

Further, for $i = 1, \ldots, 4$ define

$$C_i := \left\{ v \in \mathbf{\Psi}^* \ : \ \deg_{G_\nabla}(v, N_i) \geqslant \deg_{G_\nabla}(v, L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow})/4 \right\} \ .$$

Easy counting gives that there exists an index $i \in [4]$ such that

$$e_{G_\nabla}(C_i, N_i) \geqslant \frac{1}{16} e_{G_\nabla}(\mathbf{\Psi}^*, L'' \cap (\mathbb{XA} \cup \mathbb{XB}) \cap \mathrm{N}^{\downarrow}) \overset{\mathrm{L}7.38}{\geqslant} \frac{1}{128}\tilde{\eta}kn \ . \tag{7.107}$$

Set $Y := (\mathbb{XA} \cup \mathbb{XB}) \setminus (\mathbb{YB} \cup \mathbf{\Psi}) = (\mathbb{XA} \cup \mathbb{XB}) \setminus \mathbb{YB}$, and $\eta_{\triangleright \mathrm{L}7.27} = \eta_{\triangleright \mathrm{L}7.28} := \frac{1}{128}\tilde{\eta}$. By Lemma 7.9 we have

$$|Y| < \frac{\eta_{\triangleright \mathrm{L}7.27} n}{4\Omega^*} \ . \tag{7.108}$$

We split the rest of the proof into four subcases according to the value of i.

Subcase B, $i = 1$.

We shall apply Lemma 7.27 with $r_{\triangleright L7.27} := 2$, $\Omega^*_{\triangleright L7.27} := \Omega^*, \Omega^{**}_{\triangleright L7.27} := \sqrt{\Omega^{**}}/4$, $\delta_{\triangleright L7.27} := \frac{\eta_{\triangleright L7.27}\rho^2}{100(\Omega^*)^2}$, $\gamma_{\triangleright L7.27} := \rho$, $\eta_{\triangleright L7.27}$, $X_0 := C_1$, $X_1 := N_1$, and $X_2 := V(G_{\mathrm{exp}})$, and Y, and the graph $G_{\triangleright L7.27}$, which is formed by the vertices of G, with all edges from $E(G_\nabla)$ that are in $E(G_{\mathrm{exp}})$ or that are incident with $\mathbf{\Psi}$. We briefly verify the assumptions of Lemma 7.27. First of all the choice of $\delta_{\triangleright L7.27}$ guarantees that $\left(\frac{3\Omega^*_{\triangleright L7.27}}{\gamma_{\triangleright L7.27}}\right)^2 \delta_{\triangleright L7.27} < \frac{\eta_{\triangleright L7.27}}{10}$. Assumption 1 is given by (7.108). Assumption 2 holds since we assume that (7.107) is satisfied for $i = 1$ and by definition of $\eta_{\triangleright L7.27}$. Assumption 3 follows from the definitions of C_1 and of $\mathbf{\Psi}^*$. Assumption 4 follows from the fact that $X_1 \subseteq V(G_{\mathrm{exp}}) = X_2$, and since $\mathrm{deg}^{\mathrm{min}}(G_{\mathrm{exp}}) > \rho k$ which is guaranteed by the definition of a $(k, \Omega^{**}, \Omega^*, \Lambda, \gamma, \varepsilon', \nu, \rho)$-sparse decomposition. This definition also guarantees Assumption 5, as $Y \cup X_1 \cup X_2 \subseteq V(G) \setminus \mathbf{\Psi}$.

Lemma 7.27 outputs sets $\mathbf{\Psi}'' := X'_0$, $V_1 := X'_1$, $V_2 := X'_2$ with $\mathrm{deg^{min}}_{G_\nabla}(\mathbf{\Psi}'', V_1) \geqslant \sqrt[4]{\Omega^{**}}k/2$ (by (d)), $\mathrm{deg^{max}}_{G_{\mathrm{exp}}}(V_1, X_2 \backslash V_2) < \rho k/2$ (by (c)), $\mathrm{deg^{min}}_{G_\nabla}(V_1, \mathbf{\Psi}'') \geqslant \delta_{\triangleright L7.27}k$ (by (b)), and $\mathrm{deg^{min}}_{G_{\mathrm{exp}}}(V_2, V_1) \geqslant \delta_{\triangleright L7.27}k$ (by (b)). By (a), we have that $V_1 \subseteq \mathbb{YB} \cap L''$. As $\mathrm{deg^{min}}_{G_{\mathrm{exp}}}(V_1, X_2) \geqslant \mathrm{deg^{min}}(G_{\mathrm{exp}}) \geqslant \rho k$, we have $\mathrm{deg^{min}}_{G_{\mathrm{exp}}}(V_1, V_2) \geqslant \mathrm{deg^{min}}_{G_{\mathrm{exp}}}(V_1, X_2) - \mathrm{deg^{max}}_{G_{\mathrm{exp}}}(V_1, X_2 \setminus V_2) \geqslant \delta_{\triangleright L7.27}k$.

Since L', L'' and $\mathbf{\Psi}'$ witness Preconfiguration $(\clubsuit)(\frac{\tilde{\eta}^3\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}})$, this verifies that we have Configuration $(\diamond\mathbf{2})\left(\frac{\tilde{\eta}^3\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \sqrt[4]{\Omega^{**}}/2, \frac{\tilde{\eta}\rho^2}{12800(\Omega^*)^2}\right)$.

Subcase B, $i = 2$.

We apply Lemma 7.27 with numerical parameters $r_{\triangleright L7.27} := 2$, $\Omega^*_{\triangleright L7.27} := \Omega^*$, $\Omega^{**}_{\triangleright L7.27} := \sqrt{\Omega^{**}}/4$, $\delta_{\triangleright L7.27} := \frac{\eta_{\triangleright L7.27}\gamma^2}{100(\Omega^*)^2}$, $\gamma_{\triangleright L7.27} := \gamma$, and $\eta_{\triangleright L7.27}$. Further input to the lemma are sets $X_0 := C_2$, $X_1 := N_2$, and $X_2 := V(G) \setminus \mathbf{\Psi}$, and the set Y. The underlying graph $G_{\triangleright L7.27}$ is the graph $G_{\mathcal{D}}$ with all egdes incident with $\mathbf{\Psi}$ added. Verifying assumptions of Lemma 7.27 is analogous to Subcase B, $i = 1$ with the exception of Assumption 4. Let us therefore turn to verify it. To this end, it suffices to observe that each each vertex in X_1 is contained in at least one $(\gamma k, \gamma)$-dense spot from $\mathcal{D}$ (cf. Definition 4.6), and thus has degree at least γk in X_2.

Lemma 7.27 outputs sets X'_0, X'_1, and X'_2 which witness Configuration

$$(\diamond\mathbf{3})\left(\frac{\tilde{\eta}^3\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \sqrt[4]{\Omega^{**}}/2, \gamma/2, \frac{\tilde{\eta}\gamma^2}{12800(\Omega^*)^2}\right) .$$

In fact, the only thing not analogous to the preceding subcase is that we have to check (7.26), i.e.,

$$\mathrm{deg^{max}}_{G_{\mathcal{D}}}\big(X'_1, V(G) \setminus (X'_2 \cup \mathbf{\Psi})\big) \leqslant \frac{\gamma k}{2} .$$

As $V(G) \setminus (X'_2 \cup \mathbf{\Psi}) = X_2 \setminus X'_2$, this follows from (c) of Lemma 7.27.

Subcase B, $i = 3$.

We apply Lemma 7.27 with numerical parameters $r_{\triangleright L7.27} := 3$, $\Omega^*_{\triangleright L7.27} := \Omega^*$, $\Omega^{**}_{\triangleright L7.27} := \sqrt{\Omega^{**}}/4$, $\delta_{\triangleright L7.27} := \frac{\eta_{\triangleright L7.27}\gamma^3}{300(\Omega^*)^3}$, $\gamma_{\triangleright L7.27} := \gamma$, and $\eta_{\triangleright L7.27}$. Further inputs are the sets $X_0 := C_3$, $X_1 := N_3$, $X_2 := \mathfrak{A}$, and $X_3 := V(G) \setminus \mathbf{\Psi}$, and the set Y. The underlying graph is $G_{\triangleright L7.27} := G_\nabla \cup G_\mathcal{D}$. Verifying assumptions Lemma 7.27 is analogous to Subcase B, $i = 1$, only for Assumption 4 we observe that $\mathrm{deg^{min}}_{G_\nabla \cup G_\mathcal{D}}(X_1, X_2) \geqslant \mathrm{deg^{min}}_{G_\nabla}(X_1, X_2) \geqslant \gamma k$ by definition of $X_1 = N_3 \subseteq O$, and $\mathrm{deg^{min}}_{G_\nabla \cup G_\mathcal{D}}(X_2, X_3) \geqslant \mathrm{deg^{min}}_{G_\mathcal{D}}(X_2, X_3) \geqslant \gamma k$ for the same reason as in Subcase B, $i = 2$.

Lemma 7.27 outputs Configuration $(\diamond\mathbf{4})\left(\frac{\tilde\eta^3\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \sqrt[4]{\Omega^{**}}/2, \gamma/2, \frac{\tilde\eta\gamma^3}{38400(\Omega^*)^3}\right)$, with $\mathbf{\Psi}'' := X_0'$, $V_1 := X_1'$, $\mathfrak{A}' := X_2'$ and $V_2 := X_3'$. Indeed, all calculations are similar to the ones in the preceding two subcases, we only need to note additionally that $\mathrm{deg^{min}}_{G_\nabla \cup G_\mathcal{D}}(V_1, \mathfrak{A}') \geqslant \frac{\gamma k}{2} \frac{\tilde\eta\gamma^3 k}{38400(\Omega^*)^3}$, which follows from the definition of N_3 and of O.

Subcase B, $i = 4$.

We have $\mathbf{V} \neq \emptyset$ and $\mathfrak{c}$ is the size of an arbitrary cluster in $\mathbf{V}$. We are going apply Lemma 7.28 with $\delta_{\triangleright L7.28} := \eta_{\triangleright L7.28}/100$, $\eta_{\triangleright L7.28}$, $h_{\triangleright L7.28} := \eta_{\triangleright L7.28}\mathfrak{c}/(100\Omega^*)$, $\Omega^*_{\triangleright L7.28} := \Omega^*$, $\Omega^{**}_{\triangleright L7.28} := \sqrt{\Omega^{**}}/4$ and sets $X_0 := C_4$, $X_1 := N_4$, and Y. The underlying graph is $G_{\triangleright L7.28} := G_\nabla$, and $\mathcal{C}_{\triangleright L7.28}$ is the set of clusters $\mathbf{V}$.

The fact $e(G) \leqslant kn$ together with (7.107) and the choice of $\eta_{\triangleright L7.28}$ gives Assumption 2 of Lemma 7.28. The choice of C_4 and $\mathbf{\Psi}^*$ gives Assumption 3. The fact that $X_1 \cap \mathbf{\Psi} = \emptyset$ yields Assumption 4. With the help of (7.3) it is easy to check Assumption 1. Inequality (7.108) implies Assumption 5. To verify Assumption 6, it is enough to use that $|\mathcal{C}_{\triangleright L7.28}| \leqslant \frac{n}{\mathfrak{c}}$. We have thus verified all the assumptions of Lemma 7.28.

We claim that Lemma 7.28 outputs Configuration

$$(\diamond\mathbf{5})\left(\frac{\tilde\eta^3\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}}, \sqrt[4]{\Omega^{**}}/2, \frac{\tilde\eta}{12800}, \frac{\eta}{2}, \frac{\tilde\eta}{12800\Omega^*}\right) ,$$

with $\mathbf{\Psi}'' := X_0'$ and $V_1 := X_1'$. In fact, all conditions of the configuration, except condition (7.34), which we check below, are easy to verify. (Note that $V_1 \subseteq \mathbb{YB}$ since $V_1 \subseteq X_1 = N_4 \subseteq \mathbb{XA} \cup \mathbb{XB}$. Also, $V_1 \subseteq L''$, and thus disjoint from $\mathbf{\Psi}$. Moreover, by the conditions of Lemma 7.28, V_1 is disjoint from Y. So, $V_1 \subseteq \mathbb{YB}$.) For (7.34), observe that (7.106) implies that $\mathrm{deg^{max}}_{G_\nabla}(N_4, \mathfrak{A}) \leqslant \gamma k$. Further, we have $X_1' \subseteq N_4 \setminus Y$. So for all $x \in X_1' \subseteq \mathrm{N}^{\downarrow} \setminus Y$, we have that $\deg_{G_\nabla}(x, V(G) \setminus \mathbf{\Psi}) \geqslant \frac{9\eta k}{10}$. As $N_4 \subseteq \bigcup \mathbf{V} \setminus V(G_{\mathrm{exp}})$, we obtain $\deg_{G_{\mathrm{reg}}}(x) \geqslant \frac{9\eta k}{10} - \gamma k \geqslant \frac{\eta k}{2}$, fulfilling (7.34).

7.7.4 Proof of Lemma 7.33

Set $\mathbb{YA}_1' := \{v \in \mathbb{YA}_1 \,:\, \deg_{G_{\mathrm{exp}}}(v, \mathbb{YA}_2) \geqslant \rho k\}$. By (7.86) we have

$$e_{G_{\mathrm{exp}}}(\mathbb{YA}_1', \mathbb{YA}_2) \geqslant \rho k n . \tag{7.109}$$

Set $r_{\triangleright L7.29} := 3$, $\Omega_{\triangleright L7.29} := \Omega^*$, $\gamma_{\triangleright L7.29} := \frac{\rho\eta}{10^3}$, $\delta_{\triangleright L7.29} := \frac{\eta^3\rho^4}{10^{14}(\Omega^*)^3}$, $\eta_{\triangleright L7.29} := \rho$. Observe that by (7.3) we have that (7.71) is satisfied for these parameters. Set $Y_{\triangleright L7.29} := \bar{V}$, $X_0 := \mathbb{YA}_2$, $X_1 := \mathbb{YA}_1'$, $X_2 = X_3 := V(G_{\mathrm{exp}})^{\restriction 1}$, and $V := V(G)$. Let $E_2 := E(G_\nabla)$, and $E_1 = E_3 := E(G_{\mathrm{exp}})$. We now briefly verify conditions 1–4 of Lemma 7.29. Condition 1 follows from Definition 7.6(1). Condition 2 follows from (7.109). Condition 3 for $i = 1$ follows from the definition of $\mathbb{YA}_1'$ and from Definition 7.6(6), and for $i = 2$ from the fact that $\deg^{\min}(G_{\mathrm{exp}}) \geqslant \rho k$ and from Definition 7.6(6). Last, Condition 4 follows from the fact that $\bigcup_{i=0}^{3} X_i$ is disjoint from $\mathbf{\Psi}$.

Lemma 7.29 yields four non-empty sets $X_0', \dots, X_3'$. By assertions (a), (b), (c), and hypothesis 3 of Lemma 7.29, for all $i \in \{0, 1, 2, 3\}$, $j \in \{i-1, i+1\} \setminus \{-1, 4\}$ we have

$$\deg^{\min}{}_{H_{i,j}}(X_i', X_j') \geqslant \delta_{\triangleright L7.29} k, \tag{7.110}$$

where $H_{i,j} = G_{\mathrm{exp}}$, except for $\{i, j\} = \{1, 2\}$, where $H_{i,j} = G_\nabla$.

Thus, the sets X_0' and X_1' witness Preconfiguration $(\mathbf{exp})(\delta_{\triangleright L7.29})$. By Lemma 7.35, and by (7.87) and (7.88), the pair X_0', X_1' together with the cover $\mathcal{F}$ from (7.12) witnesses either Preconfiguration $(\heartsuit\mathbf{1})(\frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k)$ (with respect to $\mathcal{F}$) or Preconfiguration $(\heartsuit\mathbf{2})(\mathfrak{p}_2(1+\frac{\eta}{20})k)$.

Notice that (7.110) establishes (7.43)–(7.46). Thus the sets $X_0', \dots, X_3'$ witness Configuration $(\diamond\mathbf{6})(\delta_{\triangleright L7.29}, 0, 1, 1, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k)$.

7.7.5 Proof of Lemma 7.34

In Lemmas 7.39, 7.40, 7.42, 7.43, 7.44 below, we get one of the configurations $(\diamond\mathbf{6})$, $(\diamond\mathbf{7})$, $(\diamond\mathbf{8})$, $(\diamond\mathbf{9})$, or $(\diamond\mathbf{10})$, depending on whether we have case (**t1**), (**t2**), (**t3**), (**t3–t5**), or (**t5**), respectively. While the first three of these cases are resolved by a fairly straightforward application of the cleaning Lemma 7.30, the later cases require some further non-trivial computations.

Lemma 7.39. *In Case* (**t1**) *(of either* (**cA**) *or* (**cB**)*) we obtain Configuration*

$$(\diamond\mathbf{6})\Big(\frac{\eta^3\rho^4}{10^{12}(\Omega^*)^4}, 4\bar{\varepsilon}, \bar{d}/4, \beta/2, \frac{3\eta^3}{2000}, \mathfrak{p}_2(1+\frac{\eta}{20})k\Big)\ .$$

Proof. We use Lemma 7.30 with the following input parameters: $r_{\triangleright L7.30} := 3$, $\Omega_{\triangleright L7.30} := \Omega^*$, $\gamma_{\triangleright L7.30} := \eta\rho/200$, $\eta_{\triangleright L7.30} := \rho/(2\Omega^*)$, $\delta_{\triangleright L7.30} := \eta^3\rho^4/(10^{12}(\Omega^*)^4)$, $\varepsilon_{\triangleright L7.30} := \bar{\varepsilon}$, $\mu_{\triangleright L7.30} := \beta$ and $d_{\triangleright L7.30} := \bar{d}$. Note these parameters satisfy the numerical conditions of Lemma 7.30. We use the vertex sets $Y_{\triangleright L7.30} := \bar{V} \cup \mathbb{F}$, $X_0 := V_2(\mathcal{M})$, $X_1 := V_1(\mathcal{M})$, $X_2 = X_3 := V(G_{\mathrm{exp}})^{\restriction 1}$, and $V := V(G)$. The partitions of X_0 and X_1 in Lemma 7.30 are the ones induced by $\mathcal{V}(\mathcal{M})$, and the set E_1 consists of all edges from $E(\mathcal{D}_\nabla)$ between pairs from $\mathcal{M}$. Further, set $E_2 := E(G_\nabla)$ and $E_3 := E(G_{\mathrm{exp}})$.

Let us verify the conditions of Lemma 7.30. Condition 1 follows from Definition 7.6(1) and (7.16). Condition 2 holds by the assumption on $\mathcal{M}$. Condition 3 follows from Definition 7.6(6) by (7.14), and for $i = 1$ also from the definition of $\mathcal{M}$. Conditions 4 hold by the definition of $\mathcal{M}$. Finally, Condition 5 follows from the properties of the sparse decomposition ∇.

The output of Lemma 7.30 are four sets $X_0', \ldots, X_3'$. By Lemma 7.35, the sets X_0' and X_1' witness Preconfiguration $(\heartsuit\mathbf{1})(3\eta^3/(2\cdot 10^3), \mathfrak{p}_2(1+\frac{\eta}{20})k)$, or $(\heartsuit\mathbf{2})(\mathfrak{p}_2(1+\frac{\eta}{20})k)$. Further, Lemma 7.30(a) gives that (X_0', X_1') witnesses Preconfiguration $(\mathbf{reg})(4\bar{\varepsilon}, \bar{d}/4, \beta/2)$. It is now easy to verify that we have Configuration $(\diamond\mathbf{6})\big(\frac{\eta^3\rho^4}{10^{12}(\Omega^*)^4}, 4\bar{\varepsilon}, \frac{\bar{d}}{4}, \frac{\beta}{2}, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$. □

Observe that Lemma 7.39 leads to Configuration $(\diamond\mathbf{6})$ with parameters as claimed in Lemma 7.34. Indeed, no matter whether we have **(M1)** or **(M2)**, we have $4\varepsilon_\odot \geqslant 4\cdot\frac{10^5\varepsilon'}{\eta^2}$, and $\gamma^3\rho/(32\Omega^*) \leqslant \gamma^2/4$, and $\eta^2\mathfrak{c}/(8\cdot 10^3 k) \leqslant \eta^2\varepsilon'/(8\cdot 10^3) \leqslant \alpha_\odot\rho/\Omega^*$. We shall use the same monotonization of parameters also after Lemmas 7.40, Lemmas 7.42, and Lemma 7.43.

Lemma 7.40. *Case* $(\mathbf{t2})$ *(of either* $(\mathbf{cA})$ *or* $(\mathbf{cB})$*) leads to Configuration*

$$(\diamond\mathbf{7})\big(\frac{\eta^3\gamma^3\rho}{10^{12}(\Omega^*)^4}, \frac{\eta\gamma}{400}, 4\bar{\varepsilon}, \frac{\bar{d}}{4}, \frac{\beta}{2}, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)\ .$$

Proof. We use Lemma 7.30 with the following input parameters: $r_{\triangleright\mathrm{L}7.30} := 3$, $\Omega_{\triangleright\mathrm{L}7.30} := \Omega^*$, $\gamma_{\triangleright\mathrm{L}7.30} := \eta\gamma/200$, $\eta_{\triangleright\mathrm{L}7.30} := \rho/\Omega^*$, $\delta_{\triangleright\mathrm{L}7.30} := \eta^3\gamma^3\rho/(10^{12}(\Omega^*)^4)$, $\varepsilon_{\triangleright\mathrm{L}7.30} := \bar{\varepsilon}$, $\mu_{\triangleright\mathrm{L}7.30} := \beta$ and $d_{\triangleright\mathrm{L}7.30} := \bar{d}$. We use the vertex sets $Y_{\triangleright\mathrm{L}7.30} := \bar{V} \cup \mathbb{F}$, $X_0 := V_2(\mathcal{M})$, $X_1 := V_1(\mathcal{M})$, $X_2 := \mathfrak{A}^{\restriction 1}$, $X_3 := \mathfrak{P}_1$, and $V := V(G)$. The partitions of X_0 and X_1 in Lemma 7.30 are the ones induced by $\mathcal{V}(\mathcal{M})$, and the set E_1 consists of all edges from $E(\mathcal{D}_\nabla)$ between pairs from $\mathcal{M}$. Further, set $E_2 := E(G_\nabla)$ and $E_3 := E(G_{\mathcal{D}})$.

The conditions of Lemma 7.30 are verified as before, let us just note that Condition 3 follows from Definition 7.6(6) and by (7.14), and for $i = 1$ from the definition of $\mathcal{M}$, while for $i = 2$ it holds since $\mathfrak{A}$ is covered by the set $\mathcal{D}$ of $(\gamma k, \gamma)$-dense spots (cf. Definition 4.6).

It is now easy to check that the output of Lemma 7.30 are sets that witness Configuration $(\diamond\mathbf{7})\big(\frac{\eta^3\gamma^3\rho}{10^{12}(\Omega^*)^4}, \frac{\eta\gamma}{400}, 4\bar{\varepsilon}, \frac{\bar{d}}{4}, \frac{\beta}{2}, \frac{3\eta^3}{2\cdot 10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$. □

Before proceeding with dealing with cases $(\mathbf{t3})$–$(\mathbf{t5})$ and **(t3–5)** we state some properties of the matching $\bar{\mathcal{M}} := \big(\mathcal{M}_A \cup \mathcal{M}_B\big)^{\restriction 1}$.

Lemma 7.41. *Define* $V_{\mathrm{leftover}} := V(\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 1} \setminus V(\bar{\mathcal{M}})$*, and* $Y_{\bar{\mathcal{M}}} := \bar{V} \cup \mathbb{F} \cup$

$\mathbf{shadow}_{G_\mathcal{D}}(V_{\mathrm{leftover}}, \frac{\eta^2 k}{1000})$. *Then we have*

$$\begin{aligned}&\bar{\mathcal{M}} \text{ is a } (\tfrac{400\varepsilon}{\eta}, \tfrac{d}{2}, \tfrac{\eta\pi\mathfrak{c}}{200})\text{-semiregular matching absorbed by } \mathcal{M}_A \cup \mathcal{M}_B \\ &\qquad \text{and } V(\bar{\mathcal{M}}) \subseteq \mathfrak{P}_1.\end{aligned} \tag{7.111}$$

$$|Y_{\bar{\mathcal{M}}}| \leqslant \frac{3000\varepsilon\Omega^* n}{\eta^2} . \tag{7.112}$$

Proof. Property (7.111) follows from Lemma 7.8.

Observe that from properties (1) and (3) of Definition 7.6 we can calculate that

$$\begin{aligned}|V_{\mathrm{leftover}}| &\leqslant 3 \cdot k^{0.9} \cdot |\mathcal{M}_A \cup \mathcal{M}_B| + \left|\bigcup \bar{\mathcal{V}} \cup \bar{\mathcal{V}}^*\right| \\ &\leqslant 3 \cdot k^{0.9} \cdot \frac{n}{2\pi\mathfrak{c}} + 2\exp(-k^{0.1}) \overset{(7.3)}{\leqslant} 2\varepsilon n .\end{aligned} \tag{7.113}$$

Then

$$\begin{aligned}|Y_{\bar{\mathcal{M}}}| &\leqslant |\bar{V}| + |\mathbb{F}| + \left|\mathbf{shadow}_{G_\mathcal{D}}\left(V_{\mathrm{leftover}}, \frac{\eta^2 k}{1000}\right)\right| \\ {\scriptstyle\text{(by Fact 7.1)}} \quad &\leqslant |\bar{V}| + |\mathbb{F}| + |V_{\mathrm{leftover}}|\frac{1000\Omega^*}{\eta^2} \\ {\scriptstyle\text{(by (7.113), D7.6(1), (7.16))}} \quad &< \frac{3000\varepsilon\Omega^* n}{\eta^2} .\end{aligned}$$

□

Lemma 7.42. *In Case* (**t3**)(**cA**) *we get Configuration*

$$(\diamond\mathbf{8})\left(\frac{\eta^4\gamma^4\rho}{10^{15}(\Omega^*)^5}, \frac{\eta\gamma}{400}, \frac{400\varepsilon}{\eta}, 4\bar{\varepsilon}, \frac{d}{2}, \frac{\bar{d}}{4}, \frac{\eta\pi\mathfrak{c}}{200k}, \frac{\beta}{2}, \mathfrak{p}_1(1+\frac{\eta}{20})k, \mathfrak{p}_2(1+\frac{\eta}{20})k\right) .$$

Proof. We use Lemma 7.30 with the following input parameters: $r_{\triangleright\mathrm{L7.30}} := 4$, $\Omega_{\triangleright\mathrm{L7.30}} := \Omega^*$, $\gamma_{\triangleright\mathrm{L7.30}} := \eta\gamma/200$, $\eta_{\triangleright\mathrm{L7.30}} := \rho/\Omega^*$, $\delta_{\triangleright\mathrm{L7.30}} := \eta^4\gamma^4\rho/(10^{15}(\Omega^*)^5)$, $\varepsilon_{\triangleright\mathrm{L7.30}} := \bar{\varepsilon}$, $\mu_{\triangleright\mathrm{L7.30}} := \beta$ and $d_{\triangleright\mathrm{L7.30}} := \bar{d}$. We use the following vertex sets $Y_{\triangleright\mathrm{L7.30}} := Y_{\bar{\mathcal{M}}}$, $X_0 := V_2(\mathcal{M})$, $X_1 := V_1(\mathcal{M})$,

$$X_2 := (\mathbb{L}_{\eta,k}(G) \cap V_{\leadsto\mathfrak{A}})^{\restriction 0} \setminus \left(V(G_{\mathrm{exp}}) \cup \mathfrak{A} \cup V(\mathcal{M}_A \cup \mathcal{M}_B) \cup V_{\not\leadsto\mathbf{\Psi}} \cup L_\# \cup \mathtt{P}_\mathfrak{A} \cup \mathtt{P}_1\right) ,$$

$X_3 := \mathfrak{A}^{\restriction 1}$, $X_4 := \mathfrak{P}_1$, and $V := V(G)$. The partitions of X_0 and X_1 in Lemma 7.30 are the ones induced by $\mathcal{V}(\mathcal{M})$, and the set E_1 consists of all edges from $E(\mathcal{D}_\nabla)$ between pairs from $\mathcal{M}$. Further, set $E_2 = E_3 := E(G_\nabla)$ and $E_4 := E(G_\mathcal{D})$.

Most of the conditions of Lemma 7.30 are verified as before. Condition 1 follows from (7.112). Let us note that using Definition 7.6(6) and (7.14), we find that Condition 3 for $i = 2$ follows from the definition of $V_{\leadsto\mathfrak{A}}$, and Condition 3 for $i = 3$ holds as it is the same as Condition 3 for $i = 2$ in case (**t2**). Note that to prove Condition 3 for $i = 1$ we use the fact that

$$\begin{aligned}V_1(\mathcal{M}) \subseteq \mathbf{shadow}_{G_\nabla}&\left((V_{\leadsto\mathfrak{A}} \cap \mathbb{L}_{\eta,k}(G)) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B), \frac{2\eta^2 k}{10^5}\right) \\ &\setminus (\mathtt{P} \cup \mathbf{shadow}_{G_\nabla}(V(G_{\mathrm{exp}}), \rho k) \cup V_{\leadsto\mathfrak{A}}) .\end{aligned} \tag{7.114}$$

Then, for each $v \in V_1(\mathcal{M})$ we have

$$\begin{aligned}
\deg_{G_\nabla}(v, X_2) &\geqslant \mathfrak{p}_1\Big(\deg_{G_\nabla}(v, (\mathbb{L}_{\eta,k}(G) \cap V_{\rightsquigarrow\mathfrak{A}}) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) \\
&\qquad - \deg_{G_\nabla}(v, V(G_{\mathrm{exp}}) \cup \mathfrak{A} \cup V_{\not\rightsquigarrow\mathbf{\Psi}} \cup L_\# \cup \mathtt{P}_\mathfrak{A} \cup \mathtt{P}_1)\Big) - k^{0.9} \\
\text{(by (7.114))} \qquad &\geqslant \frac{\eta}{100}\left(\frac{2\eta^2 k}{10^5} - \rho k - \frac{\rho k}{100\Omega^*} - \frac{\eta^2 k}{10^5}\right) - k^{0.9} \\
&\geqslant \frac{\eta\gamma k}{200} \,,
\end{aligned}$$

which indeed verifies Condition 3.

Define $\mathcal{N} := \bar{\mathcal{M}} \setminus \{(X,Y) \in \bar{\mathcal{M}} \,:\, X \cup Y \subseteq V(\mathcal{N}_\mathfrak{A})\}$. By Lemma 7.41 we have that $\mathcal{N} \subseteq \bar{\mathcal{M}}$ is a $(\frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi\mathfrak{c}}{200})$-semiregular matching absorbed by $\mathcal{M}_A \cup \mathcal{M}_B$, and that $V(\mathcal{N}) \subseteq \mathfrak{P}_1$.

To check that the output of Lemma 7.30 together with the matching $\mathcal{N}$ leads to Configuration $(\diamond\mathbf{8})\big(\frac{\eta^4\gamma^4\rho}{10^{15}(\Omega^*)^5}, \frac{\eta\gamma}{400}, \frac{400\varepsilon}{\eta}, 4\bar{\varepsilon}, \frac{d}{2}, \frac{\bar{d}}{4}, \frac{\eta\pi\mathfrak{c}}{200k}, \frac{\beta}{2}, \mathfrak{p}_1(1+\frac{\eta}{20})k, \mathfrak{p}_2(1+\frac{\eta}{20})k\big)$ it remains to verify that (7.57) is satisfied. Take $v \in X_2'$ arbitrarily. We have

$$\begin{aligned}
\deg_{G_\mathcal{D}}(v, X_3') + \deg_{G_{\mathrm{reg}}}(v, V(\mathcal{N})) &\geqslant \deg_{G_\nabla}(v, \mathfrak{P}_1) - \deg_{G_{\mathrm{exp}}}(v) - \deg_{G_\mathcal{D}}(v, X_3 \setminus X_3') \\
&\quad - \deg_{G_{\mathrm{reg}}}(v, V(\mathcal{N}_\mathfrak{A})) - \deg_{G_{\mathrm{reg}}}(v, V_{\mathrm{leftover}}) \\
&\quad - \deg_{G_{\mathrm{reg}}}(v, V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) \,.
\end{aligned} \tag{7.115}$$

Let us now bound the terms of the right-hand side of (7.115). Definition 7.6(6) gives $\deg_{G_\nabla}(v, \mathfrak{P}_1) \geqslant \mathfrak{p}_1\left(\deg_{G_\nabla}(v) - \deg_G(v, \mathbf{\Psi})\right) - k^{0.9}$. We have that $v \notin V(G_{\mathrm{exp}})$, and thus $\deg_{G_{\mathrm{exp}}}(v) = 0$. Lemma 7.30(c) gives that $\deg_{G_\mathcal{D}}(v, X_3 \setminus X_3') \leqslant \frac{\eta\gamma k}{400}$. As $v \notin \mathtt{P}_\mathfrak{A}$, we have $\deg_{G_{\mathrm{reg}}}(v, V(\mathcal{N}_\mathfrak{A})) < \gamma k$. As $v \notin \mathbf{shadow}_{G_\mathcal{D}}\left(V_{\mathrm{leftover}}, \frac{\eta^2 k}{1000}\right)$ we have $\deg_{G_\mathcal{D}}(v, V_{\mathrm{leftover}}) \leqslant \frac{\eta^2 k}{1000}$. Last, recall that $v \notin \mathtt{P}_1 \cup V(\mathcal{M}_A \cup \mathcal{M}_B)$, and consequently $\deg_{G_{\mathrm{reg}}}(v, V(G) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) < \gamma k$. Put together, we have,

$$\begin{aligned}
\deg_{G_\mathcal{D}}(v, X_3') + \deg_{G_{\mathrm{reg}}}(v, V(\mathcal{N})) &\geqslant \mathfrak{p}_1\left(\deg_{G_\nabla}(v) - \deg_G(v, \mathbf{\Psi})\right) - \frac{2\eta^2 k}{1000} \\
\text{(as } v \in \mathbb{L}_{\eta,k}(G) \setminus (L_\# \cup V_{\not\rightsquigarrow\mathbf{\Psi}})) \qquad &\geqslant \mathfrak{p}_1\left((1+\frac{9\eta}{10})k - \frac{\eta k}{100}\right) - \frac{\eta^2 k}{500} \\
&\geqslant \mathfrak{p}_1(1+\frac{\eta}{20})k \,.
\end{aligned}$$

□

Lemma 7.43. *In case* **(t3–5)(cB)** *we get Configuration* $(\diamond\mathbf{9})\big(\frac{\rho\eta^8}{10^{26}(\Omega^*)^3}, \frac{2\eta^3}{10^3}, \mathfrak{p}_1(1+\frac{\eta}{40})k, \mathfrak{p}_2(1+\frac{\eta}{20})k, \frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi\mathfrak{c}}{200k}, 4\bar{\varepsilon}, \frac{\bar{d}}{4}, \frac{\beta}{2}\big)$.

Proof. Recall that Lemma 7.35 claims that $\mathcal{F}$ is an $(\mathcal{M}_A \cup \mathcal{M}_B)$-cover. We introduce another $(\mathcal{M}_A \cup \mathcal{M}_B)$-cover,

$$\mathcal{F}' := \mathcal{F} \cup \{X \in \mathcal{V}(\mathcal{M}_B) \,:\, X \subseteq \mathfrak{A}\} \,.$$

Claim 7.43.1. We have that $\mathrm{deg}^{\max}{}_{G_\nabla}(V_1(\mathcal{M}),\bigcup\mathcal{F}') \leqslant \frac{2\eta^3}{10^3}k$.

Proof of Claim 7.43.1. This follows from (7.92) and as $V_1(\mathcal{M})\cap V_{\leadsto\mathfrak{A}}=\emptyset$. ☐

We use Lemma 7.30 with the following input parameters: $r_{\triangleright\mathrm{L}7.30} := 2$, $\Omega_{\triangleright\mathrm{L}7.30} := \Omega^*$, $\gamma_{\triangleright\mathrm{L}7.30} := \eta^4/10^{11}$, $\eta_{\triangleright\mathrm{L}7.30} := \rho/\Omega^*$, $\delta_{\triangleright\mathrm{L}7.30} := \rho\eta^8/(10^{26}(\Omega^*)^3)$, $\varepsilon_{\triangleright\mathrm{L}7.30} := \bar{\varepsilon}$, $\mu_{\triangleright\mathrm{L}7.30} := \beta$ and $d_{\triangleright\mathrm{L}7.30} := \bar{d}$. We use the following vertex sets $Y_{\triangleright\mathrm{L}7.30} := Y_{\bar{\mathcal{M}}}$, $X_0 := V_2(\mathcal{M})$, $X_1 := V_1(\mathcal{M})$, and $X_2 := V(\bar{\mathcal{M}})\setminus\bigcup\mathcal{F}' \subseteq \bigcup\mathbf{V}^{\restriction 1}$. The partitions of X_0 and X_1 in Lemma 7.30 are the ones induced by $\mathcal{V}(\mathcal{M})$, and the set E_1 consists of all edges from $E(\mathcal{D}_\nabla)$ between pairs from $\mathcal{M}$. Further, set $E_2 := E(G_{\mathcal{D}})$.

Condition 1 of Lemma 7.30 follows from (7.112). Condition 2 follows from case **(t3–5)**. Condition 4 follows from the definition of $\mathcal{M}$. Condition 5 follows from the sparse decomposition ∇. It remains to see Condition 3. Similarly as in Lemma 7.35 one can prove that

$$\mathrm{deg}^{\min}{}_{G_\nabla}\left(\mathbb{XA}\setminus(\mathtt{P}\cup\bar{V}),V^{\restriction 1}_{\mathrm{good}}\right) \geqslant \mathfrak{p}_1(1+\frac{\eta}{20})k\ . \tag{7.116}$$

We obtain

$$\begin{aligned}
\mathrm{deg}^{\min}{}_{G_{\mathcal{D}}}(V_1(\mathcal{M})\setminus Y_{\triangleright\mathrm{L}7.30},X_2) &\geqslant \mathrm{deg}^{\min}{}_{G_{\mathcal{D}}}\big(V_1(\mathcal{M})\setminus Y_{\triangleright\mathrm{L}7.30},V(\bar{\mathcal{M}})\big)\\
&\quad -\mathrm{deg}^{\max}{}_{G_{\mathcal{D}}}(V_1(\mathcal{M}),\textstyle\bigcup\mathcal{F}')\\
\text{(by def of } \bar{\mathcal{M}}\text{, Cl 7.43.1)}\quad &\geqslant \mathrm{deg}^{\min}{}_{G_{\mathcal{D}}}\big(V_1(\mathcal{M}),V(\mathcal{M}_A\cup\mathcal{M}_B)^{\restriction 1}\big)\\
&\quad -\mathrm{deg}^{\max}{}_{G_{\mathcal{D}}}(V_1(\mathcal{M})\setminus Y_{\triangleright\mathrm{L}7.30},V_{\mathrm{leftover}})-\frac{2\eta^3k}{10^3}\\
\text{(by def of } Y_{\triangleright\mathrm{L}7.30})\quad &\geqslant \mathrm{deg}^{\min}{}_{G_\nabla}\big(V_1(\mathcal{M}),V(\mathcal{M}_A\cup\mathcal{M}_B)^{\restriction 1}\big)\\
&\quad -\mathrm{deg}^{\max}{}_{G_{\mathrm{exp}}}\big(V_1(\mathcal{M}),V(\mathcal{M}_A\cup\mathcal{M}_B)\big)\\
&\quad -\frac{\eta^2k}{1000}-\frac{2\eta^3k}{10^3}\\
\text{(by (7.8))}\quad &\geqslant \mathrm{deg}^{\min}{}_{G_\nabla}\big(\mathbb{XA}\setminus(\mathtt{P}\cup\bar{V}),V^{\restriction 1}_{\mathrm{good}}\big)\\
&\quad -\mathrm{deg}^{\max}{}_{G_\nabla}\big(V_1(\mathcal{M}),\mathfrak{A}\big)\\
&\quad -\mathrm{deg}^{\max}{}_{G_\nabla}\big(V_1(\mathcal{M}),\mathbb{L}_{\eta,k}(G)\setminus V(\mathcal{M}_A\cup\mathcal{M}_B)\big)\\
&\quad -\mathrm{deg}^{\max}{}_{G_\nabla}\big(V_1(\mathcal{M}),V(G_{\mathrm{exp}})\setminus V(\mathcal{M}_A\cup\mathcal{M}_B)\big)\\
&\quad -\mathrm{deg}^{\max}{}_{G_\nabla}\big(V_1(\mathcal{M}),V(G_{\mathrm{exp}})\cap V(\mathcal{M}_A\cup\mathcal{M}_B)\big)\\
&\quad -\frac{\eta^2k}{1000}-\frac{2\eta^3k}{10^3}\\
\text{(by (7.116), as } V_1(\mathcal{M})\cap V_{\leadsto\mathfrak{A}}=\emptyset\ \&\ \textbf{(cB)})\quad &\geqslant \mathfrak{p}_1(1+\frac{\eta}{20}k)-\frac{\rho k}{100\Omega^*}-\mathrm{deg}^{\max}{}_{G_\nabla}\big(\mathbb{XA}\setminus\mathtt{P}_3,\mathbb{XA}\big)\\
&\quad -\mathrm{deg}^{\max}{}_{G_\nabla}(V_1(\mathcal{M}),V(G_{\mathrm{exp}}))-\frac{\eta^2k}{1000}-\frac{2\eta^3k}{10^3}\\
\text{(def of } \mathtt{P}_3\ \&\text{ as } V_1(\mathcal{M})\cap\mathbf{shadow}(V(G_{\mathrm{exp}}))=\emptyset)\quad &\geqslant \mathfrak{p}_1(1+\frac{\eta}{20})k-\frac{\rho k}{100\Omega^*}-\rho k-\frac{\eta^3k}{10^3}-\frac{\eta^2k}{1000}-\frac{2\eta^3k}{10^3}\\
&\geqslant \mathfrak{p}_1(1+\frac{\eta}{30})k\ .
\end{aligned} \tag{7.117}$$

Since the last term is greater than $\gamma_{\triangleright L7.30}k = \frac{\eta^4 k}{10^{11}}$, we see that Condition 3 of Lemma 7.30 is satisfied.

The output of Lemma 7.30 are three non-empty sets X_0', X_1', X_2' disjoint from $Y_{\triangleright L7.30}$, together with $(4\bar{\varepsilon}, \frac{\bar{d}}{4})$-super-regular pairs $\{Q_0^{(j)}, Q_1^{(j)}\}_{j\in\mathcal{Y}}$ which cover (X_0', X_1') with the following properties.

$$\text{(by Lemma 7.30 (a))} \quad \min\left\{|Q_0^{(j)}|, |Q_1^{(j)}|\right\} \geqslant \frac{\beta k}{2} \text{ for each } j \in \mathcal{Y}\,, \tag{7.118}$$

$$\text{(by Lemma 7.30 (b))} \quad \mathrm{deg^{min}}_{G_{\mathcal{D}}}(X_2', X_1') \geqslant \delta_{\triangleright L7.30}k\,, \tag{7.119}$$

$$\begin{aligned}\text{(by Lemma 7.30 (c))} \quad \mathrm{deg^{min}}_{G_{\mathcal{D}}}(X_1', X_2') &\overset{(7.117)}{\geqslant} \mathfrak{p}_1(1+\frac{\eta}{30})k - \frac{\eta^4 k}{2\cdot 10^{11}} \\ &\geqslant \mathfrak{p}_1(1+\frac{\eta}{40})k\,.\end{aligned} \tag{7.120}$$

We now verify that the sets X_0', X_1', X_2', the semiregular matching $\mathcal{N}_{\triangleright D7.23} := \bar{\mathcal{M}}$ together with the $(\mathcal{M}_A\cup\mathcal{M}_B)$-cover $\mathcal{F}'$, and the family $\{(Q_0^{(j)}, Q_0^{(j)})\}_{j\in\mathcal{Y}}$ satisfy all the conditions of Configuration $(\diamond\mathbf{9})(\delta_{\triangleright L7.30}, \frac{2\eta^3}{10^3}, \mathfrak{p}_1(1+\frac{\eta}{40})k, \mathfrak{p}_2(1+\frac{\eta}{20})k, \frac{400\varepsilon}{\eta}, \frac{d}{2}, \frac{\eta\pi\mathfrak{c}}{200k}, 4\bar{\varepsilon}, \frac{\bar{d}}{4}, \frac{\beta}{2})$.

By Lemma 7.35 and by Claim 7.43.1, the pair X_0', X_1' together with the $(\mathcal{M}_A\cup\mathcal{M}_B)$-cover $\mathcal{F}'$ witnesses Preconfiguration $(\heartsuit\mathbf{1})(\frac{2\eta^3}{10^3}, \mathfrak{p}_2(1+\frac{\eta}{20})k)$. By (7.111), $\bar{\mathcal{M}}$ is as required for Configuration $(\diamond\mathbf{9})$. Property (7.42) follows from (7.118). Property (7.58) follows from (7.120). Inequality (7.119) gives (7.59). By definition of X_2, the set X_2' is as required. □

Lemma 7.44. *In Case* $(\mathbf{t5})(\mathbf{cA})$ *we get Configuration* $(\diamond\mathbf{10})\big(\varepsilon, \frac{\gamma^2 d}{2}, \pi\sqrt{\varepsilon'}\nu k, \frac{2(\Omega^*)^2 k}{\gamma^2}, \frac{\eta}{40}\big)$.

Proof. As $V(\mathcal{M}) \subseteq V(G_{\mathrm{reg}})$ we have

$$\begin{aligned}\mathrm{deg^{min}}_{G_{\mathrm{reg}}}(V(\mathcal{M}), V_{\mathrm{good}}) &\geqslant \mathrm{deg^{min}}_{G_\nabla}(V(\mathcal{M}), V_+ \setminus L_\#) - \mathrm{deg^{max}}_{G_\nabla}(V(\mathcal{M}), \mathbf{\Psi}) \\ &\quad - \mathrm{deg^{max}}_{G_\nabla}(V(\mathcal{M}), \mathfrak{A}) - \mathrm{deg^{max}}_{G_\nabla}(V(\mathcal{M}), V(G_{\mathrm{exp}})) \\ &\geqslant (1+\frac{\eta}{20})k,\end{aligned} \tag{7.121}$$

where the last line follows as $V(\mathcal{M}) \subseteq \mathbb{YA}\setminus V_{\not\rightsquigarrow\mathbf{\Psi}}$ and $V(\mathcal{M})\cap(\mathbf{shadow}_G(V(G_{\mathrm{exp}}), \rho k)\cup V_{\rightsquigarrow\mathfrak{A}}) = \emptyset$.

Define

$$\begin{aligned}\mathcal{C} &:= \left\{C \setminus \big(L_\# \cup V(\mathcal{M}_A\cup\mathcal{M}_B) \cup V_{\not\rightsquigarrow\mathbf{\Psi}} \cup \mathtt{P}_1\big) \,:\, C \in \mathbf{V}\right\}\,, \\ \mathcal{C}^- &:= \left\{C \in \mathcal{C} \,:\, |C| < \sqrt{\varepsilon'}\mathfrak{c}\right\}\,,\end{aligned}$$

We have

$$\left|\bigcup \mathcal{C}^-\right| \leqslant \sum_{C\in\mathcal{C}} \sqrt{\varepsilon'}|C| \leqslant \sqrt{\varepsilon'}n\,. \tag{7.122}$$

Set $\mathcal{V}^\circ := \mathcal{V}(\mathcal{M}_A\cup\mathcal{M}_B)\cup(\mathcal{C}\setminus\mathcal{C}^-)$ and let G° be the subgraph of G with vertex set $\bigcup\mathcal{V}^\circ$ and all edges from $E(G_{\mathrm{reg}})$ induced by $\bigcup\mathcal{V}^\circ$ plus all edges of $E(G_\nabla)$ that go between A and B for all $(A,B)\in\mathcal{M}_A\cup\mathcal{M}_B$.

Observe that from Setting 7.4 (3), Fact 4.3 and Fact 4.4, we have that for all $X \in \mathcal{V}^\circ$ we have $|\bigcup \mathrm{N}_{G^\circ}(X)| \leqslant |\bigcup \mathrm{N}_{G_{\mathcal{D}}}(X)| \leqslant \frac{\Omega^*}{\gamma} \cdot \frac{2\Omega^* k}{\gamma}$. Thus by Fact 2.7, we obtain the following statement.

Claim 7.44.1. $(G^\circ, \mathcal{V}^\circ)$ is an $(\varepsilon, \gamma^2 d/2, \pi\sqrt{\varepsilon'}\mathfrak{c}, \frac{2(\Omega^*)^2 k}{\gamma^2})$-regularized graph.

Define

$$\mathcal{L}^\circ := \left\{X \in \mathcal{V}^\circ \setminus \mathcal{V}(\mathcal{M}_A \cup \mathcal{M}_B) \,:\, \mathrm{deg^{min}}_{G^\circ}(X) \geqslant (1 + \frac{\eta}{2})k\right\}.$$

The regularized graph $(G^\circ, \mathcal{V}^\circ)$ together with $\mathcal{M}_{\triangleright\mathrm{D7.25}} := \mathcal{M}_A \cup \mathcal{M}_B$ and $\mathcal{L}^*_{\triangleright\mathrm{D7.25}} := \mathcal{L}^\circ$ will eventually turn out to witness Configuration ($\diamond$**10**). The challenge now is to find $A_{\triangleright\mathrm{D7.25}}$ and $B_{\triangleright\mathrm{D7.25}}$. To this end we shall exploit the matching $\mathcal{M}$; the relation ofbetween $\mathcal{M}$ and $(G^\circ, \mathcal{V}^\circ)$, $\mathcal{M}_A \cup \mathcal{M}_B$, and $\mathcal{L}^\circ$ is not direct. We proceed as follows. In Claim 7.44.2 we find a suitable $\mathcal{M}$-edge. In case (**M1**) this $\mathcal{M}$-edge gives readily a suitable pair $(A_{\triangleright\mathrm{D7.25}}, B_{\triangleright\mathrm{D7.25}})$ (in the final Claim 7.44.4). In case (**M2**) we have to work further with this $\mathcal{M}$-edge in Claim 7.44.3, to get a suitable $\mathbf{G}_{\mathrm{reg}}$-edge. Only then do we find $(A_{\triangleright\mathrm{D7.25}}, B_{\triangleright\mathrm{D7.25}})$ (again, in Claim 7.44.4).

Claim 7.44.2. There is an $\mathcal{M}$-edge (A, B) such that

$$\deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ) \geqslant (1 + \frac{\eta}{40})k + \frac{\eta k}{200}$$

for at least $|A|/2$ vertices $v \in A$, and at least $|B|/2$ vertices $v \in B$.

Proof of Claim 7.44.2. Set $S := \mathbf{shadow}_{G_{\mathrm{reg}}}(\bigcup \mathcal{C}^-, \frac{\eta k}{200})$, and $\mathcal{M}_S := \{(X, Y) \in \mathcal{M} : |(X \cup Y) \cap S| \geqslant |X \cup Y|/4\}$. By Fact 7.1 we obtain that $|S| \leqslant |\bigcup \mathcal{C}^-| \cdot \frac{200\Omega^*}{\eta}$. Thus $|V(\mathcal{M}_S)| \leqslant 4|S| \leqslant \frac{800\sqrt{\varepsilon'}\Omega^* n}{\eta} < \frac{\rho n}{\Omega^*}$. Consequently, $\mathcal{M} \neq \mathcal{M}_S$. We show that any $(A, B) \in \mathcal{M} \setminus \mathcal{M}_S$ satisfies the requirements of the claim. We start with an auxiliary subclaim.

Subclaim 7.44.2.1. We have

$$V_+ \cap V(G^\circ) \subseteq V(\mathcal{M}_A \cup \mathcal{M}_B) \cup V(G_{\exp}) \cup \big((V_{\rightsquigarrow\mathfrak{A}} \cap \mathbb{L}_{\eta,k}(G)) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)\big) \cup \bigcup \mathcal{L}^\circ\ .$$

Proof of Subclaim 7.44.2.1. First observe that

$$\begin{aligned} V_+ \cap V(G^\circ) &\subseteq V(G^\circ) \setminus (S^0 \setminus V(\mathcal{M}_A \cup M_B)) \\ &\subseteq V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \big(\big(V(G^\circ) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)\big) \cap \big(\mathbb{L}_{\eta,k}(G) \cup (\mathbb{S}_{\eta,k}(G) \setminus S^0)\big)\big) \\ &\subseteq V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \big(\mathbb{L}_{\frac{9\eta}{10},k}(G_\nabla) \setminus (V_{\not\rightsquigarrow\mathbf{\Psi}} \cup \mathtt{P}_1)\big) \cup V(G_{\exp})\ . \end{aligned}$$

Pick any vertex v in $\mathbb{L}_{\frac{9\eta}{10},k}(G_\nabla) \setminus (V(\mathcal{M}_A \cup \mathcal{M}_B) \cup V(G_{\mathrm{exp}}) \cup V_{\not\rightsquigarrow\mathbf{\Psi}} \cup \mathtt{P}_1 \cup V_{\rightsquigarrow\mathfrak{A}})$. Then

$$\begin{aligned}
\deg_{G_{\mathrm{reg}}}(v, V(G^\circ)) &\geqslant \deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B)) \\
{\scriptstyle (v \notin V(G_{\mathrm{exp}}))} \quad &\geqslant (1+\frac{9\eta}{10})k - \deg_G(v, \mathbf{\Psi}) - \deg_{G_{\mathcal{D}}}(v, \mathfrak{A}) \\
&\quad - \deg_{G_{\mathrm{reg}}}(v, \bigcup \mathbf{V} \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)) \\
{\scriptstyle (v \notin (V_{\not\rightsquigarrow\mathbf{\Psi}} \cup V_{\rightsquigarrow\mathfrak{A}} \cup \mathtt{P}_1 \cup V(\mathcal{M}_A \cup \mathcal{M}_B)))} \quad &\geqslant (1+\frac{9\eta}{10})k - \frac{\eta k}{100} - \frac{\rho k}{100\Omega^*} - \gamma k \\
&\geqslant (1+\frac{\eta}{2})k\ .
\end{aligned}$$

So for all $X \in \mathcal{V}^\circ \setminus \mathcal{V}(\mathcal{M}_A \cup \mathcal{M}_B)$ with $X \cap (V(G_{\mathrm{exp}}) \cup V_{\rightsquigarrow\mathfrak{A}}) = \emptyset$ we have that for all $v \in X$ that

$$\deg_{G^\circ}(v) \geqslant \deg_{G_{\mathrm{reg}}}(v, V(G^\circ)) \geqslant (1+\frac{\eta}{2})k\ .$$

That is, $X \in \mathcal{L}^\circ$. □

For all vertices $v \in A \setminus S$, we have

$$\begin{aligned}
\deg_{G_{\mathrm{reg}}}\Big(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ\Big) &\geqslant \deg_{G_{\mathrm{reg}}}(v, V_+ \setminus L_\#) - \deg_{G_{\mathrm{reg}}}\big(v, V_+ \setminus (L_\# \cup V(G^\circ)\big) \\
&\quad - \deg_{G_{\mathrm{reg}}}\big(v, (V_+ \cap V(G^\circ)) \setminus (V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \mathcal{L}^\circ)\big) \\
{\scriptstyle (v \in \mathbb{YA},\ \text{def of } \mathcal{C},\ \text{Subclaim 7.44.2.1})} \quad &\geqslant (1+\frac{\eta}{20})k - \deg_{G_{\mathrm{reg}}}(v, V_{\not\rightsquigarrow\mathbf{\Psi}} \cup \mathtt{P}_1 \cup \bigcup \mathcal{C}^-) \\
&\quad - \deg_{G_{\mathrm{reg}}}\Big(v, V(G_{\mathrm{exp}})\Big) \\
&\quad - \deg_{G_{\mathrm{reg}}}\Big(v, (V_{\rightsquigarrow\mathfrak{A}} \cap \mathbb{L}_{\eta,k}(G)) \setminus V(\mathcal{M}_A \cup \mathcal{M}_B)\Big) \\
{\scriptstyle (v \notin S \cup \mathtt{P} \cup \mathbf{shadow}(V(G_{\mathrm{exp}})) \cup R)} \quad &\geqslant (1+\frac{\eta}{20})k - \frac{\eta^2 k}{10^5} - \frac{\eta k}{200} - \rho k - \frac{2\eta^2 k}{10^5} \\
&> (1+\frac{\eta}{40})k + \frac{\eta k}{200}\ ,
\end{aligned}$$

As $|A \setminus S| \geqslant |A|/2$, the set A fulfills the requirements of the claim.

The same calculations hold for B. This finishes the proof of Claim 7.44.2. □

The next auxiliary claim is needed in our proof of Claim 7.44.4 in case **(M2)**.

Claim 7.44.3. Suppose that case **(M2)** occurs. Then there exists an edge $C_AC_B \in E(\mathbf{G}_{\mathrm{reg}})$ such that

$$\deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ) \geqslant (1+\frac{\eta}{40})k + \frac{\eta k}{400}$$

for all but at most $2\varepsilon'\mathfrak{c}$ vertices $v \in C_A$, and all but at most $2\varepsilon'\mathfrak{c}$ vertices $v \in C_B$.

Proof of Claim 7.44.3. Let $(A,B) \in \mathcal{M}$ be given as in Claim 7.44.2. Let $P_A \subseteq A$, and $P_B \subseteq B$ be the vertices which fail the assertion of Claim 7.44.2. Call a cluster $C \in \mathbf{V}$ *A-negligible* if $|C \cap (A \setminus P_A)| \leqslant \frac{\gamma^3\mathfrak{c}}{16\Omega^* k}|A|$. Let R_A be the union of all A-negligible clusters. Recall that (A,B) is entirely contained in one dense spot from $\mathcal{D}_\nabla$ (cf. **(M2)**). By

Fact 4.3, there are at most $\frac{4\Omega^* k}{\gamma^3 \mathfrak{c}}$ A-negligible clusters which contribute to $A \cap R_A$. Thus we get $|A \cap R_A| \leqslant \frac{|A|}{4}$. Similarly, we can introduce the notion B-negligible clusters, and the set R_B, and get $|B \cap R_B| \leqslant \frac{|B|}{4}$. We have

$$|A \setminus (P_A \cup R_A)| \overset{\text{C7.44.2}}{\geqslant} \frac{|A|}{2} - |A \cap R_A| \geqslant \frac{|A|}{4} .$$

Similarly, we get $|B \setminus (P_B \cup R_B)| \geqslant |B|/4$. By the regularity of the pair (A, B) there exists at least one edge $ab \in E\big(G^*[A \setminus (P_A \cup R_A), B \setminus (P_B \cup R_B)]\big)$, where $a \in A, b \in B$, and G^* is the graph formed by edges of $\mathcal{D}_\nabla$. As $V(\mathcal{M}) \subseteq V(G_{\mathrm{reg}})$ by the assumption of case **(t5)**, we have that $ab \in E(G_{\mathrm{reg}})$. Let $C_A, C_B \in \mathbf{V}$ be the clusters containing a and b, respectively. Note that $C_A C_B \in E(\mathbf{G}_{\mathrm{reg}})$.

Recall that as C_A is not A-negligible, i.e., $|C_A \cap (A \setminus P_A)| > \frac{\gamma^3 \mathfrak{c}}{16\Omega^* k} \cdot \frac{\alpha_\odot \rho k}{\Omega^*} > 2\varepsilon' \mathfrak{c}$. Note that for all the vertices $v \in C_A \cap (A \setminus P_A)$ we have $\deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ) \geqslant (1 + \frac{\eta}{40})k + \frac{\eta k}{200}$ by the definition of P_A. By Lemma 2.10, we thus have that $\deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ) \geqslant (1 + \frac{\eta}{40})k + \frac{\eta k}{400}$ for all but at most $2\varepsilon' \mathfrak{c}$ vertices v of C_A. The same calculations hold for C_B. □

Claim 7.44.4. There exist distinct $X_A, X_B \in \mathcal{V}^\circ$ with $E(G^\circ[X_A, X_B]) \neq \emptyset$ and such that $\deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ) \geqslant (1 + \frac{\eta}{40})k$ for all but at most $2\varepsilon' \mathfrak{c}$ vertices $v \in X_A$, and all but at most $2\varepsilon' \mathfrak{c}$ vertices $v \in X_B$.

Proof of Claim 7.44.4. The proof of Claim 7.44.4 depends on which of the cases **(M1)** or **(M2)** occurs.

Let us first consider the case **(M2)**. Let $C_A, C_B \in \mathbf{V}$ be given by Claim 7.44.3. We have $|C_A \setminus V(G^\circ)| \leqslant \sqrt{\varepsilon'} |C_A|$. In particular $C_A \cap V(G^\circ)$ is non-empty. Let $X_A \in \mathcal{V}^\circ$ be an arbitrary set in C_A. Similarly, we obtain a set $X_B \in \mathcal{V}^\circ$, $X_B \subseteq C_B$. The claimed properties of the pair (X_A, X_B) follow directly from Claim 7.44.3.

It remains to treat the case **(M1)**. Let (A, B) be from Claim 7.44.2. Let $(X_A, X_B) \in \mathcal{M}_{\mathrm{good}}$ be such that $X_A \supseteq A$ and $X_B \supseteq B$. Claim 7.44.2 asserts that at least

$$\frac{|A|}{2} \overset{\text{(M1)}}{\geqslant} \frac{\eta^2 \mathfrak{c}}{2 \cdot 10^4} > 2\varepsilon' \mathfrak{c}$$

vertices of A have large degree (in G_{reg}) into the set $V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ$. Therefore, by Lemma 2.10, X_A and X_B satisfy the assertion of the Claim.

□

Now we have all the ingredients to finish the proof of Lemma 7.44. It is enough to put together Claims 7.44.1, and 7.44.4 with the fact that $\deg_{G^\circ}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ) \geqslant \deg_{G_{\mathrm{reg}}}(v, V(\mathcal{M}_A \cup \mathcal{M}_B) \cup \bigcup \mathcal{L}^\circ)$ for all $v \in V(G^\circ)$, by the definition of G°, together with the definition of $\mathcal{L}^\circ$ to deduce that we are in Configuration

$$(\diamond\mathbf{10})\big(\varepsilon, \gamma^2 d/2, \pi\sqrt{\varepsilon'}\nu k, \frac{2(\Omega^*)^2 k}{\gamma^2}, \eta/40\big) ,$$

with $\tilde{G}_{\triangleright D7.25} := G^\circ$, $\mathcal{V}_{\triangleright D7.25} := \mathcal{V}^\circ$, $\mathcal{M}_{\triangleright D7.25} := \mathcal{M}_A \cup \mathcal{M}_B$, $\mathcal{L}^*_{\triangleright D7.25} := \mathcal{L}^\circ$, $A_{\triangleright D7.25} := X_A$, and $B_{\triangleright D7.25} := X_B$. □

8 Embedding trees

In this section we provide an embedding of a tree $T_{\triangleright T1.3} \in \mathbf{trees}(k)$ in the setting of the configurations introduced in Section 7. In Section 8.1 we first give a fairly detailed overview of the embedding techniques used. In Section 8.2 we introduce a class of stochastic processes which will be used for some embeddings. Section 8.3 contains a number of lemmas about embedding small trees. These lemmas will then be used for embedding knags and shrubs of a given fine partition of $T_{\triangleright T1.3}$. Embedding the entire tree $T_{\triangleright T1.3}$ is then handled in final Section 8.4 distinguishing between particular configurations. The configurations are grouped into three categories (Section 8.4.1, Section 8.4.2, and Section 8.4.3) corresponding to the similarities between the configurations.

8.1 Embedding schemes for Configurations ($\diamond\mathbf{2}$)–($\diamond\mathbf{10}$)

In Configurations ($\diamond\mathbf{2}$)–($\diamond\mathbf{10}$) we consider an τk-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of a given tree $T = T_{\triangleright T1.3} \in \mathbf{trees}(k)$ which we aim to embed in a host graph $G_{\triangleright T1.3}$ with that configuration. The total embedding of T comprises of two main ingredients. First, we need to have tools for embedding singular shrubs and knags of the τk-fine partition into various basic building bricks of the configurations: the avoiding set $\mathfrak{A}$, the expander G_{exp}, embedding into regular pairs, and embedding using the vertices of huge degree $\boldsymbol{\Psi}$. Second, we need to combine these basic techniques to embed the entire tree T. Here, the order in which different parts of T are embedded is important. Also, it is important to reserve ahead place for parts of the tree which will be embedded only later.

Below, we outline our embedding techniques, grouped into five related categories[xxv]: Configurations ($\diamond\mathbf{2}$)–($\diamond\mathbf{5}$), Configurations ($\diamond\mathbf{6}$)–($\diamond\mathbf{7}$), Configuration ($\diamond\mathbf{8}$), Configuration ($\diamond\mathbf{9}$), and Configuration ($\diamond\mathbf{10}$), in Sections 8.1.1, 8.1.2, 8.1.3, 8.1.4, 8.1.5, respectively.

[xxv] Configuration ($\diamond\mathbf{1}$) is trivial (see Section 8.4.1) and needs no outline

8.1.1 Embedding in Configurations $(\diamond\mathbf{2})$–$(\diamond\mathbf{5})$

Recall that we are working under Setting 7.4. In each of these configurations we have sets $\mathbf{\Psi}', \mathbf{\Psi}'', L'', L'$ and V_1. Further, we have some additional sets (V_2 and/or $\mathfrak{A}'$) depending on the particular configuration.

A common embedding scheme for Configurations $(\diamond\mathbf{2})$–$(\diamond\mathbf{5})$ is illustrated in Figure 8.1. There are two stages of the embedding procedure: the knags, the shrubs $\mathcal{S}_A$

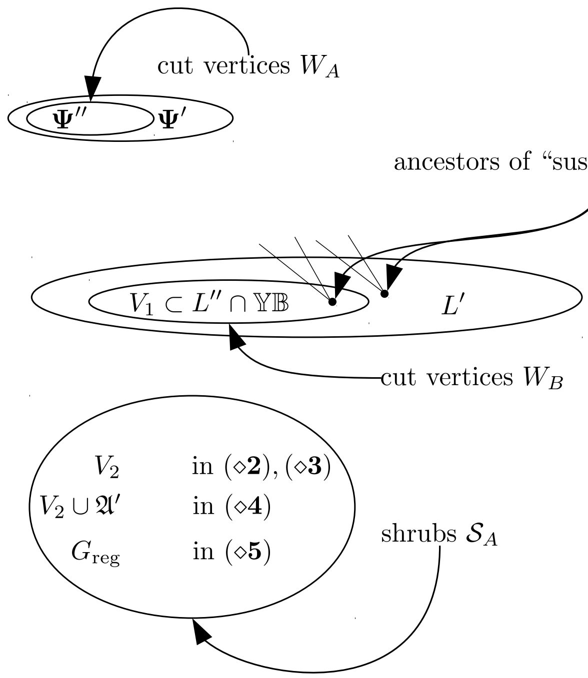

Figure 8.1: An overview of embedding of a tree $T \in \mathbf{trees}(k)$ given with its fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ using Configurations $(\diamond\mathbf{2})$–$(\diamond\mathbf{5})$. The knags are embedded between $\mathbf{\Psi}''$ and V_1, all the shrubs $\mathcal{S}_A$ are embedded into sets specific to particular configurations so that the vertices neighboring W_A are embedded in V_1. Parts of the shrubs $\mathcal{S}_B$ are embedded directly (using various embedding techniques), while the rest is "suspended", i.e., the ancestors of the unembedded remainders are embedded on vertices which have large degrees in $\mathbf{\Psi}'$. The embedding of $\mathcal{S}_B$ is then finalized in the last stage.

and some parts of the shrubs $\mathcal{S}_B$ are embedded in Stage 1, and then in Stage 2 the remainders of $\mathcal{S}_B$ are embedded. Recall that $\mathcal{S}_A$ contains both internal and end shrubs while $\mathcal{S}_B$ contains exclusively end shrubs. We note that here the shrubs $\mathcal{S}_B$ are further subdivided and some parts of them are embedded in the Stage 1 and some in Stage 2.

- In Stage 1, the knags of T are embedded in $\mathbf{\Psi}''$ and V_1 so that W_A is mapped in $\mathbf{\Psi}''$ and W_B is mapped in V_1.
- In Stage 1, the internal and end shrubs of $\mathcal{S}_A$ are embedded using the sets V_1, V_2

Main embedding lemma: Lemma 8.18		
⇑	⇑	⇑
Shrubs $\mathcal{S}_A$	Shrubs $\mathcal{S}_B$ (Stage 1)	Shrubs $\mathcal{S}_B$ (Stage 2)
($\diamond$**2**): Lemma 8.4	Lemma 8.17	Lemma 8.16
($\diamond$**3**): Lemma 8.13		
($\diamond$**4**): Lemma 8.14		
($\diamond$**5**): regularity		

Table 8.1: Embedding lemmas employed for Configurations ($\diamond$**2**)–($\diamond$**5**).

and $\mathfrak{A}'$ which are specific to the particular Configurations ($\diamond$**2**)–($\diamond$**5**). The vertices of $\mathcal{S}_A$ neighboring W_A are always embedded in V_1. Parts of the shrubs $\mathcal{S}_B$ are embedded while the ancestors of the unembedded remainders are embedded on vertices which have large degrees in $\mathbf{\Psi}'$.

- In Stage 2, the embedding of $\mathcal{S}_B$ is finalized. The remainders of $\mathcal{S}_B$ are embedded starting with embedding their roots in $\mathbf{\Psi}'$.

A hierarchy of the embedding lemmas used to resolve Configurations ($\diamond$**2**)–($\diamond$**5**) is given in Table 8.1.

8.1.2 Embedding in Configurations ($\diamond$6)–($\diamond$7)

Suppose Setting 7.4 and 7.7 (see Remark 8.1 below for a comment on the constants $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2$). Recall that we have in each of these configurations sets V_0, V_1, $V_{\mathrm{good}}^{\upharpoonright 2}$, and sets V_2, V_3.

A common embedding scheme for Configurations ($\diamond$**6**)–($\diamond$**7**) is illustrated in Figure 8.2. The embedding has three parts.

- The knags of T are embedded between V_0 and V_1 so that W_A is mapped in V_1 and W_B is mapped in V_0 using either the Preconfiguration (**exp**) or (**reg**). Note that $V_0 \cup V_1 \subseteq \mathfrak{P}_0$.

- The internal shrubs T^* of T are embedded in $V_2 \cup V_3$. The two vertices of T^* which neighbor W_A are always mapped in V_2. Note that the internal shrubs are therefore embedded in $\mathfrak{P}_1$, and thus there is no interference with embedding the knags. We need to understand why a mere degree of δk (see (7.43) and (7.47), with $\delta \ll 1$) is sufficient for embedding internal shrubs of potentially big total order, that is, how to ensure that already embedded internal trees do not cause a blockage when leaving V_1 for V_2 for embedding another internal tree. Here the

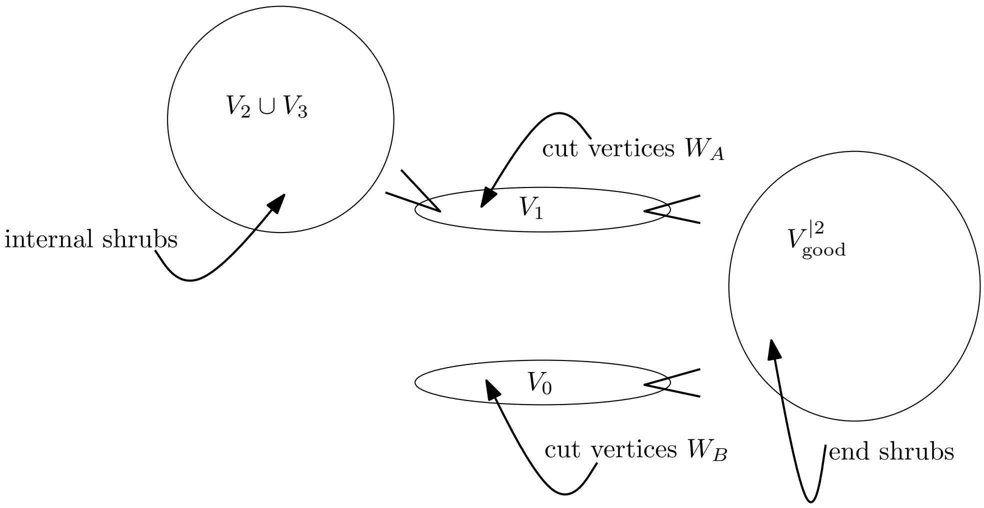

Figure 8.2: An overview of embedding a fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of a tree $T \in \mathbf{trees}(k)$ using Configurations $(\diamond\mathbf{6})$–$(\diamond\mathbf{7})$. The knags are embedded between V_0 and V_1, the internal shrubs are embedded in $V_2 \cup V_3$, and the end shrubs are embedded using $V^{\restriction 2}_{\rm good}$.

expansion[xxvi] working between the V_2 and V_3 comes into play. This property will allow that, when finished embedding an internal tree, the embedding of the follow-up knag will start in a space (in V_1) which sees almost nothing of the previously embedded internal shrubs. And properties of Preconfigurations (**exp**) and (**reg**) will allow to abide by that space during the embedding of that knag.

It is only this part of the embedding process which makes use of the specifics of the Configuration $(\diamond\mathbf{6})$ or $(\diamond\mathbf{7})$. The step of embedding the internal shrubs is the only difference to Configuration $(\diamond\mathbf{8})$, and we describe it in Section 8.1.3.

- The end shrubs are embedded in the yet unoccupied part of G. Most of the end shrubs are embedded into designated vertex set $V^{\restriction 2}_{\rm good}$, and we leave this vertex set only infrequently due to some technical reasons. Note that embedding the end shrubs is performed last and therefore leaving the set $V^{\restriction 2}_{\rm good} \subseteq \mathfrak{P}_2$ does not interfere with embedding other parts of T. Preconfiguration $(\heartsuit\mathbf{1})$ or $(\heartsuit\mathbf{2})$ is used for embedding the end shrubs.

The above embedding scheme is divided in two main steps: first the knags and the internal trees are embedded (see Lemma 8.19), and this partial embedding is then extended to end shrubs (see Lemmas 8.21 and 8.22). A more detailed hierarchy of the embedding lemmas which are used is given in Table 8.2.

[xxvi]this expansion is given by the presence of $G_{\rm exp}$ in Configurations $(\diamond\mathbf{6})$ (cf. (7.45)–(7.46)), and by the presence of the avoiding set $\mathfrak{A}$ in Configurations $(\diamond\mathbf{7})$ ($V_2 \subseteq \mathfrak{A}^{\restriction 1} \setminus \bar{V}$)

Main embedding lemma: Lemma 8.23

⇑ Internal part
- (◇**6**), (◇**7**): Lemma 8.19
- (◇**8**): Lemma 8.20

⇑ End shrubs
- (♡**1**): Lemma 8.21
- (♡**2**): Lemma 8.22

⇑ Knags (feeding Internal part)
- (**exp**): Lemma 8.4
- (**reg**): Lemma 8.7

⇑ Internal shrubs (feeding Internal part)
- (◇**6**): Lemma 8.11
- (◇**7**): Lemma 8.12
- (◇**8**): Lemmas 8.12, 8.8, 8.5

Table 8.2: Embedding lemmas employed for Configurations (◇**6**)–(◇**8**) when embedding a tree $T \in \mathbf{trees}(k)$ with a given fine partition.

Remark 8.1. *In our application of Lemma 7.33 the number* $\mathfrak{p}_1$ *will be approximately the proportion of the total order of the internal shrubs of a given fine partition* $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ *of* T *while* $\mathfrak{p}_2$ *will be approximately the proportion of the total order of the end shrubs. The number* $\mathfrak{p}_0$ *is just a small constant.*

These numbers – scaled up by k *– determine the parameter* $h_1 \approx \mathfrak{p}_1 k$ *(in Configurations* (◇**8**) *and* (◇**9**)*) and* $h_2 \approx \mathfrak{p}_2 k$ *(in Configurations* (◇**6**)*–*(◇**9**)*). The properties of these configurations will then allow to embed all the internal shrubs and end shrubs. Note that the parameter* h_1 *does not appear in Configurations* (◇**6**) *and* (◇**7**)*, while the above value of* h_2 *will allow us to embed the end shrubs. This suggests that the total order of the internal shrubs is not at all important in Configurations* (◇**6**)*–*(◇**7**)*. Indeed, we would succeed even embedding a tree with internal shrubs of total order say* $100k$*.*[xxvii]

In view of this it might be tempting to think that the end shrubs in $\mathcal{S}_A$ *could also be embedded using the same technique as the internal shrubs into the sets* $V_2 \cup V_3$ *provided by these configurations (cf. Figure 8.2). This is however not the case. Indeed, the minimum degree conditions* (7.43)*,* (7.47)*, and* (7.51) *allow embedding only a small number of shrubs from a single cut-vertex* $x \in W_A$ *while there may be many end shrubs attached to* x*; cf. Remark 3.5*(ii)*.*

8.1.3 Embedding in Configuration (◇8)

Suppose Setting 7.4 and 7.7. We are working with sets V_0, V_1, $V_{\mathrm{good}}^{\restriction 2}$, V_2, V_3 and V_4 and with semiregular matching $\mathcal{N}$ coming from the configuration.

[xxvii] Configuration (◇**8**) has this property only in part. We would succeed even embedding a tree with principal subshrubs of total order say $100k$ provided that the total order of peripheral subshrubs is somewhat smaller than h_1.

The embedding scheme follows Table 8.2, and is illustrated in Figure 8.3. Embedding

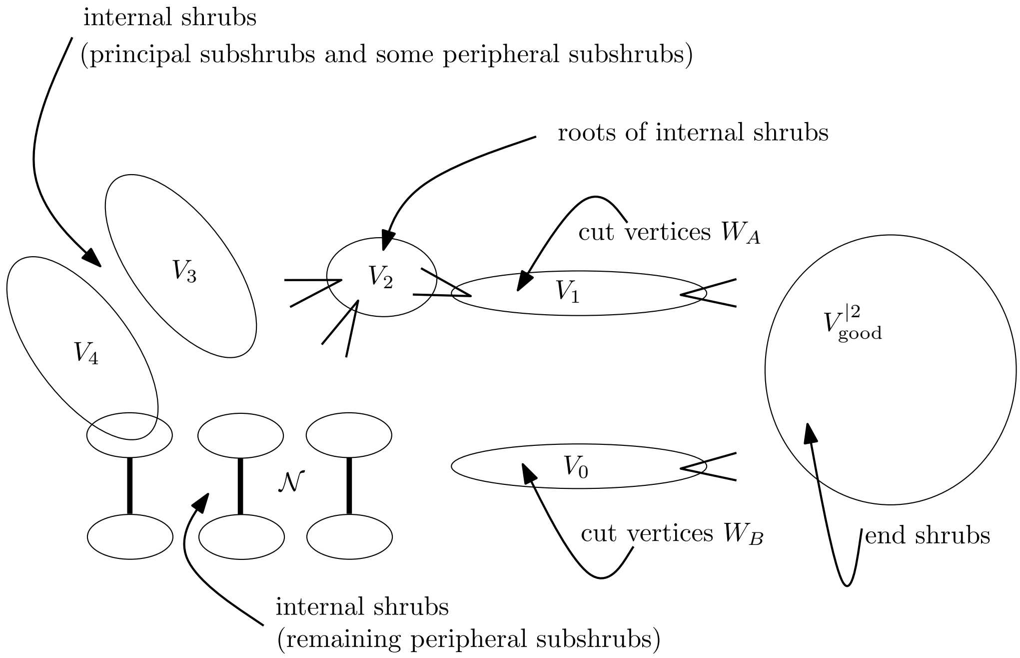

Figure 8.3: An overview of embedding a fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of a tree $T \in \mathbf{trees}(k)$ using Configuration $(\diamond\mathbf{8})$. The knags are embedded between V_0 and V_1. The roots of the internal shrubs are embedded in V_2. Some of the subshrubs of the internal shrubs are embedded in $V_3 \cup V_4$ and some in $\mathcal{N}$; principal subshrubs are always embedded in $V_3 \cup V_4$. The end shrubs are embedded in using $V_{\text{good}}^{\restriction 2}$.

of the knags and of the external shrubs is done in the same way as in Configurations $(\diamond\mathbf{6})$–$(\diamond\mathbf{7})$. We only describe here the way the internal shrubs are embedded. Their roots are embedded in V_2. From that point we proceed embedding subshrub by subshrub. Some of the subshrubs get embedded between V_3 and V_4. This pair of sets has the same expansion property as the pair V_2, V_3 in Configuration $(\diamond\mathbf{7})$. In particular, it allows to avoid the shadow of the already occupied set so that the follow-up knag can be embedded in location almost isolated from the previous images, similarly as described in Section 8.1.2. For this reason we make sure that principal subshrubs get embedded here. The degree condition from V_2 to V_3 is too weak to ensure that all remaining subshrubs are embedded between V_3 and V_4. Therefore we might have to embed some subshrubs in $\mathcal{N}$. Condition (7.57) — where h_1 is approximately the order of the internal shrubs, as in Remark 8.1 — indicates that it should be possible to accommodate all the subshrubs. For technical reasons, the order in which different types of subshrubs are embedded is very important.

8.1.4 Embedding in Configuration ($\diamond$**9**)

The embedding process in Configuration ($\diamond$**9**) follows the same scheme as in Configurations ($\diamond$**6**)–($\diamond$**8**), but the embedding of the internal shrubs follows the regularity method. Pretending the simplest situation $\mathcal{F} = \mathcal{V}_2(\mathcal{N})$ and $V_2 = V_1(\mathcal{N})$, we think of this configuration as having $\deg^{\min}{}_{G_{\mathrm{reg}}}(V_1, V_1(\mathcal{N})) \geqslant h_1$ (cf. (7.58)). See Figure 8.4 for an illustration. Similarly as above, the knags are embedded between V_0 and V_1. The

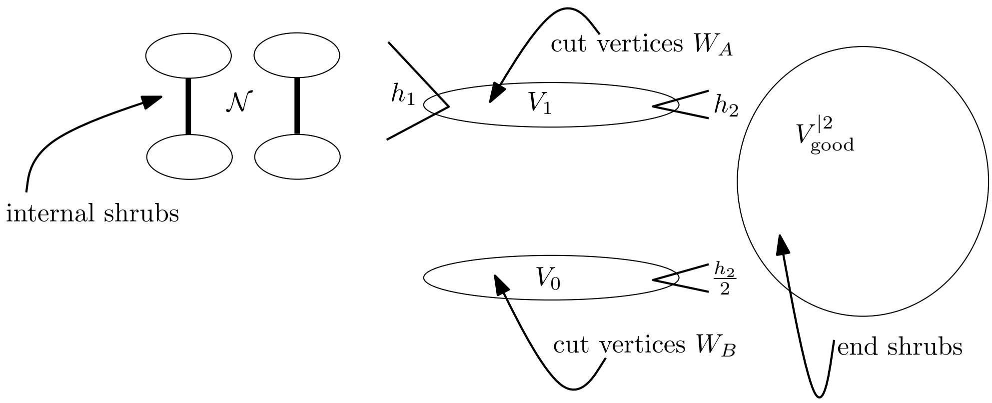

Figure 8.4: An overview of embedding a fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of a tree $T \in \mathbf{trees}(k)$ using Configuration ($\diamond$**9**). The knags are embedded between V_0 and V_1, the internal shrubs using the regularity method in $\mathcal{N}$ and the end shrubs are embedded using $V^{\restriction 2}_{\mathrm{good}}$.

internal shrubs are accommodated using the Regularity Method in $\mathcal{N}$, and the end shrubs are embedded in $V^{\restriction 2}_{\mathrm{good}}$ using Preconfiguration ($\heartsuit$**1**). The embedding lemma is stated in Lemma 8.24.

8.1.5 Embedding in Configuration ($\diamond$**10**)

Configuration ($\diamond$**10**) is very closely related to the structure obtained by Piguet and Stein [PS12] in their solution of the dense approximate case of Conjecture 1.2. Let us describe their proof first. Piguet and Stein prove that when $k > qn$ (for some fixed $q > 0$ and k sufficiently large) the cluster graph[xxviii] $\mathbf{G}_{\mathrm{reg}}$ of a graph $G \in \mathbf{LKS}(n, k, \eta)$ contains the following structure (cf. [PS12, Lemma 7]). There is a set of clusters $\mathbf{L} \subseteq \mathbf{V}$ such that each cluster in $\mathbf{L}$ contains only vertices of captured degrees at least $(1 + \frac{\eta}{2})k$. There is a matching $M \subseteq \mathbf{G}_{\mathrm{reg}}$, and an edge AB, $A, B \in \mathbf{L}$. One of the following conditions is satisfied

(H1) M covers $\mathrm{N}_{\mathbf{G}_{\mathrm{reg}}}(\{A, B\})$, or

[xxviii] ordinary, in the sense of the classic Regularity Lemma

(H2) M covers $\mathrm{N}_{\mathbf{G}_{\mathrm{reg}}}(A)$, and the vertices in B have captured degrees at least $(1+\frac{\eta}{2})\frac{k}{2}$ into $\bigcup(\mathbf{L}\cup V(M))$. Further, each edge in M has at most one endvertex in $\mathrm{N}_{\mathbf{G}_{\mathrm{reg}}}(A)$.

Piguet and Stein use structures **(H1)** and **(H2)** to embed any given tree $T\in\mathbf{trees}(k)$ into G using the Regularity Method; see Sections 3.6 and 3.7 in [PS12], respectively. Actually, a slight relaxation of **(H1)** and **(H2)** would be sufficient for the embedding to work, as can be easily seen from their proof: There is a set of clusters $\mathbf{L}\subseteq\mathbf{V}$ such that each cluster in $\mathbf{L}$ contains only vertices of captured degrees at least $(1+\frac{\eta}{2})k$. There is a matching $M\subseteq\mathbf{G}_{\mathrm{reg}}$, and an edge AB, $A,B\in\mathbf{L}$. One of the following conditions is satisfied

(H1') the vertices in $A\cup B$ have captured degrees at least $(1+\frac{\eta}{2})k$ into the vertices of $\bigcup(\mathbf{L}\cup V(M))$, or

(H2') the vertices in A have captured degrees at least $(1+\frac{\eta}{2})k$ into the vertices of $\bigcup V(M)$, and the vertices in B have captured degrees at least $(1+\frac{\eta}{2})\frac{k}{2}$ into $\bigcup(\mathbf{L}\cup V(M))$. Further, each edge in M has at most one endvertex in $\mathrm{N}_{\mathbf{G}_{\mathrm{reg}}}(A)$.

It can be seen that Configuration ($\diamond$**10**) is a direct counterpart to **(H1')**.[xxix]

The counterpart of **(H2')** is contained in Configuration ($\diamond$**9**) and the similarity is somewhat looser.

Therefore, we do not include a detailed proof of the embedding procedure in Configuration ($\diamond$**10**), referring the reader to [PS12]. The embedding lemma is formally stated in Lemma 8.25.

8.2 Stochastic process Duplicate(ℓ)

Let us introduce a class of stochastic processes, which we call Duplicate(ℓ) ($\ell\in\mathbb{N}$). These are discrete processes $(X_1,Y_1),(X_2,Y_2),\ldots,(X_q,Y_q)\in\{0,1\}^2$ (where $q\in\mathbb{N}$ is arbitrary) satisfying the following.

- For each $i\in[q]$, we have either
  - (a) $X_i=Y_i=0$ (deterministically), or
  - (b) $X_i=Y_i=1$ (deterministically), or
  - (c) exactly one of X_i and Y_i is one, and in that case $\mathbf{P}[X_i=1]=\frac{1}{2}$.

[xxix]Observe that some parts of $\mathbf{G}_{\mathrm{reg}}$ are irrelevant in the embedding process of Piguet and Stein. The objects $\mathbf{G}_{\mathrm{reg}}$, $\mathbf{L}$, and M in that structural result correspond to $(\tilde{G},\mathcal{V})$, $\mathcal{L}^*$, and $\mathcal{M}$ in Configuration ($\diamond$**10**).

- If the distribution of (X_i, Y_i) is according to (c), then the random choice is made independently of the values (X_j, Y_j) $(j < i)$.
- We have $\sum_{i=1}^{q}(X_i + Y_i) \leqslant \ell$.

Needless to say that this definition is not deep and its purpose is only to adopt the language we shall be using later. The following lemma asserts that the first and second component of a process $\mathsf{Duplicate}(\ell)$ are typically balanced.

Lemma 8.2. *Suppose that* $(X_1, Y_1), (X_2, Y_2), \ldots, (X_q, Y_q)$ *is a process in* $\mathsf{Duplicate}(\ell)$. *Then for any* $a > 0$ *we have*

$$\mathbf{P}\left[\sum_{i=1}^{q} X_q - \sum_{i=1}^{q} Y_q \geqslant a\right] \leqslant \exp\left(-\frac{a^2}{2\ell}\right) .$$

Proof. We shall be using the following version of the Chernoff bound for sums of independent random variables Z_i, with distribution $\mathbf{P}[Z_i = 1] = \mathbf{P}[Z_i = -1] = \frac{1}{2}$.

$$\mathbf{P}\left[\sum_{i=1}^{n} Z_i \geqslant a\right] \leqslant \exp\left(-\frac{a^2}{2n}\right) . \tag{8.1}$$

Let $J \subseteq [q]$ be the set of all indices i with $X_i + Y_i = 1$. By the definition of $\mathsf{Duplicate}(\ell)$, we have $|J| \leqslant \ell$. By (8.1) we have

$$\mathbf{P}\left[\sum_{J}(X_i - Y_i) \geqslant a\right] \leqslant \exp\left(-\frac{a^2}{2|J|}\right) \leqslant \exp\left(-\frac{a^2}{2\ell}\right) .$$

□

We shall use the stochastic process $\mathsf{Duplicate}$ to guarantee that certain fixed vertex sets do not get overfilled during our tree embedding procedure. $\mathsf{Duplicate}$ is used in Lemmas 8.11 and 8.12 through Lemma 8.10.

8.3 Embedding small trees

When embedding the tree $T_{\triangleright \mathrm{T}1.3}$ in our proof of Theorem 1.3 it will be important to control where different bits of $T_{\triangleright \mathrm{T}1.3}$ go. This motivates the following notation. Let $X_1, \ldots, X_\ell \subseteq V(T)$ be arbitrary vertex sets of a tree T, and let $V_1, \ldots, V_\ell \subseteq V(G)$ be arbitrary vertex sets of a graph G. Then an embedding $\phi : V(T) \to V(G)$ of T in G is an $(X_1 \hookrightarrow V_1, \ldots, X_\ell \hookrightarrow V_\ell)$*-embedding* if $\phi(X_i) \subseteq V_i$ for each $i \in [\ell]$.

We provide several sufficient conditions for embedding a small tree with additional constraints.

The first lemma deals with embedding using an avoiding set.

Lemma 8.3. *Let $\Lambda, k \in \mathbb{N}$ and let $\varepsilon, \gamma \in (0, \frac{1}{2})$ with $\gamma^2 > \varepsilon$. Suppose $\mathfrak{A}$ is a $(\Lambda, \varepsilon, \gamma, k)$-avoiding set with respect to a set $\mathcal{D}$ of $(\gamma k, \gamma)$-dense spots in a graph H. Suppose that $(T_1, r_1), \ldots, (T_\ell, r_\ell)$ are rooted trees with $|\bigcup_i T_i| \leqslant \gamma k/2$. Let $U \subseteq V(H)$ with $|U| \leqslant \Lambda k$, and let $U^* \subseteq \mathfrak{A}$ with $|U^*| \geqslant \varepsilon k + \ell$. Then there are mutually disjoint $(r_i \hookrightarrow U^*, V(T_i) \setminus \{r_i\} \hookrightarrow V(H) \setminus U)$-embeddings of the trees (T_i, r_i) in H.*

Proof. Since $\mathfrak{A}$ is $(\Lambda, \varepsilon, \gamma, k)$-avoiding, there exists a set $Y \subseteq \mathfrak{A}$ with $|Y| \leqslant \varepsilon k$, such that each vertex v in $\mathfrak{A} \setminus Y$ has degree at least γk into some $(\gamma k, \gamma)$-dense spot $D \in \mathcal{D}$ with $|U \cap V(D)| \leqslant \gamma^2 k$. In particular, $U^* \setminus Y$ is large enough so that we can embed all vertices r_i there. We extend this embedding successively to an embedding of $\bigcup_i T_i$, in each step finding a suitable image in $V(D) \setminus U$ for one neighbour of an already embedded vertex $v \in \bigcup_i V(T_i)$. This is possible since the image of v has degree at least $\gamma k - |U \cap V(D)| > \gamma k/2 \geqslant \sum_i v(T_i)$ into $V(D) \setminus U$. □

The next lemma deals with embedding a tree into a nowhere-dense graph, a primal example of which is the graph G_{exp}.

Lemma 8.4. *Let $k \in \mathbb{N}$, let $Q \geqslant 1$ and let $\gamma, \zeta \in (0, 1)$ be such that $128Q\gamma \leqslant \zeta^2$. Let H be a $(\gamma k, \gamma)$-nowhere-dense graph. Let $(T_1, r_1), \ldots, (T_\ell, r_\ell)$ be rooted trees of total order less than $\zeta k/4$. Let $V_1, V_2, U, U^* \subseteq V(H)$ be four sets with $U^* \subseteq V_1$, $|U| < Qk$, $|U^*| > \frac{32Q^2\gamma}{\zeta}k + \ell$, and $\mathrm{deg}^{\mathrm{min}}{}_H(V_j, V_{3-j}) \geqslant \zeta k$ for $j = 1, 2$. Then there are mutually disjoint $(r_i \hookrightarrow U^*, V_{\mathrm{even}}(T_i) \hookrightarrow V_1 \setminus U, V_{\mathrm{odd}}(T_i) \hookrightarrow V_2 \setminus U)$-embeddings of the trees (T_i, r_i) in H.*

Proof. Set $B := \mathbf{shadow}_H(U, \zeta k/2)$. By Fact 7.2, we have $|B| \leqslant \frac{32Q^2\gamma}{\zeta}k \leqslant \frac{\zeta}{4}k$. In particular, $U^* \setminus B$ is large enough to accommodate the images $\phi(r_i)$ of all vertices r_i.

Successively, extend ϕ, in each step mapping a neighbour u of some already embedded vertex $v \in \bigcup_i V(T_i)$ to a yet unused neighbour of $\phi(v)$ in $V_j \setminus (B \cup U)$, where j is either 1 or 2, depending on the parity of $\mathrm{dist}_T(r, v)$. This is possible as $\phi(v)$, lying outside B, has at least $\zeta k/2$ neighbours in $V_i \setminus U$. Thus $\phi(v)$ has at least $\zeta k/4$ neighbours in $V_i \setminus (U \cup B)$, which is more than $\sum_i v(T_i)$. □

The next three standard lemmas deal with embedding a tree in a regular or a super-regular pair. We omit their proofs.

Lemma 8.5. *Let $\varepsilon > 0$ and $\beta > 2\varepsilon$. Let (C, D) be an ε-regular pair in a graph H, with $|C| = |D| =: \ell$, and with density $\mathrm{d}(C, D) \geqslant 3\beta$. Suppose that there are sets $X \subseteq C$, $Y \subseteq D$, and $X^* \subseteq X$ satisfying $\min\{|X|, |Y|\} \geqslant 4\frac{\varepsilon}{\beta}\ell$ and $|X^*| > \frac{\beta}{2}\ell$. Let (T, r) be a rooted tree of order $v(T) \leqslant \varepsilon\ell$. Then there exists an $(r \hookrightarrow X^*, V_{\mathrm{even}}(T) \hookrightarrow X, V_{\mathrm{odd}}(T) \hookrightarrow Y)$-embedding of T in H.*

Lemma 8.6. *Let $\beta, \varepsilon > 0$ and $\ell \in \mathbb{N}$ be such that $\beta > 2\varepsilon$. Let (C, D) be an ε-regular pair with $|C| = |D| = \ell$ of density $\mathrm{d}(C, D) \geqslant 3\beta$ in a graph H. Let $(T_1, r_1), (T_2, r_2), \ldots, (T_s, r_s)$ be rooted trees with $v(T_i) \leqslant \varepsilon\ell$ for all $i \in [s]$. Let $U \subseteq V(H)$ fulfill $|C \cap U| = |D \cap U|$, and let $X^* \subseteq (C \cup D) \setminus U$ be such that*

$$|X^*| \geqslant \sum_{i=1}^{s} v(T_i) + 50\beta\ell \ . \tag{8.2}$$

Then there are mutually disjoint $(r_i \hookrightarrow X^*, V(T_i) \hookrightarrow (C \cup D) \setminus U)$-embeddings of the trees (T_i, r_i) in H.

Lemma 8.7. *Let $d > 10\varepsilon > 0$. Suppose that (A, B) forms an (ε, d)-super-regular pair with $|A|, |B| \geqslant \ell$. Let $U_A \subseteq A$, $U_B \subseteq B$ be such that $|U_A| \leqslant |A|/2$ and $|U_B| \leqslant d|B|/4$. Let (T, r) be a rooted tree of order at most $d\ell/4$, and let $v \in A \setminus U_A$ be arbitrary. Then there exists an $(r \hookrightarrow v, V_{\mathrm{even}}(T, r) \hookrightarrow A \setminus U_A, V_{\mathrm{odd}}(T, r) \hookrightarrow B \setminus U_B)$-embedding of T.*

The next lemma says that each regular pair can be filled-up in a balanced way by trees.

Lemma 8.8. *Let G be a graph, $v \in V(G)$ be a vertex, $\mathcal{M}$ be an $(\varepsilon, d, \nu k)$-semiregular matching in G, and $\{f_{CD}\}_{(C,D)\in\mathcal{M}}$ a family of integers between $-\tau k$ and τk. Suppose (T, r) is a rooted tree,*

$$v(T) \leqslant \left(1 - \frac{4(\varepsilon + \frac{\tau}{\nu})}{d - 2\varepsilon}\right) |V(\mathcal{M})| \ ,$$

with the property that each component of $T - r$ has order at most τk. If $V(\mathcal{M}) \subseteq \mathrm{N}_G(v)$ then there exists an $(r \hookrightarrow v, V(T - r) \hookrightarrow V(\mathcal{M}))$-embedding ϕ of T such that for each $(C, D) \in \mathcal{M}$ we have $|C \cap \phi(T)| + f_{CD} = |D \cap \phi(T)| \pm \tau k$.

The proof of Lemma 8.8 is standard, and is given for example in [HP, Lemma 6.6].

Lemma 8.8 suggests the following definitions. A *discrepancy* of a set X with respect to a pair of sets (C, D) is the number $|C \cap X| - |D \cap X|$. X is *s-balanced* with respect to a semiregular matching $\mathcal{M}$ if the discrepancy of X with respect to each $(C, D) \in \mathcal{M}$ is at most s in absolute value.

Lemma 8.9. *Let G be a graph, $v \in V(G)$ be a vertex, $\mathcal{M}$ be an $(\varepsilon, d, \nu k)$-semiregular matching in G with an $\mathcal{M}$-cover $\mathcal{F}$, and $U \subseteq V(G)$. Suppose (T, r) is a rooted tree with*

$$v(T) + |U| \leqslant \deg_G\left(v, V(\mathcal{M}) \setminus \bigcup \mathcal{F}\right) - \frac{4(\varepsilon + \frac{\tau}{\nu})}{d - 2\varepsilon}|V(\mathcal{M})| \ ,$$

such that each component of $T - r$ has order at most τk. Then there exists an $(r \hookrightarrow v, V(T - r) \hookrightarrow V(\mathcal{M}) \setminus U)$-embedding ϕ of T.

The proof of Lemma 8.9 is again standard and we again omit it.

The following lemma uses a probabilistic technique to embed a shrub while reserving a set of vertices in the host graph for later use. We wish the reserved set to use about as much space inside certain given sets P_i as the image of our shrub does. (In later applications the sets P_i correspond to neighbourhoods of vertices which are still 'active'.)

Lemma 8.10 will find an immediate application in all the remaining lemmas of this subsection. However it is only really needed for Lemmas 8.11–8.12, which deal with embedding shrubs in the presence of one of the Configurations ($\diamond$**6**)–($\diamond$**8**). For Lemmas 8.13 and 8.14, which are for Configurations ($\diamond$**3**) and ($\diamond$**4**) a simpler auxiliary lemma (without reservations) would suffice.

Lemma 8.10. *Let H be a graph, let $X^*, X_1, X_2, P_1, P_2, \ldots, P_L \subseteq V(H)$, and let (T_1, r_1), $\ldots$, (T_ℓ, r_ℓ) be rooted trees, such that $L \leqslant k$, $|P_j| \leqslant k$ for each $j \in [L]$, and $|X^*| \geqslant 2\ell$. Suppose that $\deg^{\min}(X_1 \cup X^*, X_2) \geqslant 2\sum v(T_i)$ and $\deg^{\min}(X_2, X_1) \geqslant 2\sum v(T_i)$.*

Then there exist pairwise disjoint $(r_i \hookrightarrow X^*, V_{\mathrm{even}}(T_i, r_i) \setminus \{r_i\} \hookrightarrow X_1, V_{\mathrm{odd}}(T_i, r_i) \hookrightarrow X_2)$-embeddings ϕ_i of T_i in G and a set $C \subseteq V(H) \setminus \bigcup \phi_i(T_i)$ of size $\sum v(T_i)$ such that for each $j \in [L]$ we have

$$|P_j \cap \bigcup \phi_i(T_i)| \leqslant |P_j \cap C| + k^{3/4}. \tag{8.3}$$

Proof. Let $m := \sum v(T_i)$.

We construct pairwise disjoint random $(r_i \hookrightarrow X^*, V_{\mathrm{even}}(T_i, r_i) \setminus \{r_i\} \hookrightarrow X_1, V_{\mathrm{odd}}(T_i, r_i) \hookrightarrow X_2)$-embeddings ϕ_i and a set $C \subseteq V(H) \setminus \bigcup \phi_i(T_i)$ which satisfies (8.3) with positive probability. Then the statement follows.

Enumerate the vertices of $\bigcup T_i$ as $\bigcup V(T_i) = \{v_1, \ldots, v_m\}$ such that $v_i = r_i$ for $i = 1, \ldots, \ell$, and such that for each $j > \ell$ we have that the parent of v_j lies is the set $\{v_1, \ldots, v_{j-1}\}$. Pick pairwise disjoint sets $A_1, \ldots, A_\ell \subseteq X^*$ of size two. Uniformly at random denote one element of A_j as x_j and the other as y_j.

Now, successively for $i = \ell + 1, \ldots, m$, we shall define vertices x_i and y_i. Let r denote the root of the tree in which v_i lies, and let $v_s = \mathrm{Par}(v_i)$. We shall choose $x_i, y_j \in X_{j_i}$ where $j_i = \mathrm{dist}(r, v_i) \mod 2 + 1$. In step i, proceed as follows. Since $x_s \in X_{j_s}$ (or since $x_s \in X^*$), we have

$$\deg(x_s, X_{j_i} \setminus \bigcup_{h<i} \{x_h, y_h\}) \geqslant 2.$$

Hence, we may take an arbitrary subset $A_i \subseteq (\mathrm{N}(x_s) \cap X_{j_i}) \setminus \bigcup_{h<i} \{x_h, y_h\}$ of size exactly two. As above, randomly label its elements as x_i and y_i independently of all other choices.

The choices of the maps $(v_j \mapsto x_j)_{j=1}^m$ determine $\phi_1, \ldots, \phi_\ell$. Then $C := \{y_1, \ldots, y_m\}$ has size exactly m and avoids $\bigcup \phi_i(T_i)$.

For each $j \in [L]$ we set up a stochastic process $\mathfrak{S}^{(j)} = \left((X_i^{(j)}, Y_i^{(j)})\right)_{i=1}^m$, defined by $X_i^{(j)} = \mathbf{1}_{\{x_i \in P_j\}}$ and $Y_i^{(j)} = \mathbf{1}_{\{y_i \in P_j\}}$. Note that $\mathfrak{S}^{(j)} \in \mathsf{Duplicate}(|P_j|) \subseteq \mathsf{Duplicate}(k)$. Thus, for a fixed $j \in [L]$, by Lemma 8.2, the probability that $|P_j \cap (\bigcup \phi_i(T_i))| > |P_j \cap C| + k^{3/4}$ is at most $\exp(-\sqrt{k}/2)$. Using the union bound over all $j \in [L]$ we get that Property 8.4 holds with probability at least

$$1 - L \cdot \exp\left(-\frac{\sqrt{k}}{2}\right) > 0 \, .$$

This finishes the proof. □

We now get to the first application of Lemma 8.10.

Lemma 8.11. *Assume we are in Setting 7.4. Suppose that the sets $V_0, \ldots, V_3$ witness Configuration $(\diamond\mathbf{6})(\delta, 1, 0, 0, 0)$, where $300/\delta < k$. Suppose that $U, U^*, P_1, P_2, \ldots, P_L \subseteq V(G)$, and $L \leqslant k$, are such that $|U| \leqslant \frac{\delta}{24\sqrt{\gamma}}k$, $U^* \subseteq V_2$, $|U^*| \geqslant \frac{\delta}{4}k$, and $|P_j| \leqslant k$ for each $j \in [L]$. Let (T, r) be a rooted tree of order at most $\delta k/8$.*

Then there exists a $(r \hookrightarrow U^*, V_{\mathrm{even}}(T, r) \setminus \{r\} \hookrightarrow V_2 \setminus U, V_{\mathrm{odd}}(T, r) \hookrightarrow V_3 \setminus U)$-embedding ϕ of T in G and a set $C \subseteq V(G - \phi(T))$ of size $v(T)$ such that for each $j \in [L]$ we have

$$|P_j \cap \phi(T)| \leqslant |P_j \cap C| + k^{3/4}. \tag{8.4}$$

Proof. Set $B := \mathbf{shadow}_{G_{\exp}}(U, \delta k/4)$. By Fact 7.2, we have that $|B| \leqslant 64\frac{\gamma}{\delta}(\frac{\delta}{24\sqrt{\gamma}})^2 k \leqslant \frac{\delta}{4}k - 2$. In particular, $X^* := U^* \setminus B$ has size at least 2. Set $X_1 := V_2 \setminus (U \cup B)$ and set $X_2 := V_3 \setminus (U \cup B)$. Using (7.45) and (7.46), we have for $j = 1, 2$ that

$$\mathrm{deg^{min}}_{G_{\exp}}(X_j, X_{3-j}) \geqslant \delta k - \mathrm{deg^{max}}_{G_{\exp}}(X_j, U) - |B| \geqslant \delta k - \frac{\delta}{4}k - \frac{\delta}{4}k \geqslant 2v(T).$$

We may thus apply Lemma 8.10 to obtain the desired embedding ϕ. □

Lemma 8.12. *Assume Setting 7.4 and Setting 7.7. Suppose that we are given sets $Y_1, Y_2 \subseteq \mathfrak{P}_1 \setminus \bar{V}$ with $Y_1 \subseteq \mathfrak{A}$, $\mathrm{deg^{max}}_{G_{\mathcal{D}}}(Y_1, \mathfrak{P}_1 \setminus Y_2) \leqslant \frac{\eta\gamma}{400}$, and $\mathrm{deg^{min}}_{G_{\mathcal{D}}}(Y_2, Y_1) \geqslant \delta k$.*

Suppose that $U, U^*, P_1, P_2, \ldots, P_L \subseteq V(G)$ are sets such that $|U| \leqslant \frac{\Lambda\delta}{2\Omega^*}k$, $U^* \subseteq Y_1$, with $|U^*| \geqslant \frac{\delta}{4}k$, $|P_j| \leqslant k$ for each $j \in [L]$, and $L \leqslant k$. Suppose $(T_1, r_1), \ldots, (T_\ell, r_\ell)$ are rooted trees of total order at most $\delta k/1000$. Suppose further that $\delta < \eta\gamma/100$, $\varepsilon' < \delta/1000$, and $k > 1000/\delta$.

Then there exist pairwise disjoint $(r_i \hookrightarrow U^*, V_{\mathrm{even}}(T_i, r_i) \hookrightarrow Y_1 \setminus U, V_{\mathrm{odd}}(T_i, r_i) \hookrightarrow Y_2 \setminus U)$-embeddings ϕ_i of T_i in G and a set $C \subseteq V(G - \bigcup \phi_i(T_i))$ of size $\sum v(T_i)$ such that for each $j \in [L]$ we have that

$$|P_j \cap \bigcup \phi_i(T_i)| \leqslant |P_j \cap C| + k^{3/4}. \tag{8.5}$$

Proof. Set $U' := \mathbf{shadow}_{G_{\mathcal{D}}}(U, \delta k/2) \cup U$. By Fact 7.1, we have $|U'| \leqslant \Lambda k$. As Y_1 is a $(\Lambda, \varepsilon', \gamma, k)$-avoiding set, by Definition 4.6 there exists a set $B \subseteq Y_1$, $|B| \leqslant \varepsilon' k$ such that for all $v \in Y_1 \setminus B$ there exists a dense spot $D_v \in \mathcal{D}$ with $v \in V(D_v)$ and $|V(D_v) \cap U'| \leqslant \gamma^2 k$. As Y_1 is disjoint from $\bar{V}$, by Definition 7.6(4) and by (7.14), we have that $\deg_{D_v}(v, V(D_v)^{\restriction 1}) \geqslant \frac{\eta\gamma}{200}k$. We have that $\deg_{G_{\mathcal{D}}}(v, V(D_v)^{\restriction 1} \setminus V_3) < \frac{\eta\gamma}{400}k$, and hence,

$$\deg_{G_{\mathcal{D}}}\left(v, (V(D_v)^{\restriction 1} \cap Y_2) \setminus U'\right) \geqslant \frac{\eta\gamma k}{400} - \gamma^2 k \geqslant \frac{\eta\gamma k}{800} .$$

Thus,

$$\mathrm{deg^{min}}_{G_{\mathcal{D}}}(Y_1 \setminus B, Y_2 \setminus U') \geqslant \frac{\eta\gamma k}{800} \geqslant 2 \sum v(T_i) . \tag{8.6}$$

Further, by the definition of U' and by (7.50), we have

$$\mathrm{deg^{min}}_{G_{\mathcal{D}}}(Y_2 \setminus U', Y_1 \setminus U) \geqslant \frac{\delta k}{2} \geqslant 2 \sum v(T_i) . \tag{8.7}$$

Set $X^* := U^* \setminus B$, and note that $|U^* \setminus B| \geqslant \delta k/4 - \varepsilon' k \geqslant 2\ell$. Set $X_1 := Y_1 \setminus (U \cup B)$ and $X_2 := Y_2 \setminus (U' \cup B)$. Inequalities (8.6) and (8.7) guarantee that we may apply Lemma 8.10 to obtain the desired embeddings ϕ_i. □

Lemma 8.13. *Assume Setting 7.4. Suppose that the sets $L', L'', \mathbf{\Psi}', \mathbf{\Psi}'', V_1, V_2$ witness Configuration $(\diamond\mathbf{3})(0, 0, \gamma/4, \delta)$. Suppose that $U, U^* \subseteq V(G)$ are sets such that $|U| \leqslant k$, $U^* \subseteq V_1$, $|U^*| \geqslant \frac{\delta}{4}k$. Suppose (T, r) is a rooted tree of order at most $\delta k/1000$. Suppose further that $\delta \leqslant \gamma/100$, $\varepsilon' < \delta/1000$, and $4\Omega^*/\delta \leqslant \Lambda$.*

Then there is an $(r \hookrightarrow U^*, V_{\mathrm{even}}(T, r) \setminus \{r\} \hookrightarrow V_1 \setminus U, V_{\mathrm{odd}}(T, r) \hookrightarrow V_2 \setminus U)$-embedding of T in G.

Proof. The proof of this lemma is very similar to the one of Lemma 8.12 (in fact, even easier). Set $U' := \mathbf{shadow}_{G_{\mathcal{D}}}(U, \delta k/2) \cup U$ and note that $|U'| \leqslant \Lambda k$ by Fact 7.1. As V_1 is $(\Lambda, \varepsilon', \gamma, k)$-avoiding, by Definition 4.6 there is a set $B \subseteq V_1$, $|B| \leqslant \varepsilon' k$ such that for all $v \in V_1 \setminus B$ there exists a dense spot $D_v \in \mathcal{D}$ with $\deg_{D_v}(v, V(D_v) \setminus U') \geqslant \gamma k/2$. By (7.26), we know that $\deg_{G_{\mathcal{D}}}(v, V(D_v) \setminus V_2) \leqslant \gamma k/4$, and hence, $\deg_{G_{\mathcal{D}}}\left(v, (V(D_v) \cap V_2) \setminus U'\right) \geqslant \gamma k/4$. Thus,

$$\mathrm{deg^{min}}_{G_{\mathcal{D}}}(V_1 \setminus B, V_2 \setminus U') \geqslant \frac{\gamma k}{4} \geqslant 2v(T) . \tag{8.8}$$

Further, by the definition of U' and by (7.27), we have

$$\mathrm{deg^{min}}_{G_{\mathcal{D}}}(V_2 \setminus U', V_1 \setminus U) \geqslant \frac{\delta k}{2} \geqslant 2(T) . \tag{8.9}$$

Set $X^* := U^* \setminus B$, and note that $|X^*| \geqslant \delta k/4 - \varepsilon' k \geqslant 2$. Set $X_1 := V_1 \setminus (U \cup B)$ and $X_2 := V_2 \setminus (U' \cup B)$. Inequalities (8.8) and (8.9) guarantee that we may apply Lemma 8.10 (with empty sets P_i) to obtain the desired embedding ϕ. □

Lemma 8.14. *Assume Setting 7.4. Suppose that the sets $L', L'', \mathbf{\Psi}', \mathbf{\Psi}'', V_1, \mathfrak{A}', V_2$ witness Configuration $(\diamond\mathbf{4})(0, 0, \gamma/4, \delta)$. Suppose that $U \subseteq V(G)$, $U^* \subseteq V_1$ are sets such that $|U| \leqslant k$ and $|U^*| \geqslant \frac{\delta}{4}k$. Suppose (T, r) is a rooted tree of order at most $\delta k/20$ with a fruit r'. Suppose further that $4\varepsilon' \leqslant \delta \leqslant \gamma/100$, and $\Lambda \geqslant 300(\frac{\Omega^*}{\delta})^3$.*

Then there exists an $(r \hookrightarrow U^*, r' \hookrightarrow V_1 \setminus U, V(T) \setminus \{r, r'\} \hookrightarrow (\mathfrak{A}' \cup V_2) \setminus U)$-embedding of T in G.

Proof. Set $U' := \tilde{U} \cup \mathbf{shadow}_{G_\nabla - \mathbf{\Psi}}(U, \delta k/4) \cup \mathbf{shadow}^{(2)}_{G_\nabla - \mathbf{\Psi}}(\tilde{U}, \delta k/4)$ and $U'' := \tilde{U} \cup \mathbf{shadow}_{G_\mathcal{D}}(U', \delta k/2)$ and use Fact 7.1 to see that $|U'| \leqslant \frac{\delta}{4\Omega^*}\Lambda k$ and $|U''| \leqslant \Lambda k$. We then use Definition 4.6 and (7.31) to see that there is a set $B \subseteq \mathfrak{A}'$ of size at most $\varepsilon' k$ such that

$$\deg^{\min}{}_{G_\mathcal{D}}(\mathfrak{A}' \setminus B, V_2 \setminus U'') \geqslant 2v(T)\ . \tag{8.10}$$

Using (8.10), and employing (7.28) and (7.30), we see that we may apply Lemma 8.10 with $X^*_{\triangleright \mathrm{L}8.10} := U^*$, $X_{1,\triangleright \mathrm{L}8.10} := \mathfrak{A}' \setminus (B \cup U')$ and $X_{2,\triangleright \mathrm{L}8.10} := V_2 \setminus U''$ (and with empty sets P_i) in order to embed the tree $T - T(r, \uparrow r')$ rooted at r. Then embed $T(r, \uparrow r')$, by applying Lemma 8.10 a second time, using (7.28) and (7.29). □

8.4 Main embedding lemmas

For this section, we need to introduce the notion of ghost. Given a semiregular matching $\mathcal{N}$, we call an involution $\mathfrak{d} : V(\mathcal{N}) \to V(\mathcal{N})$ with the property that $\mathfrak{d}(S) = T$ for each $(S, T) \in \mathcal{N}$ a *matching involution.*

Assume Setting 7.4 and fix a matching involution $\mathfrak{b}$ for $\mathcal{M}_A \cup \mathcal{M}_B$. For any set $U \subseteq V(G)$ we then define by

$$\mathbf{ghost}(U) := U \cup \mathfrak{b}\big(U \cap V(\mathcal{M}_A \cup \mathcal{M}_B)\big)\ .$$

Clearly, we have that $|\mathbf{ghost}(U)| \leqslant 2|U|$, and $|\mathbf{ghost}(U) \cap S| = |\mathbf{ghost}(U) \cap T|$ for each $(S, T) \in \mathcal{M}_A \cup \mathcal{M}_B$.

The notion of ghost extends to other semiregular matchings. If $\mathcal{N}$ is a semiregular matching and $\mathfrak{d}$ a matching involution for $\mathcal{N}$ then we write $\mathbf{ghost}_\mathfrak{d}(U) := U \cup \mathfrak{d}\big(U \cap V(\mathcal{N})\big)$.

8.4.1 Dealing with Configuration $(\diamond\mathbf{1})$

This subsection contains an easy observation that $\mathbf{trees}(k) \subseteq G$ in case G contains Configuration $(\diamond\mathbf{1})$.

Lemma 8.15. *Let G be a graph, and let $A, B \subseteq V(G)$ be such that $\deg^{\min}(G[A, B]) \geqslant k/2$, and $\deg^{\min}(A) \geqslant k$. Then $\mathbf{trees}(k) \subseteq G$.*

Proof. Let $T \in \mathbf{trees}(k)$ have colour classes X and Y, with $|X| \geqslant k/2 \geqslant |Y|$. By Fact 2.1, for the set W of those leaves of T that lie in X, we have $|X \setminus W| \leqslant k/2$. We embed $T - W$ greedily in G, mapping Y to A and $X \setminus W$ to B. We then embed W using the fact that $\deg^{\min}(A) \geqslant k$. □

8.4.2 Dealing with Configurations (⋄2)–(⋄5)

In this section we show how to embed $T_{\triangleright T1.3}$ in the presence of configurations (⋄**2**)–(⋄**5**). As outlined in Section 8.1.1 our main embedding lemma, Lemma 8.18, builds on Lemma 8.17 which handles Stage 1 of the embedding, and Lemma 8.16 which handles Stage 2.

Lemma 8.16. *Assume we are in Setting 7.4. Suppose L'', L' and $\mathbf{\Psi}'$ witness Preconfiguration* (♣)$(\frac{10^5\Omega^*}{\eta})$*. Let (T, r) be a rooted tree of order at most $\gamma^2\nu k/6$. Let $U \subseteq V(G)$ with $|U|+v(T) \leqslant k$, and let $v \in \mathbf{\Psi}'\setminus U$. Then there exists an $(r \hookrightarrow v, V(T) \hookrightarrow V(G)\setminus U)$-embedding of (T, r).*

Proof. We proceed by induction on the order of T. The base $v(T) \leqslant 2$ obviously holds. Let us assume Lemma 8.16 is true for all trees T' with $v(T') < v(T)$.

Let $U_1 := \mathbf{shadow}_{G_\nabla}(U - \mathbf{\Psi}, \eta k/200)$, and $U_2 := \bigcup\{C \in \mathbf{V} : |C \cap U| \geqslant \frac{1}{2}|C|\}$. We have $|U_1| \leqslant \frac{200\Omega^*}{\eta}k$ by Fact 7.1, and $|U_2| \leqslant 2|U|$. Set

$$
\begin{aligned}
L_{\mathfrak{A}} &:= L'' \cap \mathbf{shadow}_{G_\nabla}(\mathfrak{A}, \frac{\eta k}{50}),\\
L_{\mathbf{\Psi}} &:= L'' \cap \mathbf{shadow}_{G_\nabla}\left(\mathbf{\Psi}, |U \cap \mathbf{\Psi}| + \frac{\eta k}{50}\right), \text{ and}\\
L_{\mathbf{V}} &:= L'' \cap \mathbf{shadow}_{G_{\mathrm{reg}}}\left(V(G_{\mathrm{reg}}), (1 + \frac{\eta}{50})k - |U \cap \mathbf{\Psi}|\right).
\end{aligned}
$$

Observe that $L_{\mathbf{V}} \subseteq \bigcup \mathbf{V}$ and that since $L'' \subseteq \mathbb{L}_{\frac{9}{10}\eta,k}(G_\nabla) \setminus \mathbf{\Psi}$, we have

$$L'' \subseteq V(G_{\exp}) \cup \mathfrak{A} \cup L_{\mathbf{\Psi}} \cup L_{\mathfrak{A}} \cup L_{\mathbf{V}}.$$

As by (7.25), we have $\deg_G(v, L'') \geqslant \frac{10^5\Omega^* k}{\eta} > 5(|U \cup U_1 \cup U_2| + v(T) + \eta k)$, one of the following five cases must occur.

<u>*Case I:* $\deg_G(v, V(G_{\exp}) \setminus U) > v(T) + \eta k$.</u> Lemma 8.4 gives an embedding of the forest $T - r$ (whose components are rooted at neighbours of r). The input sets/parameters of Lemma 8.4 are $Q_{\triangleright L8.4} := 1$, $\zeta_{\triangleright L8.4} := 12\sqrt{\gamma}$, $U^*_{\triangleright L8.4} := (\mathrm{N}_G(v) \cap V(G_{\exp})) \setminus U$, $U_{\triangleright L8.4} := U$, $V_1 = V_2 := V(G_{\exp})$.

<u>*Case II:* $\deg_G(v, \mathfrak{A} \setminus U) > v(T) + \eta k$.</u> Lemma 8.3 gives an embedding of the forest $T - r$ (whose components are rooted at neighbours of r). The input sets/parameters of Lemma 8.3 are $U^*_{\triangleright L8.3} := (\mathrm{N}_G(v) \cap \mathfrak{A}) \setminus U$, $U_{\triangleright L8.3} := U$ and $\varepsilon_{\triangleright L8.3} := \varepsilon' \leqslant \eta$. Here,

and below, we tacitly implicitly assume parameters of the same name to be the same, i.e. $\gamma_{\triangleright L8.3} := \gamma$.

<u>*Case III:* $\deg_G(v, L_{\mathfrak{A}} \setminus (U \cup U_1)) > v(T) + \eta k$.</u> We only outline the strategy. Embed the children of r in $L_{\mathfrak{A}} \setminus (U \cup U_1)$ using a map $\phi : \mathrm{Ch}_T(r) \to L_{\mathfrak{A}} \setminus (U \cup U_1)$. By definition of $L_{\mathfrak{A}}$, and U_1, we have $\deg_{G_\nabla}(\phi(w), \mathfrak{A} \setminus U) > \frac{\eta k}{100}$ for each $w \in \mathrm{Ch}_T(r)$. Now, for every $w \in \mathrm{Ch}_T(r)$ we can proceed as in Case II to extend this embedding to the rooted tree $\big(T(r, \uparrow w), w\big)$. That is, Case III is "Case II with an extra step in the beginning".

<u>*Case IV:* $\deg_G(v, L_{\mathbf{\Psi}} \setminus U) > v(T) + \eta k$.</u> We embed the children $\mathrm{Ch}_T(r)$ of r in distinct vertices of $L_{\mathbf{\Psi}} \setminus U$. This is possible by the assumption of Case IV.

Now, (7.23) implies that ${\deg^{\min}}_{G_\nabla}(L_{\mathbf{\Psi}}, \mathbf{\Psi'}) \geqslant |U \cap \mathbf{\Psi}| + \frac{\eta k}{100}$. Consequently,

$$ {\deg^{\min}}_{G_\nabla}(L_{\mathbf{\Psi}}, \mathbf{\Psi'} \setminus U) \geqslant \frac{\eta k}{100} \;. $$

Therefore, for each $w \in \mathrm{Ch}_T(r)$ embedded in $L_{\mathbf{\Psi}} \setminus U$ we can find an embedding of $\mathrm{Ch}_T(w)$ in $\mathbf{\Psi'} \setminus U$ such that the images of grandchildren of r are disjoint. We fix such an embedding. We can now apply induction. More specifically, for each grandchild u of r we embed the rooted tree $\big(T(r, \uparrow u), u\big)$ using Lemma 8.16 (employing induction) using the updated set U, to which the images of the newly embedded vertices were added.

<u>*Case V:* $\deg_G(v, L_{\mathbf{V}} \setminus (U \cup \cup U_1 \cup U_2)) \geqslant v(T)$.</u> Let $u_1, \ldots, u_\ell$ be the children of r. Let us consider arbitrary distinct neighbours $x_1, \ldots, x_\ell \in L_{\mathbf{V}} \setminus (U \cup U_1 \cup U_2)$ of v. Let $T_i := T(r, \uparrow u_i)$. We sequentially embed the rooted trees (T_i, u_i), $i = 1, \ldots, \ell$, writing ϕ for the embedding. In step i, consider the set $W_i := \Big(U \cup \bigcup_{j<i} \phi(T_j)\Big) \setminus \mathbf{\Psi}$. Let $D_i \in \mathbf{V}$ be the cluster containing x_i. By definition of $L_{\mathbf{V}}$ and of U_1,

$$ \deg_{G_{\mathrm{reg}}}(x_i, V(G_{\mathrm{reg}}) \setminus W_i) \geqslant \frac{\eta k}{50} - \frac{\eta k}{200} \geqslant \frac{\eta k}{100} \;. $$

Fact 4.11 yields a cluster $C_i \in \mathbf{V}$ such that

$$ \deg_{G_{\mathrm{reg}}}(x_i, C_i \setminus W_i) \geqslant \frac{\eta}{100} \cdot \frac{\gamma \mathfrak{c}}{2(\Omega^*)^2} > \frac{\gamma^2 \mathfrak{c}}{2} + v(T) > \frac{12\varepsilon' \mathfrak{c}}{\gamma^2} + v(T) \;. $$

In particular there is at least one edge from $E(G_{\mathrm{reg}})$ between C_i and D_i, and therefore, (C_i, D_i) forms an ε'-regular pair of density at least γ^2 in G_{reg}. Map u_i to x_i and let $F_1, \ldots, F_m$ be the components of the forest $T_i - u_i$. We now sequentially embed the trees F_j in the pair (D_i, C_i) using Lemma 8.5, with $X_{\triangleright L8.5} := C_i \setminus (W_i \cup \bigcup_{q<j} \phi(F_q))$, $X^*_{\triangleright L8.5} := \mathrm{N}_{G_{\mathrm{reg}}}(x_i, X_{\triangleright L8.5})$, $Y_{\triangleright L8.5} := D_i \setminus (W_i \cup \{x_i\} \cup \bigcup_{q<j} \phi(F_q))$, $\varepsilon_{\triangleright L8.5} := \varepsilon'$, and $\beta_{\triangleright L8.5} := \gamma^2/3$. □

We are now ready for the lemma that will handle Stage 1 in configurations $(\diamond\mathbf{2})$–$(\diamond\mathbf{5})$.

Lemma 8.17. *Assume we are in Setting 7.4, with $L'', L', \mathbf{\Psi}'$ witnessing $(\clubsuit)(\Omega^\star)$ in G. Let $U \subseteq V(G) \setminus \mathbf{\Psi}$ and let (T, r) be a rooted tree with $v(T) \leqslant k/2$ and $|U| + v(T) \leqslant k$. Suppose that each component of $T - r$ has order at most τk. Let $x \in (L'' \cap \mathbb{YB}) \setminus \bigcup_{i=0}^{2} \mathbf{shadow}^{(i)}_{G_\nabla}(\mathbf{ghost}(U), \eta k/1000)$.*

Then there is a subtree T' of T with $r \in V(T')$ which has an $(r \hookrightarrow x, V(T') \setminus \{r\} \hookrightarrow V(G) \setminus \mathbf{\Psi})$-embedding ϕ. Further, the components of $T - T'$ can be partitioned into two (possibly empty) sets $\mathcal{C}_1$, $\mathcal{C}_2$, such that the following two assertions hold.

(a) If $\mathcal{C}_1 \neq \emptyset$, then ${\deg^{\min}}_{G_\nabla}(\phi(\mathrm{Par}(V(\bigcup \mathcal{C}_1))), \mathbf{\Psi}') > k + \frac{\eta k}{100} - v(T')$,

(b) $\mathrm{Par}(V(\bigcup \mathcal{C}_2)) \subseteq \{r\}$, *and* $\deg_{G_\nabla}(x, \mathbf{\Psi}') > \frac{k}{2} + \frac{\eta k}{100} - v(T' \cup \bigcup \mathcal{C}_1)$.

Proof. Let $\mathcal{C}$ be the set of all components of $T - r$. We start by defining $\mathcal{C}_2$. Then, we have to distribute $T - \bigcup \mathcal{C}_2$ between T' and $\mathcal{C}_1$. First, we find a set $\mathcal{C}_M \subseteq \mathcal{C} \setminus \mathcal{C}_2$ which fits into the matching $\mathcal{M}_A \cup \mathcal{M}_B$ (and thus will form part of T'). Then, we consider the remaining components of $\mathcal{C} \setminus \mathcal{C}_2$: some of these will be embedded entirely, of others we only embed the root, and leave the rest for $\mathcal{C}_1$. Everything embedded will become a part of T'.

Throughout the proof we write **shadow** for $\mathbf{shadow}_{G_\nabla}$.

Set $\overline{V_{\mathrm{good}}} := V_{\mathrm{good}} \setminus \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000})$, and choose $\tilde{\mathcal{C}} \subseteq \mathcal{C}$ such that

$$\deg_{G_\nabla}(x, \overline{V_{\mathrm{good}}}) - \frac{\eta k}{30} \; < \; \sum_{S \in \tilde{\mathcal{C}}} v(S) \; \leqslant \; \max\left\{0, \deg_{G_\nabla}\left(x, \overline{V_{\mathrm{good}}}\right) - \frac{\eta k}{40}\right\}. \tag{8.11}$$

Set $\mathcal{C}_2 := \mathcal{C} \setminus \tilde{\mathcal{C}}$. Note that this choice clearly satisfies the first part of (b). Let us now verify the second part of (b). For this, we calculate

$$\begin{aligned}
\deg_{G_\nabla}(x, \mathbf{\Psi}') &\geqslant \deg_{G_\nabla}(x, V_+ \setminus L_\#) - \deg_{G_\nabla}(x, \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000})) \\
&\quad - \deg_{G_\nabla}(x, V_+ \setminus (L_\# \cup \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000}) \cup \mathbf{\Psi})) \\
&\quad - \deg_{G_\nabla}(x, \mathbf{\Psi} \setminus \mathbf{\Psi}') \\
\text{((*), see below)} \quad &\geqslant \left(\frac{k}{2} + \frac{\eta k}{20}\right) - \frac{\eta k}{1000} - \left(\sum_{S \in \tilde{\mathcal{C}}} v(S) + \frac{\eta k}{30}\right) - \frac{\eta k}{100} \\
&> \frac{k}{2} - \sum_{S \in \tilde{\mathcal{C}}} v(S) + \frac{\eta k}{20} \\
&\geqslant \frac{k}{2} - v(T' \cup \bigcup \mathcal{C}_1) + \frac{\eta k}{100},
\end{aligned}$$

as desired for (b). (Here, (*) follows by (7.10), by the fact that $x \notin \mathbf{shadow}^{(2)}(\mathbf{ghost}(U), \frac{\eta k}{1000})$, by (8.11), and by (7.23).)

Now, set

$$\mathcal{M} := \left\{(X_1, X_2) \in \mathcal{M}_A \cup \mathcal{M}_B \; : \; \deg_{G_\mathcal{D}}(x, (X_1 \cup X_2) \setminus \mathfrak{A}) > 0\right\}. \tag{8.12}$$

Claim 8.17.1. We have $|V(\mathcal{M})| \leqslant \frac{4(\Omega^*)^2}{\gamma^2}k$.

Proof of Claim 8.17.1. Indeed, let $(X_1, X_2) \in \mathcal{M}$, i.e. $(X_1, X_2) \in \mathcal{M}_A \cup \mathcal{M}_B$ with $\deg_{G_\mathcal{D}}(x, (X_1 \cup X_2) \setminus \mathfrak{A}) > 0$. Then, using Property 4 of Setting 7.4, we see that there exists a cluster $C_{(X_1,X_2)} \in \mathbf{V}$ such that $\deg_{G_\mathcal{D}}(x, C_{(X_1,X_2)}) > 0$, and either $X_1 \subseteq C_{(X_1,X_2)}$ or $X_2 \subseteq C_{(X_1,X_2)}$. In particular, there exists a dense spot $(A_{(X_1,X_2)}, B_{(X_1,X_2)}; F_{(X_1,X_2)}) \in \mathcal{D}$ such that $x \in A_{(X_1,X_2)}$, and $X_1 \subseteq B_{(X_1,X_2)}$ or $X_2 \subseteq B_{(X_1,X_2)}$. By Fact 4.4, there are at most $\frac{\Omega^*}{\gamma}$ such dense spots, let Z denote the union of all vertices contained in these spots. Fact 4.3 implies that $|Z| \leqslant \frac{2(\Omega^*)^2}{\gamma^2}k$. Thus $|V(\mathcal{M})| \leqslant 2|V(\mathcal{M}) \cap Z| \leqslant 2|Z| \leqslant \frac{4(\Omega^*)^2}{\gamma^2}k$. □

First we shall embed as many components from $\tilde{\mathcal{C}}$ as possible in $\mathcal{M}$. To this end, consider an inclusion-maximal subset $\mathcal{C}_M$ of $\tilde{\mathcal{C}}$ with

$$\sum_{S \in \mathcal{C}_M} v(S) \leqslant \deg_{G_\nabla}(x, V(\mathcal{M})) - \frac{\eta k}{1000} . \tag{8.13}$$

We aim to utilize the degree of x to $V(\mathcal{M})$ to embed $\mathcal{C}_M$ in $V(\mathcal{M})$ using the regularity method.

Remark 8.17.2. There is a seeming inconsistency of the defining formulas (8.12) for $\mathcal{M}$, and (8.13) for $\mathcal{C}_M$. That is, (8.12) involves the degree in $G_\mathcal{D}$ and excludes the set $\mathfrak{A}$, while (8.13) involves the degree in G_∇. The setting in (8.12) was chosen so that it allows us to control the size of $\mathcal{M}$ in Claim 8.17.1, crucially relying on Property 4 of Setting 7.4. Such a control is necessary to make the regularity method work. Indeed, in each regular pair there may be a small number of atypical vertices[xxx], and we must avoid these vertices when embedding the components by the regularity method. Thus without the control on $|\mathcal{M}|$ it might happen that the degree of x is unusable because x sees very small numbers of atypical vertices in an enormous number of sets corresponding to $\mathcal{M}$-vertices. On the other hand, the edges x sends to $\mathfrak{A}$ can be utilized by other techniques in later stages. Once we have defined $\mathcal{M}$ we want to use the full degree to $V(\mathcal{M})$ to ensure we can embed the shrubs as balanced as possible into the $\mathcal{M}$-edges. This is necessary as otherwise part of the degree of x might be unusable for embedding, e.g. because it might go to $\mathcal{M}$-vertices whose partners are already full.

For each $(C, D) \in \mathcal{M}$ we choose $\mathcal{C}_{CD} \subseteq \mathcal{C}_M$ maximal such that

$$\sum_{S \in \mathcal{C}_{CD}} v(S) \leqslant \deg_{G_\nabla}(x, (C \cup D) \setminus \mathbf{ghost}(U)) - (\frac{\gamma}{\Omega^*})^3|C| , \tag{8.14}$$

[xxx]The issue of atypicality itself could be avoided by preprocessing each pair (S, T) of $\mathcal{M}_A \cup \mathcal{M}_B$ and making it super-regular. However this is not possible for atypicality with respect to a given (but unknown in advance) subpair (S', T').

and further, we require $\mathcal{C}_{CD}$ to be disjoint from families $\mathcal{C}_{C'D'}$ defined in previous steps. We claim that $\{\mathcal{C}_{CD}\}_{(C,D)\in\mathcal{M}}$ forms a partition of $\mathcal{C}_M$, i.e., all the elements of $\mathcal{C}_M$ are used. Indeed, otherwise, by the maximality of $\mathcal{C}_{CD}$ and since the components of $T-r$ have size at most τk, we obtain

$$\begin{aligned}\sum_{S\in\mathcal{C}_{CD}} v(S) &\geqslant \deg_{G_\nabla}(x,(C\cup D)\setminus \mathbf{ghost}(U)) - (\frac{\gamma}{\Omega^*})^3|C| - \tau k \\ &\overset{(7.3)}{\geqslant} \deg_{G_\nabla}(x,(C\cup D)\setminus \mathbf{ghost}(U)) - 2(\frac{\gamma}{\Omega^*})^3|C|\ ,\end{aligned} \tag{8.15}$$

for each $(C,D)\in\mathcal{M}$. Then we have

$$\begin{aligned}\sum_{S\in\mathcal{C}_M} v(S) &> \sum_{(C,D)\in\mathcal{M}}\ \sum_{S\in\mathcal{C}_{CD}} v(S) \\ \text{(by (8.15))}\quad &\geqslant \sum_{(C,D)\in\mathcal{M}} \left(\deg_{G_\nabla}(x,(C\cup D)\setminus \mathbf{ghost}(U)) - 2(\frac{\gamma}{\Omega^*})^3|C|\right) \\ \text{(by Claim 8.17.1 and Fact 5.5)}\quad &\geqslant \deg_{G_\nabla}(x,V(\mathcal{M})\setminus \mathbf{ghost}(U)) - 2(\frac{\gamma}{\Omega^*})^3\cdot\frac{2(\Omega^*)^2}{\gamma^2}k \\ \text{(as } x\notin \mathbf{shadow}(\mathbf{ghost}(U))\text{)}\quad &\geqslant \deg_{G_\nabla}(x,V(\mathcal{M})) - \frac{\eta k}{1000} \\ \text{(by (8.13))}\quad &\geqslant \sum_{S\in\mathcal{C}_M} v(S)\ ,\end{aligned}$$

a contradiction.

We use Lemma 8.6 to embed the components of $\mathcal{C}_{CD}$ in $(C\cup D)\setminus\mathbf{ghost}(U)$ with the following setting: $C_{\triangleright\mathrm{L}8.6} := C$, $D_{\triangleright\mathrm{L}8.6} := D$, $U_{\triangleright\mathrm{L}8.6} := \mathbf{ghost}(U)$, $X^*_{\triangleright\mathrm{L}8.6} := (\mathrm{N}_{G_\nabla}(x)\cap(C\cup D))\setminus U_{\triangleright\mathrm{L}8.6}$, and (T_i,r_i) are the rooted trees from $\mathcal{C}_{CD}$ with the roots being the neighbours of r. The constants in Lemma 8.6 are $\varepsilon_{\triangleright\mathrm{L}8.6} := \varepsilon'$, $\beta_{\triangleright\mathrm{L}8.6} := \sqrt{\varepsilon'}$, and $\ell_{\triangleright\mathrm{L}8.6} := |C| \geqslant \nu\pi k$. The rooted trees in $\mathcal{C}_{CD}$ are smaller than $\varepsilon_{\triangleright\mathrm{L}8.6}\ell_{\triangleright\mathrm{L}8.6}$ by (7.3). Condition (8.2) is satisfied by (8.14), and since $(\gamma/\Omega^*)^3 \geqslant 50\sqrt{\varepsilon'}$.

It remains to deal with the components $\tilde{\mathcal{C}}\setminus\mathcal{C}_M$. In the sequel we shall assume that $\tilde{\mathcal{C}}\setminus\mathcal{C}_M\neq\emptyset$ (otherwise skip this step and go directly to the definition of T' and $\mathcal{C}_1$, with $p=0$). Thus, by our choice of $\mathcal{C}_M$, we have

$$\sum_{S\in\mathcal{C}_M} v(S) \geqslant \deg_{G_\nabla}(x,V(\mathcal{M})) - \frac{\eta k}{900}\ . \tag{8.16}$$

Let $T_1,T_2,\ldots,T_p$ be the trees of $\tilde{\mathcal{C}}\setminus\mathcal{C}_M$ rooted at the vertices $r_i\in\mathrm{Ch}(r)\cap V(T_i)$. We shall sequentially extend our embedding of $\mathcal{C}_M$ to subtrees $T'_i\subseteq T_i$. Let $U_i\subseteq V(G)$ be the union of the images of $\bigcup\mathcal{C}_M\cup\{r\}$ and of $T'_1,\ldots,T'_i$ under this embedding.

Suppose that we have embedded the trees $T'_1,\ldots,T'_i$ for some $i=0,1,\ldots,p-1$. We claim that at least one of the following holds.

(V1) $\deg_{G_\nabla}(x,V(G_{\mathrm{exp}})\setminus(U\cup U_i)) \geqslant \frac{\eta k}{1000}$,

(V2) $\deg_{G_\nabla}(x, \mathfrak{A} \setminus (U \cup U_i)) \geqslant \frac{\eta k}{1000}$, or

(V3) $\deg_{G_\nabla}(x, L' \setminus (V(G_{\rm exp}) \cup \mathfrak{A} \cup U \cup U_i \cup \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000}))) \geqslant \frac{\eta k}{1000}$.

Indeed, suppose that none of **(V1)**–**(V3)** holds. Then, first note that since $U \subseteq \mathbf{ghost}(U)$ and since $x \notin \mathbf{shadow}(\mathbf{ghost}(U), \eta k/1000)$, we have

$$\deg_{G_\nabla}(x, U) \leqslant \eta k/1000. \tag{8.17}$$

Also,

$$\deg_{G_\mathcal{D}}(x, V(\mathcal{M}_A \cup \mathcal{M}_B)) \leqslant \deg_{G_\mathcal{D}}(x, V(\mathcal{M}) \cup \mathfrak{A}). \tag{8.18}$$

Thus, writing $R_1 := (V(\mathcal{M}) \cup V(G_{\rm exp}) \cup \mathfrak{A} \cup L') \setminus (U \cup \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000}))$, and $R_2 := (V(G_{\rm exp}) \cup \mathfrak{A} \cup L') \setminus (V(\mathcal{M}) \cup U \cup \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000}))$, we have

$$\begin{aligned}
&\deg_{G_\nabla}\left(x, V_{\rm good} \setminus \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000})\right) \\
\text{(by (8.17) and (8.18), def of } V_{\rm good}) \quad &\leqslant \deg_{G_\nabla}(x, R_1) \\
&\quad + \deg_{G_\nabla}\left(x, \mathbb{L}_{\frac{9}{10}\eta, k}(G_\nabla) \setminus (\mathbf{\Psi} \cup L')\right) + \frac{\eta k}{1000} \\
\text{(by (7.25))} \quad &\leqslant \deg_{G_\nabla}(x, R_2) + \deg_{G_\nabla}(x, V(\mathcal{M})) + \frac{\eta k}{100} + \frac{\eta k}{1000} \\
\text{(by } \neg\textbf{(V1)}, \neg\textbf{(V2)}, \neg\textbf{(V3)}, \text{ by (8.16))} \quad &\leqslant 3 \cdot \frac{\eta k}{1000} + \sum_{j=1}^{i} v(T'_j) + \sum_{S \in \mathcal{C}_M} v(S) + \frac{\eta k}{900} + \frac{\eta k}{100} + \frac{\eta k}{1000} \\
&< \sum_{S \in \tilde{\mathcal{C}}} v(S) + \frac{\eta k}{40},
\end{aligned}$$

a contradiction to (8.11).

In cases **(V1)**–**(V2)** we shall embed the entire tree $T'_{i+1} := T_{i+1}$. In case **(V3)** we either embed the entire tree $T'_{i+1} := T_{i+1}$, or embed only one vertex $T'_{i+1} := r_{i+1}$ (that will only happen in case **(V3c)**). In the latter case, we keep track of the components of $T_{i+1} - r_{i+1}$ in the set $\mathcal{C}_{1,i+1}$ (we tacitly assume we set $\mathcal{C}_{1,i+1} := \emptyset$ in all cases other than **(V3c)**). The union of the sets $\mathcal{C}_{1,i}$ will later form the set $\mathcal{C}_1$. Let us go through our three cases in detail.

In case **(V1)** we embed T_{i+1} rooted at r_{i+1} using Lemma 8.4 for one tree (i.e. $\ell_{\triangleright\mathrm{L}8.4} := 1$) with the following sets/parameters: $H_{\triangleright\mathrm{L}8.4} := G_{\rm exp}$, $U_{\triangleright\mathrm{L}8.4} := U \cup U_i$, $U^*_{\triangleright\mathrm{L}8.4} := \mathrm{N}_{G_\nabla}(x) \cap (V(G_{\rm exp}) \setminus (U \cup U_i))$, $V_1 = V_2 := V(G_{\rm exp})$, $Q_{\triangleright\mathrm{L}8.4} := 1$, $\zeta_{\triangleright\mathrm{L}8.4} := \rho$, and $\gamma_{\triangleright\mathrm{L}8.4} := \gamma$. Note that $|U \cup U_i| < k$, that $|\mathrm{N}_{G_\nabla}(x) \cap (V(G_{\rm exp}) \setminus (U \cup U_i))| \geqslant \eta k/1000 > 32\gamma k/\rho + 1$, that $v(T_{i+1}) \leqslant \tau k < \rho k/4$ and that $128\gamma < \rho^2$.

In case **(V2)** we embed T_{i+1} rooted at r_{i+1} using Lemma 8.3 for one tree (i.e. $\ell_{\triangleright\mathrm{L}8.3} := 1$) with the following setting: $H_{\triangleright\mathrm{L}8.3} := G - \mathbf{\Psi}$, $\mathfrak{A}_{\triangleright\mathrm{L}8.3} := \mathfrak{A}$, $U_{\triangleright\mathrm{L}8.3} := U \cup U_i$, $U^*_{\triangleright\mathrm{L}8.3} := \mathrm{N}_{G_\nabla}(x) \cap (\mathfrak{A} \setminus (U \cup U_i))$, $\Lambda_{\triangleright\mathrm{L}8.3} := \Lambda$, $\gamma_{\triangleright\mathrm{L}8.3} := \gamma$, $\varepsilon_{\triangleright\mathrm{L}8.3} := \varepsilon'$. Note that

$|U \cup U_i| \leqslant k < \Lambda k$, that $|\mathrm{N}_{G_\nabla}(x) \cap (\mathfrak{A} \setminus (U \cup U_i))| \geqslant \eta k/1000 > 2\varepsilon' k$, and that $v(T_{i+1}) \leqslant \tau k < \gamma k/2$.

We commence case **(V3)** with an auxiliary claim.

Claim 8.17.3. There exists $C_0 \in \mathbf{V}$ such that

$$\deg_{G_\mathcal{D}}\left(x, (C_0 \cap L') \setminus (V(G_{\mathrm{exp}}) \cup U \cup U_i \cup \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000}))\right) \geqslant \frac{\varepsilon'}{\gamma^2}\mathfrak{c}\ .$$

Proof of Claim 8.17.3. Observe that $L' \setminus (V(G_{\mathrm{exp}}) \cup \mathfrak{A} \cup \mathbf{\Psi} \cup U \cup U_i) \subseteq \bigcup \mathbf{V}$ and that (since $x \in \bigcup \mathbf{V}$)

$$E_{G_\nabla}[x, L' \setminus (V(G_{\mathrm{exp}}) \cup \mathfrak{A} \cup U \cup U_i \cup \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000}))] \subseteq E(G_\mathcal{D})\ .$$

By Fact 4.11, there are at most $\frac{2(\Omega^*)^2 k}{\gamma^2 \mathfrak{c}}$ clusters $C \in \mathbf{V}$ such that $\deg_{G_\mathcal{D}}(x, C) > 0$. Using the assumption **(V3)**, there exists a cluster $C_0 \in \mathbf{V}$ such that

$$\begin{aligned}\deg_{G_\mathcal{D}}\left(x, (C_0 \cap L') \setminus (V(G_{\mathrm{exp}}) \cup U \cup U_i \cup \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000}))\right) &\geqslant \frac{\eta k}{1000} \cdot \frac{\gamma^2 \mathfrak{c}}{2(\Omega^*)^2 k} \\ &\overset{(7.3)}{\geqslant} \frac{\varepsilon'}{\gamma^2}\mathfrak{c}\ ,\end{aligned}$$

as desired. □

Let us take a cluster C_0 from Claim 8.17.3. We embed the root r_{i+1} of T_{i+1} in an arbitrary neighbour y of x in $(C_0 \cap L') \setminus (V(G_{\mathrm{exp}}) \cup U \cup U_i \cup \mathbf{shadow}(\mathbf{ghost}(U), \frac{\eta k}{1000}))$.

Let $H \subseteq G$ be the subgraph of G consisting of all edges in dense spots $\mathcal{D}$, and all edges incident with $\mathbf{\Psi}'$. As by (7.23), y has at most $\eta k/100$ neighbours in $\mathbf{\Psi} \setminus \mathbf{\Psi}'$, and since $y \in L' \subseteq \mathbb{L}_{9\eta/10,k}(G_\nabla)$ and $y \notin \mathbf{shadow}(U, \frac{\eta k}{100})$, we find that

$$\begin{aligned}\deg_H\left(y, V(G) \setminus ((U \cup U_i) \cup (\mathbf{\Psi} \setminus \mathbf{\Psi}'))\right) &\geqslant \left(1 + \frac{9\eta}{10}\right)k - \frac{\eta k}{1000} - |U_i| - \frac{\eta k}{100} \\ &> k - |U_i| + \frac{\eta k}{2}\ .\end{aligned}$$

Therefore, one of the three following subcases must occur. (Recall that $y \notin \mathfrak{A}$ as $y \in C_0 \in \mathbf{V}$.)

(V3a) $\deg_{G_\nabla}(y, \mathfrak{A} \setminus (U \cup U_i)) \geqslant \frac{\eta k}{6}$,

(V3b) $\deg_{G_{\mathrm{reg}}}(y, \bigcup \mathbf{V} \setminus (U \cup U_i)) \geqslant \frac{\eta k}{6}$, or

(V3c) $\deg_{G_\nabla}(y, \mathbf{\Psi}') \geqslant k - |U_i| + \frac{\eta k}{6}$.

In case **(V3a)** we embed the components of $T_{i+1} - r_{i+1}$ (as trees rooted at the children of r_{i+1}) using the same technique as in case **(V2)**, with Lemma 8.3.

In **(V3b)** we embed the components of $T_{i+1} - r_{i+1}$ (as trees rooted at the children of r_{i+1}). By Fact 4.11 there exists a cluster $D \in \mathbf{V}$ such that

$$\deg_{G_{\mathrm{reg}}}(y, D \setminus (U \cup U_i)) \geqslant \frac{\eta k}{6} \cdot \frac{\gamma^2 \mathfrak{c}}{2(\Omega^*)^2 k} > \frac{\gamma^2}{2} \mathfrak{c}. \tag{8.19}$$

We use Lemma 8.5 with input $\varepsilon_{\triangleright \mathrm{L8.5}} := \varepsilon'$, $\beta_{\triangleright \mathrm{L8.5}} := \gamma^2$, $C_{\triangleright \mathrm{L8.5}} := D$, $D_{\triangleright \mathrm{L8.5}} := C_0$, $X^*_{\triangleright \mathrm{L8.5}} = X_{\triangleright \mathrm{L8.5}} := D \setminus (U \cup U_i)$ and $Y_{\triangleright \mathrm{L8.5}} := C_0 \setminus (U \cup U_i \cup \{y\})$ to embed the tree T_{i+1} into the pair (C_0, D), by embedding the components of $T_{i+1} - r_{i+1}$ one after the other. The numerical conditions of Lemma 8.5 hold because of Claim (8.17.3) and because of (8.19).

In case **(V3c)** we set $T'_{i+1} := r_{i+1}$ and define $\mathcal{C}_{1,i+1}$ as set of all components of $T_{i+1} - r_{i+1}$. Then $\phi(\mathrm{Par}(\bigcup \mathcal{C}_{1,i+1}) \cap V(T'_{i+1})) = \{y\}$ and

$$\deg_{G_\nabla}(y, \mathbf{\Psi}') \geqslant k - |U_i| + \frac{\eta k}{6} \; . \tag{8.20}$$

When all the trees $T_1, \ldots, T_p$ are processed, we define $T' := \{r\} \cup \bigcup \mathcal{C}_M \cup \bigcup_{i=1}^p T'_i$, and set $\mathcal{C}_1 := \bigcup_{i=1}^p \mathcal{C}_{1,i}$. Thus also (a) is satisfied by (8.20) for $i = p$, since $|T'| = |U_p|$. This finishes the proof of the lemma. □

It turns out that our techniques for embedding a tree $T \in \mathbf{trees}(k)$ for Configurations ($\diamond$**2**)–($\diamond$**5**) are very similar. In Lemma 8.18 below we resolve these tasks at once. The proof of Lemma 8.18 follows the same basic strategy for each of the configurations ($\diamond$**2**)–($\diamond$**5**) and deviates only in the elementary procedures of embedding shrubs of T.

Lemma 8.18. *Suppose that we are in Setting 7.4, and one of the following configurations can be found in G:*

a) Configuration ($\diamond$**2**) $\big((\Omega^*)^2, 5(\Omega^*)^9, \rho^3\big)$,

b) Configuration ($\diamond$**3**) $\big((\Omega^*)^2, 5(\Omega^*)^9, \gamma/2, \gamma^3/100\big)$,

c) Configuration ($\diamond$**4**) $\big((\Omega^*)^2, 5(\Omega^*)^9, \gamma/2, \gamma^4/100\big)$, *or*

d) Configuration ($\diamond$**5**) $\Big((\Omega^*)^2, 5(\Omega^*)^9, \varepsilon', 2/(\Omega^*)^3, \frac{1}{(\Omega^*)^5}\Big)$,

Let (T, r) be a rooted tree of order k with a (τk)-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$. Then $T \subseteq G$.

Proof. First observe that each of the configurations given by a)–d) contains two sets $\mathbf{\Psi}'' \subseteq \mathbf{\Psi}$ and $V_1 \subseteq V(G) \setminus \mathbf{\Psi}$ with

$${\deg^{\min}}_{G_\nabla}(\mathbf{\Psi}'', V_1) \geqslant 5(\Omega^*)^9 k \; , \tag{8.21}$$

$${\deg^{\min}}_{G_\nabla}(V_1, \mathbf{\Psi}'') \geqslant \varepsilon' k \; . \tag{8.22}$$

For any vertex $z \in W_A \cup W_B$ we define $T(z)$ as the forest consisting of all components of $T - (W_A \cup W_B)$ that contain children of z. Throughout the proof, we write ϕ for the current partial embedding of T into G.

Overview of the embedding procedure. As outlined in Section 8.1.1 the embedding scheme is the same for Configurations $(\diamond\mathbf{2})$–$(\diamond\mathbf{5})$. The embedding ϕ is defined in two stages. In Stage 1, we embed $W_A \cup W_B$, all the internal shrubs, all the end shrubs of $\mathcal{S}_A$, and a part[xxxi] of the end shrubs of $\mathcal{S}_B$. In Stage 2 we embed the rest of $\mathcal{S}_B$. Which part of $\mathcal{S}_B$ are embedded in Stage 1 and which part in Stage 2 will be determined during Stage 1. We first give a rough outline of both stages listing some conditions which we require to be met, and then we describe each of the stages in detail.

Stage 1 is defined in $|W_A \cup \{r\}|$ steps. First we map r to any vertex in $\mathbf{\Psi}''$. Then in each step we pick a vertex $x \in W_A$ for which the embedding ϕ has already been defined but such that ϕ is not yet defined for any of the children of x. In this step we embed $T(x)$, together with all the children and grandchildren of x in the knag which contains x. For each $y \in W_B \cap \mathrm{Ch}(x)$, Lemma 8.17 determines a subforest $T'(y) \subseteq T(y)$ which is embedded in Stage 1, and sets $\mathcal{C}_1(y)$ and $\mathcal{C}_2(y)$, which will be embedded in Stage 2.

The embedding in each step of Stage 1 will be defined so that the following properties hold.

(*1) All vertices from W_A are mapped to $\mathbf{\Psi}''$.

(*2) All vertices except for W_A are mapped to $V(G) \setminus \mathbf{\Psi}$.

(*3) For each $y \in W_B$, for each $v \in \mathrm{Par}(V(\bigcup \mathcal{C}_1(y)))$ it holds that

$$\deg_G(\phi(v), \mathbf{\Psi}') \geqslant k + \tfrac{\eta k}{100} - v(T'(y)) \ .$$

(*4) For each $y \in W_B$, for each $v \in \mathrm{Par}(V(\bigcup \mathcal{C}_2(y)))$ it holds that

$$\deg_G(\phi(v), \mathbf{\Psi}') \geqslant \tfrac{k}{2} + \tfrac{\eta k}{100} - v(T'(y) \cup \bigcup \mathcal{C}_1(y)) \ .$$

In Stage 2, we shall utilize properties (*3) and (*4) to embed $T_B^* := \bigcup \mathcal{S}_B - \bigcup_{y \in W_B} T'(y)$. Stage 2 is substantially simpler than Stage 1; this is due to the fact that T_B^* consists only of end shrubs.

The embedding step of Stage 1. The embedding step is the same for Configurations $(\diamond\mathbf{2})$–$(\diamond\mathbf{5})$, except for the embedding of internal shrubs. The order of the embedding steps is illustrated in Figure 8.5.

[xxxi] in the sense that individual shrubs $\mathcal{S}_B$ may be embedded only in part

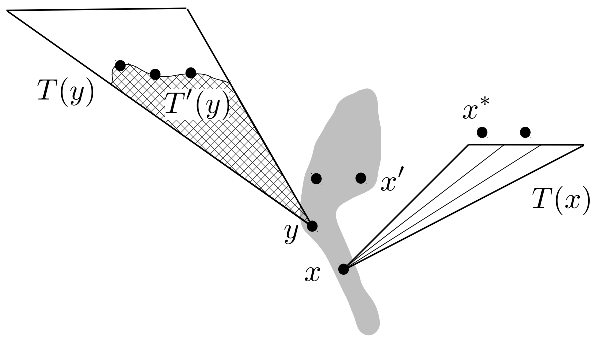

Figure 8.5: Stage 1 of the embedding in the proof of Lemma 8.18. Starting from an already embedded vertex $x \in W_A$ we extend the embedding to (in this order)
(1) all the children $y \in W_B$ of x in the same knag (in grey),
(2) a part $T'(y)$ of the forest $T(y)$,
(3) all the grandchildren $x' \in W_A$ of x in the same knag,
(4) the forest $T(x)$ together with the bordering cut-vertices $x^* \in W_A$.

In each step we have picked $x \in W_A$ already embedded in G but such that none of $\mathrm{Ch}(x)$ are embedded. By (*1), or by the choice of $\phi(r)$, we have $\phi(x) \in \mathbf{\Psi}''$. So by (8.21) we have

$$\deg_{G_\nabla}(\phi(x), V_1 \setminus U) \geqslant 5(\Omega^*)^9 k - k. \tag{8.23}$$

First, we embed successively in $|W_B \cap \mathrm{Ch}(x)|$ steps the vertices $y \in W_B \cap \mathrm{Ch}(x)$ together with components $T'(y) \subseteq T(y)$ which will be determined on the way. Suppose that in a certain step we are to embed $y \in W_B \cap \mathrm{Ch}(x)$ and the (to be determined) tree $T'(y)$. Let $F := \bigcup_{i=0}^{2} \mathbf{shadow}^{(i)}_{G_\nabla - \mathbf{\Psi}}(\mathbf{ghost}(U), \frac{\eta k}{10^5})$, where U is the set of vertices used by the embedding ϕ in previous steps, so $|U| \leqslant k$. By Fact 7.1, $|F| \leqslant \frac{10^{10}(\Omega^*)^2}{\eta^2} k$. We embed y anywhere in $(\mathrm{N}_G(\phi(x)) \cap V_1) \setminus F$, cf. (8.21). Note that then (*2) holds for y. We use Lemma 8.17 in order to embed $T'(y) \subseteq T(y)$ (the subtree $T'(y)$ is determined by Lemma 8.17). Lemma 8.17 ensures that (*3) and (*4) hold and that we have $\phi(V(T'(y))) \subseteq V(G) \setminus \mathbf{\Psi}$.

Also, we map the vertices $x' \in W_A \cap \mathrm{Ch}(y)$ to $\mathbf{\Psi}'' \setminus U$. To justify this step, employing (*2), it is enough to prove that

$$\deg(\phi(y), \mathbf{\Psi}'') \geqslant |W_A| \,. \tag{8.24}$$

Indeed, on one hand, we have $|W_A| \leqslant 336/\tau$ by Definition 3.1(c). On the other hand, we have that $\phi(y) \in V_1$, and thus (8.22) applies. We can thus embed x' as planned, ensuring (*1), and finishing the step for y.

Next, we sequentially embed the components $\tilde{T}$ of $T(x)$. In the following, we describe such an embedding procedure only for an internal shrub $\tilde{T}$, with x^* denoting the other neighbour of $\tilde{T}$ in W_A (cf. (*1)). The case when $\tilde{T}$ is an end shrub is analogous:

actually it is even easier as we do not have to worry about placing x^* well. The actual embedding of $\tilde{T}$ together with x^* depends on the configuration we are in. We shall slightly abuse notation by letting U now denote everything embedded before the tree $\tilde{T}$.

For Configuration $(\diamond\mathbf{2})$, we use Lemma 8.4 for one tree, namely $\tilde{T} - x^*$, using the following setting: $Q_{\triangleright L8.4} := 1, \gamma_{\triangleright L8.4} := \gamma, \zeta_{\triangleright L8.4} := \rho^3, H_{\triangleright L8.4} := G_{\mathrm{exp}}$, $U_{\triangleright L8.4} := U$, and $U^*_{\triangleright L8.4} := (\mathrm{N}_{G_\nabla}(\phi(x)) \cap V_1) \setminus U$ (this last set is large enough by (8.23)). The child of x gets embedded in $(\mathrm{N}_{G_\nabla}(\phi(x)) \cap V_1) \setminus U$, the vertices at odd distance from x get embedded in V_1, and the vertices at even distance from x get embedded in V_2. In particular, $\mathrm{Par}_T(x^*)$ gets embedded in V_1. After this, we accomodate x^* in a vertex in $\mathbf{\Psi}'' \setminus U$ which is adjacent to $\phi(\mathrm{Par}_T(x^*))$. This is possible by the same reasoning as in (8.24).

For Configuration $(\diamond\mathbf{3})$, we use Lemma 8.13 to embed $\tilde{T}$ with the setting $\gamma_{\triangleright L8.13} := \gamma, \delta_{\triangleright L8.13} := \gamma^3/100, U_{\triangleright L8.13} := U$ and $U^*_{\triangleright L8.13} := (\mathrm{N}_{G_\nabla}(\phi(x)) \cap V_1) \setminus U$ (this last set is large enough by (8.23)). Then the child of x gets embedded in $(\mathrm{N}_{G_\nabla}(\phi(x)) \cap V_1) \setminus U$, vertices of $\tilde{T}$ of odd distance to x (i.e. of even distance to the root of $\tilde{T}$) get embedded in $V_1 \setminus U$, and vertices of even distance get embedded in $V_2 \setminus U$. We extend the embedding by mapping x^* to a suitable vertex in $\mathbf{\Psi}'' \setminus U$ adjacent to $\phi(\mathrm{Par}_T(x^*))$ in the same way as above.

For Configuration $(\diamond\mathbf{4})$, we use Lemma 8.14 to embed $\tilde{T}$ with the setting $\gamma_{\triangleright L8.14} := \gamma, \delta_{\triangleright L8.14} := \gamma^4/100, U_{\triangleright L8.14} := U$ and $U^*_{\triangleright L8.14} := (\mathrm{N}_{G_\nabla}(\phi(x)) \cap V_1) \setminus U$ (this last set is large enough by (8.23)). The fruit $r'_{\triangleright L8.14}$ in the lemma is chosen as $\mathrm{Par}_T(x^*)$, note that this is indeed a fruit (in $\tilde{T}$) because of Definition 3.1 (i). Then the child of x gets embedded in $(\mathrm{N}_{G_\nabla}(\phi(x)) \cap V_1) \setminus U$, the vertex $r'_{\triangleright L8.14} = \mathrm{Par}_T(x^*)$ gets embedded in $V_1 \setminus U$, and the rest of $\tilde{T}$ gets embedded in $(\mathfrak{A}' \cup V_2) \setminus U$. This allows us to extend the embedding to x^* as above.

In Configuration $(\diamond\mathbf{5})$, let $\mathbf{W} \subseteq \mathbf{V}$ denote the set of those clusters, which have at least an $\frac{1}{2(\Omega^*)^5}$-fraction of their vertices contained in the set $U' := U \cup \mathbf{shadow}_{G_{\mathrm{reg}}}(U, k/(\Omega^*)^3)$. We get from Fact 7.1 that $|U'| \leqslant 2(\Omega^*)^4 k$, and consequently $|U' \cup \bigcup \mathbf{W}| \leqslant 4(\Omega^*)^9 k$. By (8.23) we can find a vertex $v \in (\mathrm{N}_G(\phi(x)) \cap V_1) \setminus (U' \cup \bigcup \mathbf{W})$.

We use the fact that $v \notin \mathbf{shadow}_{G_{\mathrm{reg}}}(U, k/(\Omega^*)^3)$ together with inequality (7.34) to see that $\deg_{G_{\mathrm{reg}}}(v, V(G_{\mathrm{reg}}) \setminus U) \geqslant k/(\Omega^*)^3$. Now, since there are only boundedly many clusters seen from v (cf. Fact 4.11), there must be a cluster $D \in \mathbf{V}$ such that

$$\deg_{G_{\mathrm{reg}}}(v, D \setminus U) \geqslant \frac{\gamma^2}{2 \cdot (\Omega^*)^5} |D| \geqslant \gamma^3 |D| . \tag{8.25}$$

Let C be the cluster containing v. We have $|(C \cap V_1) \setminus U| \geqslant \frac{1}{2(\Omega^*)^5} |C| \geqslant \gamma^3 |C|$ because of (7.35) and since $C \notin \mathbf{W}$. Thus, by Fact 2.7, $\big((C \cap V_1) \setminus U, D \setminus U\big)$ is an $2\varepsilon'/\gamma^3$-regular pair of density at least $\gamma^2/2$. We can therefore embed $\tilde{T}$ in this pair using the regularity method. Moreover, by (8.25), we can do so by mapping the child z of x to v. Thus the parent of x^* (lying at even distance to z) will be embedded in $(C \cap V_1) \setminus U$. We can then extend our embedding to x^* as above.

This finishes our embedding of $T(x)$. Note that in all cases we have $\phi(x^*) \in \mathbf{\Psi''}$ and $\phi(V(\tilde{T})) \subseteq V(G) \setminus \mathbf{\Psi}$, as required by (*1) and (*2).

The embedding steps of Stage 2. For $i = 1, 2$, set $Z_i := \bigcup_{y \in W_B} \mathrm{Ch}(T'(y)) \cap \bigcup \mathcal{C}_i(y)$.

First, we embed all the vertices $z \in Z_2$ in $\mathbf{\Psi'}$. By (*2), until now, only vertices of $W_A \cup Z_2$ are mapped to $\mathbf{\Psi'}$, and using (*4) and the properties (c), (k) and (l) of Definiton 3.1, we see that

$$\begin{aligned}\deg_G(\phi(\mathrm{Par}(z)), \mathbf{\Psi'}) &\geqslant \frac{\eta k}{100} + (\frac{k}{2} - \bigcup_{y \in W_B} (T'(y) \cup \bigcup \mathcal{C}_1(y)) \\ &> |W_A| + |Z_2| \,.\end{aligned}$$

So there is space for the vertex z in $\mathbf{\Psi'} \cap \phi(\mathrm{N}_G(\mathrm{Par}(z)))$.

Next, we embed all the vertices $z \in Z_1$ in $\mathbf{\Psi'}$. By (*2), until now, only vertices of $W_A \cup Z_2 \cup Z_1$ are mapped to $\mathbf{\Psi'}$, and by (*3) we have, similarly as above,

$$\deg_G(\phi(\mathrm{Par}(z)), \mathbf{\Psi'}) > |W_A| + |Z_2| + |Z_1| \,.$$

So z can be embedded in $\mathbf{\Psi'} \cap \mathrm{N}_G(\phi(\mathrm{Par}(z)))$ as planned.

Finally, for $z \in Z_1 \cup Z_2$, denote by T_z the component of $\mathcal{C}_1 \cup \mathcal{C}_2$ that contains z. We use Lemma 8.16 to embed the rest of the rooted tree (T_z, z). (Note that our parameters work because of (7.3).) Once all rooted trees (T_z, z), $z \in Z_1 \cup Z_2$ have been processed, we have finished Stage 2 and thus the proof of the lemma. □

8.4.3 Dealing with Configurations (⋄**6**)–(⋄**10**)

We follow the schemes outlined in Sections 8.1.2,8.1.3, 8.1.4, and 8.1.5.

Embedding a tree $T_{\triangleright \mathrm{T1.3}} \in \mathbf{trees}(k)$ using Configurations (⋄**6**), (⋄**7**), (⋄**8**) has two parts: first the internal part of $T_{\triangleright \mathrm{T1.3}}$ is embedded, and then this partial embedding is extended to end shrubs of $T_{\triangleright \mathrm{T1.3}}$ as well. Lemma 8.19 (for configurations (⋄**6**) and (⋄**7**)) and Lemma 8.20 (for configuration (⋄**8**)) are used for the former part, and Lemmas 8.21 and 8.22 (depending on whether we have (♡**1**) or (♡**2**)) for the latter. Lemma 8.23 then combines these two pieces together.

Embedding using Configurations (⋄**9**) and (⋄**10**) is resolved in Lemmas 8.24 and 8.25, respectively.

Lemma 8.19. *Suppose we are in Setting 7.4 and 7.7, and we have one of the following two configurations:*

- *Configuration* $(\diamond\mathbf{6})(\delta_6, \tilde{\varepsilon}, d', \mu, 1, 0)$*, or*
- *Configuration* $(\diamond\mathbf{7})(\delta_7, \eta\gamma/400, \tilde{\varepsilon}, d', \mu, 1, 0)$*,*

with $10^3\sqrt{\gamma}(\Omega^*)^2 \leqslant \delta_6^3$*,* $10^2(\Omega^*)^3/\Lambda \leqslant \delta_7^3 < \eta^3\gamma^3/10^6$*,* $d' > 10\tilde{\varepsilon} > 0$*, and* $d'\mu\tau k \geqslant 4\cdot 10^3$*. (In either of these two configurations we have two distinguished sets* V_0 *and* V_1*.)*

Suppose that $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ *is an* (τk)*-fine partition of a rooted tree* (T, r) *of order at most* k*. Let* T' *be the tree induced by all the cut-vertices* $W_A \cup W_B$ *and all the internal shrubs.*

Then there exists an embedding ϕ *of* T' *such that* $\phi(W_A) \subseteq V_1$*,* $\phi(W_B) \subseteq V_0$*, and* $\phi(T' - (W_A \cup W_B)) \subseteq \mathfrak{P}_1$*.*

Proof. Consider an arbitrary ordered skeleton $(P_0^*, T_1^*, P_1^*, \ldots, T_m^*, P_m^*)$ of $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$. Such an ordered skeleton exists by Lemma 3.7. For $i = 1, \ldots, m$, let $y_i \in W_A$ be the seed of the internal shrub T_i^*, and $r_i \in \mathrm{N}_T(y_i) \cap V(T_i^*)$ its root. In the rooted tree (T_i^*, r_i) there is a unique vertex $f_i \in V(T_i^*)$ in $\mathrm{N}_T(W_A \cup W_B) \setminus \{r_i\}$. Note that f_i is a fruit of (T_i^*, r_i).

Our strategy is as follows. We sequentially embed the knags and the internal shrubs in the order given by the ordered skeleton. That is, we alternate between embedding knags and internal trees. For embedding the knags we use Lemma 8.4, and Lemma 8.7, respectively, depending on whether we have Preconfiguration (**exp**) or (**reg**). For embedding the internal shrubs, we use Lemmas 8.11 and 8.12 if we have Configurations $(\diamond\mathbf{6})$, and $(\diamond\mathbf{7})$, respectively. However the settings in Lemma 8.11–8.12 is almost the same and the embedding can be done in an almost unified way. To ease orientation in this case distinction below we mark underlined each of the steps.

Throughout, ϕ denotes the current (partial) embedding of $(P_0^*, T_1^*, P_1^* \ldots, T_m^*, P_m^*)$. In steps, we extend ϕ. We write $C_i := \phi(T_i^* - r_i)$, $C_{<i} := C_1 \cup \ldots \cup C_{i-1}$, $C_{\leqslant i} := C_1 \cup \ldots \cup C_i$. When embedding T_i^* we shall introduce an auxiliary set $D_i \subseteq V(G)$ of size $v(T_i^*)$ which is disjoint from $C_{\leqslant i}$ and from $D_{<i}$ (with the obvious definition). Note that thus we have $|C_{<i} \cup D_{<i}| \leqslant 2k$ in every step i. We define $W_{A,i} := W_A \cap (V(P_0^*) \cup \ldots \cup V(P_{i-1}^*))$, and $W_{B,i} := W_B \cap (V(P_0^*) \cup \ldots \cup V(P_{i-1}^*))$.

Further three important families of sets $F_i, U_i, W_i \subseteq V(G)$ are introduced below. Knag P_i^* will be embedded outside F_i. Also, when embedding the knags the embedding will be such that $\phi(W_A) \subseteq V_1$ and $\phi(W_B) \subseteq V_0$.

The value $h = 6$ or $h = 7$ indicates whether we have configuration $(\diamond\mathbf{6})$ or $(\diamond\mathbf{7})$.

Define

$$F_i := \mathbf{shadow}_{G-\mathbf{\Psi}}\left(C_{<i} \cup D_{<i}, \frac{\delta_h k}{8}\right) . \tag{8.26}$$

By Fact 7.1, we have

$$|F_i| \leqslant \frac{9(\Omega^*)}{\delta_h} k . \tag{8.27}$$

Define $U_i := F_i$ if we have Preconfiguration (**exp**) (note that in that case we have Configuration ($\diamond$**6**)). To define U_i in case of Preconfiguration (**reg**) we make use of the super-regular pairs $(Q_0^{(j)}, Q_1^{(j)})$ $(j \in \mathcal{Y})$.

$$U_i := F_i \cup \bigcup \left\{ Q_1^{(j)} \ : \ j \in \mathcal{Y}, |Q_1^{(j)} \cap F_i| \geqslant \frac{|Q_1^{(j)}|}{2} \right\} . \tag{8.28}$$

In any case, we have $|U_i| \leqslant 2|F_i|$.

Finally, set

$$W_i := \mathbf{shadow}_{G-\mathbf{\Psi}}\left(U_i, \frac{\delta_h k}{2}\right) . \tag{8.29}$$

We have

$$|W_i| \leqslant \frac{36(\Omega^*)^2}{\delta_h^2} k . \tag{8.30}$$

In the following we assume that $r \in W_A$. The case when $r \in W_B$ is similar.

Embedding knag P_0^* in Preconfiguration (**exp**). Again, recall that we are in Configuration ($\diamond$**6**). We use Lemma 8.4 to embed the single tree P_0^* ($\ell_{\triangleright \text{L8.4}} := 1$) with the following setting $U^*_{\triangleright \text{L8.4}} := V_1, U_{\triangleright \text{L8.4}} := U_0 = F_0 = \emptyset$, $Q_{\triangleright \text{L8.4}} := \frac{18\Omega^*}{\delta_6}$, $\zeta_{\triangleright \text{L8.4}} := \delta_6$. Lemma 8.4 outputs an embedding of the tree P_0^* such that $\phi(V_{\text{even}}(P_0^*, r)) \subseteq V_1 \setminus F_0$ and $\phi(V_{\text{odd}}(P_0^*, r)) \subseteq V_0$.

Embedding knag P_0^* in Preconfiguration (**reg**). Take an arbitrary $j \in \mathcal{Y}$. We shall use Lemma 8.7 to embed P_0^* in $(Q_0^{(j)}, Q_1^{(j)})$. More precisely, we use Lemma 8.7 with $A_{\triangleright \text{L8.7}} := Q_1^{(j)}$, $B_{\triangleright \text{L8.7}} := Q_0^{(j)}$, $\varepsilon_{\triangleright \text{L8.7}} := \tilde{\varepsilon}$, $d_{\triangleright \text{L8.7}} := d'$, $\ell_{\triangleright \text{L8.7}} := \mu k$, $U_A := U_0 = \emptyset$, $U_B := \emptyset$. Lemma 8.7 outputs a $(V_{\text{even}}(P_0^*, r) \hookrightarrow V_1 \setminus F_0, V_{\text{odd}}(P_0^*, r) \hookrightarrow V_0)$-embedding of P_0^*.

After embedding knag P_i^* $(i = 0, \ldots, m)$. For each $y \in W_A \cap V(P_i^*)$ let $S_y := (V_2 \cap \mathrm{N}_G(\phi(y))) \setminus (C_{<i} \cup D_{<i})$. As $\deg_G(\phi(y), V_2) \geqslant \delta_h k$ (see (7.43), (7.47)), and as y was embedded outside of F_i, we have $|S_y| \geqslant \frac{7\delta_h k}{8}$. We inductively claim that for each step $i' \leqslant m \leqslant |W_A| \leqslant k^{0.1}$ we have

$$|S_y \setminus C_{<i'}| \geqslant |S_y \setminus D_{<i'}| - i' k^{0.75} \geqslant |S_y \setminus D_{<i'}| - k^{0.85} . \tag{8.31}$$

This is certainly true for the initial step $i' = i$. Note that as $C_{<i'} \cap D_{<i'} = \emptyset$, we get as a consequence that

$$|S_y \setminus C_{<i'}| \geqslant \frac{7\delta_h k}{16} - k^{0.85} > \frac{\delta_h k}{3} . \tag{8.32}$$

Embedding internal shrub T_i^*. Assume that in step $i = 1, 2, \dots, m$ we embedded shrubs and knags $P_0^*, T_1^*, P_1^*, \dots, T_{i-1}^*, P_{i-1}^*$. Assume that the parent y of the root r_i of T_i^* is mapped on a vertex $\phi(y) \in V_1$. By (8.32), we have for the set $U^* := S_y \setminus (C_{<i-1} \cup C_{i-1} \cup \phi(\{r_1, \dots, r_{i-1}\})$ that $|U^*| \geqslant \frac{\delta_h k}{3} - \tau k \geqslant \frac{\delta_h k}{4}$. If we have Configuration ($\diamond$**6**) or ($\diamond$**7**) we use one of the embedding Lemmas 8.11–8.12, respectively, with parameters $U_{\triangleright L8.11-8.12} := W_i, U^*_{\triangleright L8.11-8.12} := U^*, L_{\triangleright L8.11-8.12} := |W_{A,i}|$, the family $\{P_t\}_{\triangleright L8.11-8.12} := \{S_y\}_{y \in W_{A,i}}$, and the rooted tree (T_i^*, r_i) with fruit f_i (further for Configuration ($\diamond$**7**), set $\ell := 1$, $Y_1 := V_2$ and $Y_2 := V_3$). The corresponding embedding lemma outputs an embedding ϕ of T_i^* and a set $D_{i+1} := C_{\triangleright L8.11-8.12}$ with the following properties:

(p1) $D_i \cap \phi(T_i^*) = \emptyset$.

(p2) $(C_{<i} \cup D_{<i}) \cap (\phi(T_i^* - r_i) \cup D_i) = \emptyset$.

(p3) $\phi(T_i^*) \cup D_i \subseteq \mathfrak{P}_1$.

(p4) For each $y \in W_{A,i}$ we have $|S_y \cap \phi(T_i^*)| \leqslant |S_y \cap D_i| + k^{3/4}$. In particular, this (together with **(p1)**) gives (8.31) even for i.

(p5) We have $\phi(f_i) \in V_2 \setminus U$.

Embedding knags P_i^* ($i \geqslant 1$) in Preconfiguration (**exp**). Let s_i be the root of P_i^* (i.e., its $\preceq_r$-minimal vertex), and let $q_i := \mathrm{Par}_T(s_i)$. We use Lemma 8.4 to embed the single tree P_i^* ($\ell_{\triangleright L8.4} := 1$) with the following setting $U^*_{\triangleright L8.4} := (\mathrm{N}_G(\phi(q_i)) \cap V_1) \setminus U_i = (\mathrm{N}_G(\phi(q_i)) \cap V_1) \setminus F_i, U_{\triangleright L8.4} := U_i = F_i, Q_{\triangleright L8.4} := \frac{18\Omega^*}{\delta_6}, \zeta_{\triangleright L8.4} := \delta_6$. Lemma 8.4 outputs an embedding of the tree P_i^* such that $\phi(V_{\mathrm{even}}(P_i^*, s_i)) \subseteq V_1 \setminus F_i$ and $\phi(V_{\mathrm{odd}}(P_i^*, s_i)) \subseteq V_0$.

Embedding knags P_i^* ($i \geqslant 1$) in Preconfiguration (**reg**). Let s_i be the root of P_i^*, and let $q_i := \mathrm{Par}_T(s_i)$. Let $j \in \mathcal{Y}$ be such that $(\mathrm{N}_G(\phi(q_i)) \cap Q_1^{(j)}) \setminus U_i \neq \emptyset$. We shall use Lemma 8.7 to embed P_i^* in $(Q_0^{(j)}, Q_1^{(j)})$. More precisely, we use Lemma 8.7 with $A_{\triangleright L8.7} := Q_1^{(j)}$, $B_{\triangleright L8.7} := Q_0^{(j)}$, $\varepsilon_{\triangleright L8.7} := \tilde{\varepsilon}$, $d_{\triangleright L8.7} := d'$, $\ell_{\triangleright L8.7} := \mu k$, $U_A := U_i \cap A$, $U_B := \phi(W_{B,<i})$. Lemma 8.7 outputs a $(V_{\mathrm{even}}(P_i^*, r) \hookrightarrow V_1 \setminus F_i, V_{\mathrm{odd}}(P_i^*, r) \hookrightarrow V_0)$-embedding of P_i^*. $\square$

Lemma 8.20. *Suppose we are in Setting 7.4 and 7.7 and we have Configuration*

$$(\diamond\mathbf{8})(\delta, \frac{\eta\gamma}{400}, \varepsilon_1, \varepsilon_2, d_1, d_2, \mu_1, \mu_2, h_1, 0)$$

with $2 \cdot 10^5 (\Omega^*)^6/\Lambda \leqslant \delta^6 < \eta^6\gamma^6/10^{12}$, $d_2 > 10\varepsilon_2 > 0$, $d_2\mu_2\tau k \geqslant 4 \cdot 10^3$, *and* $\max\{\varepsilon_1, \tau/\mu_1\} \leqslant \eta^2\gamma^2 d_1/(10^{10}(\Omega^*)^3)$. *(Recall that then we have two distinguished sets V_0 and V_1.)*

Suppose that $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ is an (τk)-fine partition of a rooted tree (T, r) of order at most k. Let T' be the tree induced by all the cut-vertices $W_A \cup W_B$ and all the internal shrubs. Suppose that

$$v(T') < h_1 - \frac{\eta^2 k}{10^5} \, . \tag{8.33}$$

Then there exists an embedding ϕ of T' such that $\phi(W_A) \subseteq V_1$, $\phi(W_B) \subseteq V_0$, and $\phi(T') \subseteq \mathfrak{P}_0 \cup \mathfrak{P}_1$.

Proof. Consider an arbitrary ordered skeleton $(P_0^*, T_1^*, P_1^*, \ldots, T_m^*, P_m^*)$ of $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$. Such an ordered skeleton exists by Lemma 3.7. For $i = 1, \ldots, m$, let $y_i \in W_A$ be the seed of the internal shrub T_i^*, and $r_i \in \mathrm{N}_T(y_i) \cap V(T_i^*)$ its root. In the rooted tree (T_i^*, r_i) there is a unique vertex $f_i \in V(T_i^*)$ in $\mathrm{N}_T(W_A \cup W_B) \setminus \{r_i\}$. Note that f_i is a fruit of (T_i^*, r_i). Let $R_i := \mathrm{Ch}_{T'}\big(V(P_i^*)\big)$. The set R_i contains roots of internal shrubs attached to P_i^*. Let M_i be the principal subshrub of T_i^*, let m_i be its root, and let $\mathcal{L}_i$ be the peripheral subshrubs of T_i^*.

We now outline our strategy. We also give pointers to steps $(\mathbf{S}_i\mathbf{a})$–$(\mathbf{S}_i\mathbf{d})$ and $(\mathbf{F})$ which are marked underline in the text below. We sequentially for $i = 0, \ldots, m$ embed the knags P_i^* in $V_0 \cup V_1$ (step $(\mathbf{S}_i\mathbf{b})$) and the roots R_i in V_2 (step $(\mathbf{S}_i\mathbf{c})$). Then for every vertex $r_j \in R_i$, we embed the peripheral subshrubs $\mathcal{L}_j^*$ in $\mathcal{N}$ if the degree $\deg_{G_{\mathrm{reg}}}(\phi(r_j), V(\mathcal{N}))$ allows (step $(\mathbf{S}_i\mathbf{d})$). Denote the union of all these peripheral subshrubs by $\mathcal{Q}_i$ (note that subshrubs from $\mathcal{Q}_i$ may belong to shrubs $T_{i+1}^*, \ldots, T_m^*$). Next, in step $(\mathbf{S}_{i+1}\mathbf{a})$ we embed the principal subshrub M_{i+1} in $V_3 \cup V_4$ (with $f_{i+1} \hookrightarrow V_2$). The embedding of unembedded subshrubs from $\bigcup_{j=1}^m \mathcal{L}_j$ is postponed to the very end of the entire embedding process (step $(\mathbf{F})$). The order of these steps is displayed below.

$$(\mathbf{S}_0\mathbf{b}) \to (\mathbf{S}_0\mathbf{c}) \to (\mathbf{S}_0\mathbf{d}) \to (\mathbf{S}_1\mathbf{a}) \to (\mathbf{S}_1\mathbf{b}) \to \ldots$$
$$\to (\mathbf{S}_{m-1}\mathbf{c}) \to (\mathbf{S}_{m-1}\mathbf{d}) \to (\mathbf{S}_m\mathbf{a}) \to (\mathbf{S}_m\mathbf{b}) \to (\mathbf{F})$$

For embedding the knags we use Lemma 8.7. For embedding subshrubs in $\mathcal{N}$ we use Lemmas 8.8 and 8.5. For embedding subshrubs in $V_3 \cup V_4$ we use Lemma 8.12.

Throughout, ϕ denotes the current (partial) embedding of $(P_0^*, T_1^*, P_1^*, \ldots, T_m^*, P_m^*)$. In steps, we extend ϕ. We write $C_i := \phi\big(T_i^* - r_i - \mathrm{Ch}(r_i)\big) \cup \phi\big(V(\mathcal{Q}_i)\big)$, $C_{<i} := C_1 \cup \ldots \cup C_{i-1}$, $C_{\leqslant i} := C_1 \cup \ldots \cup C_i$. When embedding T_i^* we shall introduce an auxiliary set $D_i \subseteq V_3$ of size at most $|\phi\big(V(T_i^*)\big)|$ which is disjoint from $C_{\leqslant i}$ and from $D_{<i}$ (with the obvious definition). Note that thus we have $|C_{<i} \cup D_{<i}| \leqslant 2k$ in every step i. We define $W_{B,i} := W_B \cap (V(P_0^*) \cup \ldots \cup V(P_{i-1}^*))$.

Fix a matching involution $\mathfrak{d}$ for $\mathcal{N}$. Further three important families of sets $F_i, U_i, W_i \subseteq V(G)$ are introduced below. Knag P_i^* will be embedded outside F_i. Also, when embedding the knags, the roots R_i and the fruits f_i, the embedding will be such that

$\phi(W_A) \subseteq V_1$, $\phi(W_B) \subseteq V_0$, $\phi(\bigcup R_i) \subseteq V_2$, and $\phi(f_1, f_2, \ldots) \subseteq V_2$. The knags, the roots R_i and the fruits f_i will be the only vertices embedded in $\mathfrak{P}_0$. Thus we will at each stage have

$$|\mathrm{Im}(\phi) \cap \mathfrak{P}_0| \leqslant 3 \cdot (|W_A| + |W_B|) \overset{\text{D3.1(c)}}{\leqslant} \frac{2016}{\tau} < \frac{\delta k}{8} . \tag{8.34}$$

Define

$$F_i := \mathbf{shadow}^{(2)}_{G-\mathbf{\Psi}} \left(\mathbf{ghost}_{\mathfrak{d}}(C_{<i} \cup D_{<i}), \frac{\delta k}{8} \right) . \tag{8.35}$$

By Fact 7.1, we have

$$|F_i| \leqslant \frac{65(\Omega^*)^2}{\delta^2} k . \tag{8.36}$$

We now use the super-regular pairs $(Q_0^{(j)}, Q_1^{(j)})$ $(j \in \mathcal{Y})$ to define

$$U_i := F_i \cup \bigcup \left\{ Q_1^{(j)} \ : \ j \in \mathcal{Y}, |Q_1^{(j)} \cap F_i| \geqslant \frac{|Q_1^{(j)}|}{2} \right\} . \tag{8.37}$$

We have $|U_i| \leqslant 2|F_i|$. Finally, set

$$W_i := \mathbf{shadow}^{(2)}_{G-\mathbf{\Psi}} \left(U_i, \frac{\delta k}{2} \right) . \tag{8.38}$$

We have

$$|W_i| \leqslant \frac{520(\Omega^*)^4}{\delta^4} k . \tag{8.39}$$

In the following we assume that $r \in W_A$. The case when $r \in W_B$ is similar.

<u>($\mathbf{S_0 b}$): *Embedding knag P_0^*.*</u> Take an arbitrary $j \in \mathcal{Y}$. We shall use Lemma 8.7 to embed P_0^* in $(Q_0^{(j)}, Q_1^{(j)})$. More precisely, we use Lemma 8.7 with $A_{\triangleright \text{L8.7}} := Q_1^{(j)}$, $B_{\triangleright \text{L8.7}} := Q_0^{(j)}$, $\varepsilon_{\triangleright \text{L8.7}} := \varepsilon_2$, $d_{\triangleright \text{L8.7}} := d_2$, $\ell_{\triangleright \text{L8.7}} := \mu_2 k$, $U_A := U_0 = \emptyset$, $U_B := \emptyset$. Lemma 8.7 outputs a $(V_{\text{even}}(P_0^*, r) \hookrightarrow V_1 \setminus F_0, V_{\text{odd}}(P_0^*, r) \hookrightarrow V_0)$-embedding of P_0^*.

<u>($\mathbf{S_i c}$): *After embedding P_i^*: Embedding R_i.*</u> We embed the vertices of R_i in V_2. For a given vertex $y \in R_i$, let $x \in W_A$ be the parent of y. Combining (7.51) with the fact that $\phi(x) \notin F_i$, we have that

$$\left| \mathrm{N}_G \left(\phi(x), V_2 \setminus \mathbf{shadow}_{G-\mathbf{\Psi}} \left(\mathbf{ghost}_{\mathfrak{d}}(C_{<i} \cup D_{<i}), \frac{\delta k}{8} \right) \right) \right| \geqslant \frac{7\delta k}{8} .$$

Thus by (8.34) we can accommodate the vertices of R_i in

$$V_2 \setminus \mathbf{shadow}_{G-\mathbf{\Psi}} \left(\mathbf{ghost}_{\mathfrak{d}}(C_{<i} \cup D_{<i}), \frac{\delta k}{8} \right) .$$

For each $y \in R_i$ let $S_y := (V_3 \cap \mathrm{N}_G(\phi(y))) \setminus (C_{<i} \cup D_{<i})$. We inductively claim that for each step $i' = i, \ldots, m$ we have

$$|S_y \setminus C_{<i'}| \geqslant |S_y \setminus D_{<i'}| - i' k^{0.75} \tag{8.40}$$

$$\overset{(\text{as } m \leqslant |W_A| \leqslant k^{0.1})}{\geqslant} |S_y \setminus D_{<i'}| - k^{0.85} . \tag{8.41}$$

This is certainly true for the initial step $i' = i$. The induction step is established in Claim 8.20.1.

As y was embedded outside of $\mathbf{shadow}_{G-\Psi}\left(C_{<i} \cup D_{<i}, \frac{\delta k}{8}\right)$, we have by (7.53) that $|S_y| \geqslant \frac{7\delta k}{8}$. Note that as $C_{<i'} \cap D_{<i'} = \emptyset$, we get as a consequence of (8.41) that

$$|S_y \setminus C_{<i'}| \geqslant \frac{3\delta k}{8} . \tag{8.42}$$

<u>($\mathbf{S}_i\mathbf{d}$): *After embedding P_i^*: Embedding peripheral subshrubs in $\mathcal{N}$.*</u> This step is further divided in two substeps. First we shall aim to embed as many subshrubs as possible in $\mathcal{N}$ in a balanced way going through all the parents $y \in R_i$. For the embedding we shall use Lemma 8.8. When it is not possible to embed any further subshrub in a balanced way in $\mathcal{N}$, we aim to embed in $\mathcal{N}$ as many of the leftover subshrubs as possible. For the embedding we use Lemma 8.5. All the peripheral subshrubs embedded in step ($\mathbf{S}_i\mathbf{d}$) are denoted by $\mathcal{Q}_i$. This rough description will be enough to prove the claim below.

Claim 8.20.1. If (8.40) holds for $i' = i$ and for $y \in \bigcup_{j \leqslant i} R_j$ then it also holds for y, and $i' = i + 1$.

Proof of Claim 8.20.1. Observe that as $V(\mathcal{N}) \cap \mathfrak{A} = \emptyset$, we shall not embed anything in V_3 in this step. In other words, $S_y \setminus C_{<i} = S_y \setminus (C_{<i} \cup \phi(\mathcal{Q}_i))$.

The claim is now trivial for $i = 0$ as $C_{<1} = \phi(\mathcal{Q}_0)$. For $i > 0$ the claim follows from (8.45) for step ($\mathbf{S}_i\mathbf{a}$) (that step precedes the current step). □

Let $R_i = \{y_{q_1}, \ldots, y_{q_\ell}\}$. In step j, suppose that we have embedded peripheral shrubs $\mathcal{F}_p \in \{\emptyset, \mathcal{L}_{q_p}\}$ ($p < j$). We also inductively assume that

$$\phi\Big(\bigcup_{p<j} \mathcal{F}_p\Big) \text{ is } (\tau k)\text{-balanced with respect to } \mathcal{N}. \tag{8.43}$$

We are now going to determine peripheral shrubs $\mathcal{F}_j \in \{\emptyset, \mathcal{L}_{q_j}\}$ which we will embed in $\mathcal{N}$ in this step. To this end, we first construct a semiregular matching $\mathcal{N}_j$ absorbed by $\mathcal{N}$ defined by $\mathcal{N}_j := \{(X'_1, X'_2) : (X_1, X_2) \in \mathcal{N}\}$, where for $(X_1, X_2) \in \mathcal{N}$ we define (X'_1, X'_2) as the maximal balanced unoccupied subpair seen from $\phi(y_{q_j})$, i.e., for $b = 1, 2$, we take

$$X'_b \subseteq \left(X_b \cap \mathrm{N}_{G_{\mathrm{reg}}}(\phi(y_{q_j}))\right) \setminus \left(\phi(\bigcup_{p<j} \mathcal{F}_p) \cup \mathbf{ghost}_{\mathfrak{d}}(C_{<i})\right)$$

maximal subject to $|X'_1| = |X'_2|$. If $|V(\mathcal{N}_j)| \geqslant \frac{\eta^2 k}{10^7}$ then take $\mathcal{F}_j := \mathcal{L}_{q_j}$. Otherwise, let $\mathcal{F}_j := \emptyset$. We now show how to embed $\mathcal{F}_j$ in $\mathcal{N}_j$. Recall that the total order of $\mathcal{F}_j$ is at most τk. Using the same argument as in Claim 8.17.1 we have

$$\left|\bigcup\{X \cup Y : (X, Y) \in \mathcal{N}, \deg_{G_{\mathcal{D}}}(\phi(y_{q_j}), X \cup Y) > 0\right| \leqslant \frac{4(\Omega^*)^2}{\gamma^2} k .$$

Thus, there exists a subpair $(X_1', X_2') \in \mathcal{N}_j$ of some $(X_1, X_2) \in \mathcal{N}$ with

$$\frac{|X_1'|}{|X_1|} \geqslant \frac{\frac{\eta^2}{10^7\Omega^*}k}{\frac{4(\Omega^*)^2}{\gamma^2}k} . \tag{8.44}$$

In particular, (X_1', X_2') forms an $\frac{10^8\varepsilon_1(\Omega^*)^3}{\gamma^2\eta^2}$-regular pair of density at least $d_1/2$. We use Lemma 8.8 to embed $\mathcal{F}_j$ in $\mathcal{M}_{\triangleright \text{L8.8}} := \{(X_1', X_2')\}$. The family $\{f_{CD}\}_{\triangleright \text{L8.8}}$ comprises of a single number $f_{(X_1', X_2')}$ which is the discrepancy of $\phi\big(\bigcup_{p<j}\mathcal{F}_p\big)$ with respect to (X_1, X_2). This guarantees that (8.43) is preserved. This finishes the j-th step. We now repeat this step for $j+1, \ldots, \ell$.

Next, we proceed with embedding some additional peripheral shrubs in $\mathcal{N}$ in an unbalanced way. In the j-th step suppose that we embedded subshrubs $\mathcal{H}_1, \ldots, \mathcal{H}_{j-1}$ in this way. That is, for those $j = 1, \ldots, \ell$ for which $\mathcal{F}_j = \emptyset$ we take $\mathcal{H}_j := \mathcal{L}_{q_j}$ if

$$\deg_{G_{\text{reg}}}\Big(\phi(y_{q_j}), V(\mathcal{N}) \setminus \phi(\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_\ell)\Big) - v(\mathcal{H}_1 \cup \ldots \cup \mathcal{H}_{j-1}) > \frac{\eta^2 k}{10^6} ,$$

and $\mathcal{H}_j := \emptyset$ otherwise. Let

$$\tilde{\mathcal{N}} := \left\{(X,Y) \in \mathcal{N} \ : \ |(X \cup Y) \cap (C_{<i} \cup D_{<i})| < \frac{\eta^2}{10^8\Omega^*}|X|\right\} .$$

As y_{q_j} was embedded outside of $\mathbf{shadow}_{G-\mathbf{\Psi}}\left(C_{<i} \cup D_{<i}, \frac{\delta k}{8}\right)$, we have

$$\deg_{G_{\text{reg}}}(\phi(y_{q_j}), V(\tilde{\mathcal{N}})) \geqslant \deg_{G_{\text{reg}}}(\phi(y_{q_j}), V(\mathcal{N})) - \frac{10^8\delta\Omega^*}{\eta^2}k \geqslant \deg_{G_{\text{reg}}}(\phi(y_{q_j}), V(\mathcal{N})) - \frac{\eta^2}{10^7}k .$$

Similar calculations as in (8.44) give that there is a pair $(X, Y) \in \tilde{\mathcal{N}}$ with

$$\deg_{G_{\text{reg}}}\big(\phi(y_{q_j}), (X \cup Y) \setminus \phi(\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_\ell)\big) - |(X \cup Y) \cap \phi(\mathcal{H}_1 \cup \ldots \cup \mathcal{H}_{j-1})| \geqslant \frac{\gamma^2\eta^2}{10^7(\Omega^*)^3}|X \cup Y| .$$

Suppose for example that

$$\deg_{G_{\text{reg}}}\big(\phi(y_{q_j}), X \setminus \phi(\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_\ell)\big) \geqslant \deg_{G_{\text{reg}}}\big(\phi(y_{q_j}), Y \setminus \phi(\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_\ell)\big) .$$

Then by the definition of $\tilde{\mathcal{N}}$ we get that

$$\begin{aligned}
&\deg_{G_{\text{reg}}}\big(\phi(y_{q_j}), X \setminus (C_{<i} \cup D_{<i} \cup \phi(\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_\ell \cup \mathcal{H}_1 \cup \ldots \cup \mathcal{H}_{j-1}))\big) \\
&\qquad \geqslant \frac{\gamma^2\eta^2}{10^7(\Omega^*)^3}|X| - \tau k \text{ and} \\
&|Y \setminus (C_{<i} \cup D_{<i} \cup \phi(\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_\ell \cup \mathcal{H}_1 \cup \ldots \cup \mathcal{H}_{j-1}))| \\
&\qquad \geqslant \frac{\gamma^2\eta^2}{10^7(\Omega^*)^3}|Y| - \tau k .
\end{aligned}$$

We can then embed $\mathcal{H}_j$ into the unoccupied part (X, Y) using Lemma 8.5. We now repeat this step for $j+1, \ldots, \ell$.

After this step, the set $C_{<i+1}$ is defined. We now observe that if $\mathcal{F}_j \cup \mathcal{H}_j \neq \mathcal{L}_{q_j}$ then the vertex y_{q_j} must have an insufficient degree in $\mathcal{N}$. More precisely, $\deg_{G_{\text{reg}}}(\phi(y_{q_j}), V(\mathcal{N})) - |C_{<i+1} \cap V(\mathcal{N})| < \frac{\eta^2 k}{10^6}$. Combined with (7.57), we have the following.

Claim 8.20.2. Suppose that $y \in R_i$ is such that the peripheral subshrubs whose parent is y were not embedded in $\mathcal{N}$. Then we have

$$\deg_{G_\mathcal{D}}(\phi(y), V_3) \geqslant h_1 - |C_{<i+1} \cap V(\mathcal{N})| - \frac{\eta^2 k}{10^6} \ .$$

This finishes the embedding in step i.

($\mathbf{S}_i\mathbf{a}$): *Embedding the principal subshrub M_i ($i \geqslant 1$).* We shall embed the principal subshrub M_i in $V_3 \cup V_4$ with the fruit f_i mapped to V_2.

The embedding has three stages. First we embed $M_i - M_i(\uparrow f_i)$, then we embed f_i, and finally we embed the forest $M_i(\uparrow f_i) - f_i$. The embedding of $M_i - M_i(\uparrow f_i)$ is an application of Lemma 8.12 analogous to the case of Configuration ($\diamond\mathbf{7}$) in the previous Lemma 8.19. That is, set $Y_1 := V_3$, $Y_2 := V_4$, $U^*_{\triangleright \mathrm{L}8.12} := S_{r_i} \setminus (C_{<i} \cup \phi(m_1, \ldots, m_{i-1}))$, and

$$U_{\triangleright \mathrm{L}8.12} := C_{<i} \cup D_{<i} \cup \mathbf{shadow}^{(2)}_{G-\mathbf{\Psi}}\left(C_{<i} \cup D_{<i}, \frac{\delta k}{8}\right) \cup W_i \ .$$

Note that

$$|U_{\triangleright \mathrm{L}8.12}| \leqslant \frac{10^5(\Omega^*)^5}{\delta^5} k \leqslant \frac{\delta\Lambda}{2\Omega^*} k \ , \text{ and}$$
$$|U_{\triangleright \mathrm{L}8.12}| \overset{(8.42)}{\geqslant} \frac{3\delta k}{8} - i \geqslant \frac{\delta k}{4} \ .$$

The family $\{P_1, \ldots, P_L\}_{\triangleright \mathrm{L}8.12}$ is $\{S_y\}_{y \in R_{<i}}$. There is only one tree to be embedded, $M_i - M_i(\uparrow f_i)$, which is rooted at the child of r_i. All the conditions of Lemma 8.12 are fulfilled. Lemma 8.12 gives an embedding of $M_i - M_i(\uparrow f_i)$ in $V_3 \cup V_4 \subseteq \mathfrak{P}_1$ with the property that $\mathrm{Par}(f_i)$ is mapped to $V_3 \setminus \left(\mathbf{shadow}^{(2)}_{G-\mathbf{\Psi}}(C_{<i} \cup D_{<i}, \frac{\delta k}{8}) \cup W_i\right)$. The lemma further gives a set $D' := C_{\triangleright \mathrm{L}8.12}$.

Using the degree condition (7.54) we can embed f_i to

$$V_2 \setminus \left(\mathbf{shadow}^{(1)}_{G-\mathbf{\Psi}}\left(C_{<i} \cup D_{<i}, \frac{\delta k}{8}\right) \cup \mathbf{shadow}^{(1)}_{G-\mathbf{\Psi}}(U_i, \frac{\delta k}{2})\right)$$

(recall that (8.34) asserts that only very little space in V_2 is occupied). To embed $M_i(\uparrow f_i) - f_i$ we use again Lemma 8.12. The parameters are this time $Y_1 := V_3$, $Y_2 := V_4$,

$$U^*_{\triangleright \mathrm{L}8.12} := (\mathrm{N}_G(\phi(f_i)) \cap V_3) \setminus (C_{<i} \cup D_{<i} \cup \phi(M_i - M_i(\uparrow f_i)) \cup \phi(m_1, \ldots, m_{i-1})) \ , \text{ and}$$
$$U_{\triangleright \mathrm{L}8.12} := C_{<i} \cup D_{<i} \cup \phi(M_i - M_i(\uparrow f_i) \cup D') \cup \phi(m_1, \ldots, m_{i-1}) \ .$$

Note that $|U^*_{\triangleright \mathrm{L}8.12}| \geqslant \frac{\delta k}{4}$ by (7.54), by the fact that $\phi(f_i) \notin \mathbf{shadow}^{(1)}_{G-\mathbf{\Psi}}\left(C_{<i} \cup D_{<i}, \frac{\delta k}{8}\right)$, and as $v(T_i) + i < \delta k/8$. The family $\{P_1, \ldots, P_L\}_{\triangleright \mathrm{L}8.12}$ is $\{S_y\}_{y \in R_{<i}}$. The trees to be embedded are the components of $M_i(\uparrow f_i) - f_i$ rooted at the children of f_i. All the conditions of Lemma 8.12 are fulfilled. The lemma provides an embedding in $V_3 \cup V_4 \subseteq \mathfrak{P}_1$.

It further gives a set $D'' := C_{\triangleright\mathrm{L}8.12}$. Set $D_i := V_3 \cap (D' \cup D'')$. The set D_i is such that for each $y \in R_i$,

$$|S_y \cap \phi(M_i)| \leqslant |S_y \cap D_i| + k^{0.75} , \tag{8.45}$$

as $S_y \subseteq V_3$.

($\mathbf{S}_i\mathbf{b}$): *Embedding knag P_i^* ($i \geqslant 1$).* Let s_i be the root of P_i^*. Note that $f_i = \mathrm{Par}_T(s_i)$. Let $j \in \mathcal{Y}$ be such that $(\mathrm{N}_G(\phi(f_i)) \cap Q_1^{(j)}) \setminus U_i \neq \emptyset$. Such a j exists by (7.52) and the fact that $\phi(f_i) \notin \mathbf{shadow}_{G-\mathbf{\Psi}}(U_i, \frac{\delta k}{2})$. We shall use Lemma 8.7 to embed P_i^* in $(Q_0^{(j)}, Q_1^{(j)})$. More precisely, we use Lemma 8.7 with $A_{\triangleright\mathrm{L}8.7} := Q_1^{(j)}$, $B_{\triangleright\mathrm{L}8.7} := Q_0^{(j)}$, $\varepsilon_{\triangleright\mathrm{L}8.7} := \tilde{\varepsilon}$, $d_{\triangleright\mathrm{L}8.7} := d'$, $\ell_{\triangleright\mathrm{L}8.7} := \mu k$, $U_A := (U_i \cup \phi(R_{<i} \cup \{f_1, \ldots, f_i\})) \cap A_{\triangleright\mathrm{L}8.7}$, $U_B := \phi(W_{B,<i} \cup R_{<i} \cup \{f_1, \ldots, f_i\}) \cap B_{\triangleright\mathrm{L}8.7}$. Lemma 8.7 outputs a $(V_{\mathrm{even}}(P_i^*, s_i) \hookrightarrow V_1 \setminus F_i, V_{\mathrm{odd}}(P_i^*, s_i) \hookrightarrow V_0)$-embedding of P_i^*.

($\mathbf{F}$): *Embedding the leftover subshrubs from $\bigcup \mathcal{L}_i$.* Recall that the sets V_3 and $V(\mathcal{N})$ are disjoint.

Let us take sequentially those $i \in [m]$ for which the subshrubs $\mathcal{L}_i$ were not embedded. By Claim 8.20.2 we have thus

$$\deg_{G_{\mathcal{D}}}(\phi(r_i), V_3 \setminus \mathrm{Im}(\phi)) \geqslant h_1 - |\phi^{-1}(V(\mathcal{N}))| - |\phi^{-1}(V_3)| - \frac{\eta^2 k}{10^6} \overset{(8.33)}{\geqslant} \frac{\eta^2 k}{10^6} .$$

An application of Lemma 8.12 in which $Y_1 := V_3$, $Y_2 := V_4$, $U_{\triangleright\mathrm{L}8.12} := \mathrm{Im}(\phi)$, $U^*_{\triangleright\mathrm{L}8.12} := \mathrm{N}_{G_{\mathcal{D}}}(\phi(r_i)) \cap V_3 \setminus \mathrm{Im}(\phi)$, and $\{P_1, \ldots, P_L\}_{\triangleright\mathrm{L}8.12} := \emptyset$ gives an embedding of $\mathcal{L}_i$ in $V_3 \cup V_4 \subseteq \mathfrak{P}_1$. $\square$

Lemma 8.21. *Suppose we are in Setting 7.4 and 7.7, and that the sets V_0 and V_1 witness Preconfiguration $(\heartsuit\mathbf{1})(2\eta^3 k/10^3, h)$. Suppose that $U \subseteq \mathfrak{P}_0 \cup \mathfrak{P}_1$. Suppose that $\{x_j\}_{j=1}^{\ell} \subseteq V_0$ and $\{y_j\}_{j=1}^{\ell'} \subseteq V_1$ are mutually distinct vertices. Let $\{(T_j, r_j)\}_{j=1}^{\ell}$ and $\{(T'_j, r'_j)\}_{j=1}^{\ell'}$ be families of rooted trees such that each component of $T_j - r_j$ and of $T'_j - r'_j$ has order at most τk.*

If

$$\sum_j v(T_j) \leqslant \frac{h}{2} - \frac{\eta^2 k}{1000} , \tag{8.46}$$

$$\sum_j v(T'_j) \leqslant h - \frac{\eta^2 k}{1000} , \text{ and} \tag{8.47}$$

$$|U| + \sum_j v(T_j) + \sum_j v(T'_j) \leqslant k , \tag{8.48}$$

then there exist mutually disjoint $(r_j \hookrightarrow x_j, V(T_j) \setminus \{r_j\} \hookrightarrow V(G) \setminus U)$-embeddings of T_j and $(r'_j \hookrightarrow y_j, V(T'_j) \setminus \{r'_j\} \hookrightarrow V(G) \setminus U)$-embeddings of T'_j in G.

Proof. The embedding has three stages. In Stage I we embed some components of $T_j - r_j$ (for all $j = 1, \dots, \ell$) in the parts of $(\mathcal{M}_A \cup \mathcal{M}_B)$-edges which are "seen in a balanced way from x_j". In Stage II we embed the remaining components of $T_j - r_j$. Last, in Stage III we embed all the components $T'_j - r'_j$ (for all $j = 1, \dots, \ell'$).

Remark 8.21.1. In order to embed the trees T_j and T'_j we shall exploit the fact that the vertices x_j and y_j have large degrees into the set $V_{\mathrm{good}}^{\restriction 2}$ (cf. (7.36) and (7.37)). This will enable us to extend the mappings of $\{r_j \mapsto x_j\}_j$ and $\{r'_j \mapsto y_j\}_j$ to embeddings of T_j and T'_j by exploiting the properties of the set V_{good}. The following four techniques will be used to this end (see (**U1**), ..., (**U4**) below):

- embedding into the $(\mathcal{M}_A \cup \mathcal{M}_B)$-edges using the regularity method,
- embedding using the nowhere-dense graph G_{exp},
- embedding in cluster graph $\mathbf{G}_{\mathrm{reg}}$, and
- embedding using $\mathfrak{A}$.

We will argue that since the degree of the root vertices x_j into the sets allowing these embedding techniques is bigger than the size of the image U of previously embedded parts of the trees (cf. (7.36)–(7.37), and (8.46)–(8.47)) we can indeed find placements for the roots r_j avoiding U. This itself is however insufficient. Indeed, if one $\mathcal{M}_A \cup \mathcal{M}_B$-vertex X is occupied by U then not only X but also its partner in $\mathcal{M}_A \cup \mathcal{M}_B$ are unsuitable for embedding using the regularity of $\mathcal{M}_A \cup \mathcal{M}_B$. This phenomenon of "doubling of the forbidden set" is illustrated in Figure 8.6. The trees T_j are embedded

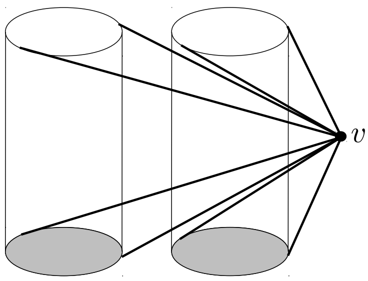

Figure 8.6: Even though the degree of v into the depicted semiregular matching is twice as much as the forbidden set (in gray) one cannot embed any tree from v into the semiregular matching.

in two stages (Stage I and Stage II) instead of just one to circumvent this issue.

The issue does not arise with the trees T'_j, as the vertices y_j are guaranteed to see only one part of the matching $\mathcal{M}_A \cup \mathcal{M}_B$ (cf. (7.38)).

Set $\alpha := \frac{\eta^3 \gamma^2}{10^{10} (\Omega^*)^2}$.

Let us first give a bound on the total size of $(\mathcal{M}_A\cup\mathcal{M}_B)$-vertices $C\in\mathcal{V}(\mathcal{M}_A\cup\mathcal{M}_B)$, $C\subseteq\bigcup\mathbf{V}$ seen from a given vertex via edges of $G_{\mathcal{D}}$. This bound will be used repeatedly.

Claim 8.21.2. Consider an arbitrary vertex $v\in V(G)$. Then for the set $\mathcal{U}:=\{C\in\mathcal{V}(\mathcal{M}_A\cup\mathcal{M}_B)\,:\,C\subseteq\bigcup\mathbf{V},\deg_{G_{\mathcal{D}}}(x,C)>0\}$ we have

$$|\bigcup\mathcal{U}|\leqslant\frac{2(\Omega^*)^2k}{\gamma^2}\text{ , and} \tag{8.49}$$

$$|\mathcal{U}|\leqslant\frac{2(\Omega^*)^2k}{\gamma^2\pi\mathfrak{c}}\ . \tag{8.50}$$

Proof of Claim 8.21.2. Let $\mathbf{U}\subseteq\mathbf{V}$ be those clusters which intersect $\mathrm{N}_{G_{\mathcal{D}}}(x_j)$. Using the same argument as in the proof of Claim 8.17.1 we get that $|\bigcup\mathbf{U}|\leqslant\frac{2(\Omega^*)^2k}{\gamma^2}$. Then (8.49) follows. Also, (8.50) follows since $\mathcal{M}_A\cup\mathcal{M}_B$ is $(\varepsilon,d,\pi\mathfrak{c})$-semiregular. □

Stage I: We proceed inductively for $j=1,\dots,\ell$. Suppose that we embedded some components $\mathcal{F}_1,\dots,\mathcal{F}_{j-1}$ of the forests $T_1-r_1,\dots,T_{j-1}-r_{j-1}$. We write F_{j-1} for the partial images of this embedding. We inductively assume that

$$F_{j-1}\text{ is }\tau k\text{-balanced w.r.t. }\mathcal{M}_A\cup\mathcal{M}_B. \tag{8.51}$$

For each $(A,B)\in\mathcal{M}_A\cup\mathcal{M}_B$ such that $\deg_{G_{\mathcal{D}}}(x_j,(A\cup B)\setminus\mathfrak{A})>0$ take a subpair (A',B'),

$$A'\subseteq(A\cap\mathrm{N}_{G_{\mathcal{D}}\cup G_\nabla}(x_j))^{\restriction 2}\setminus F_{j-1}\quad\text{and}\quad B'\subseteq(B\cap\mathrm{N}_{G_{\mathcal{D}}\cup G_\nabla}(x_j))^{\restriction 2}\setminus F_{j-1}\ ,$$

such that

$$|A'|=|B'|=\min\left\{|(A\cap\mathrm{N}_{G_{\mathcal{D}}\cup G_\nabla}(x_j))^{\restriction 2}\setminus F_{j-1}|,|(B\cap\mathrm{N}_{G_{\mathcal{D}}\cup G_\nabla}(x_j))^{\restriction 2}\setminus F_{j-1}|\right\}\ .$$

These pairs comprise a semiregular matching $\mathcal{N}_j$. Pairs $(A,B)\in\mathcal{M}_A\cup\mathcal{M}_B$ such that $\deg_{G_{\mathcal{D}}}(x_j,(A\cup B)\setminus\mathfrak{A})=0$ are not considered for the construction of $\mathcal{N}_j$. Let $\mathcal{M}_j:=\{(A',B')\in\mathcal{N}_j:|A'|>\alpha|A|\}$. By Fact 2.7 $\mathcal{M}_j$ is an $(2\varepsilon/\alpha,d/2,\alpha\pi\mathfrak{c})$-semiregular matching.

Claim 8.21.3. We have $|V(\mathcal{M}_j)|\geqslant|V(\mathcal{N}_j)|-\frac{\eta^3k}{10^9}$.

Proof of Claim 8.21.3. By (8.49), and by Property 4 of Setting 7.4, we have $|V(\mathcal{M}_j)|\geqslant|V(\mathcal{N}_j)|-\alpha\times\frac{2\Omega^2k}{\gamma^2}$. □

Let $\mathcal{F}_j$ be a maximal set of components of T_j-r_j such that

$$v(\mathcal{F}_j)\leqslant|V(\mathcal{M}_j)|-\frac{\eta^3k}{10^9}\ . \tag{8.52}$$

Observe that when $\mathcal{F}_j$ does not contain all the components of T_j-r_j then

$$v(\mathcal{F}_j)\geqslant|V(\mathcal{M}_j)|-\frac{\eta^3k}{10^9}-\tau k\geqslant|V(\mathcal{M}_j)|-\frac{2\eta^3k}{10^9}\ . \tag{8.53}$$

Lemma 8.8 guarantees that we can find an embedding of $\mathcal{F}_j$ in $\mathcal{M}_j$. Further the lemma together with the induction hypothesis (8.51) implies that the embedding can be chosen so that the new image set F_j is τk-balanced w.r.t. $\mathcal{M}_A \cup \mathcal{M}_B$. We fix this embedding. If $\mathcal{F}_j$ does not contain all the components of $T_j - r_j$ then (8.53) gives

$$|V(\mathcal{M}_j) \setminus F_j| \leqslant \frac{2\eta^3 k}{10^9} . \tag{8.54}$$

<u>After Stage I:</u> Let $\mathcal{N}^*$ be a maximal semiregular matching contained by $(\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 2}$ excluding F_ℓ. The first claim below says that the matching $\mathcal{N}^*$ exhausts the free space in $(\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 2} \cap \bigcup \mathbf{V}$ while the second claim says that for those $j \in [\ell]$ for which $\mathcal{F}_j$ is not all the components of $T_j - r_j$ we have that x_j (roughly) sees at most one $\mathcal{N}^*$-vertex of each $\mathcal{N}^*$-edge.

Claim 8.21.4. We have

$$\deg^{\max}{}_{G_\mathcal{D}} \Big(V_0 \cup V_1, V(\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 2} \setminus (V(\mathcal{N}^*) \cup F_\ell \cup \mathfrak{A}) \Big) < \frac{\eta^3 k}{10^9} .$$

Proof of Claim 8.21.4. Let us consider an arbitrary vertex $v \in V_0 \cup V_1$. By (8.50) the number of $(\mathcal{M}_A \cup \mathcal{M}_B)$-vertices $C \subseteq \bigcup \mathbf{V}$ such that $\deg_{G_\mathcal{D}}(x, C) > 0$ is at most $\frac{2(\Omega^*)^2 k}{\gamma^2 \pi \mathfrak{c}}$.

Observe now that for each $(\mathcal{M}_A \cup \mathcal{M}_B)$-edge (A, B) we have due to (8.51) that

$$\Big| (A \cup B)^{\restriction 2} \setminus (V(\mathcal{N}^*) \cup F_\ell) \Big| \leqslant \tau k . \tag{8.55}$$

Thus summing (8.55) over all $(\mathcal{M}_A \cup \mathcal{M}_B)$-edges (A, B) with $\deg_{G_\mathcal{D}}(x, (A \cup B) \setminus \mathfrak{A}) > 0$ we get

$$\deg_{G_\mathcal{D}} \Big(x, V(\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 2} \setminus (V(\mathcal{N}^*) \cup F_\ell \cup \mathfrak{A}) \Big) \leqslant \frac{4(\Omega^*)^2 k}{\gamma^2 \pi \mathfrak{c}} \cdot \tau k .$$

The claim now follows by (7.3). □

Claim 8.21.5. Let $j \in [\ell]$ be such that $\mathcal{F}_j$ is not all the components of $T_j - r_j$. Then there exists an $\mathcal{N}^*$-cover $\mathcal{X}_j$ such that $\deg_{G_\mathcal{D}}(x_j, \bigcup \mathcal{X}_j) \leqslant \frac{3\eta^3 k}{10^9}$.

Proof of Claim 8.21.5. First, we define an $(\mathcal{M}_A \cup \mathcal{M}_B)$-cover $\mathcal{R}_j$ as follows. For an $(\mathcal{M}_A \cup \mathcal{M}_B)$-edge (A, B) let $\mathcal{R}_j$ contain A if

$$|(A \cap \mathrm{N}_{G_\mathcal{D} \cup G_\nabla}(x_j))^{\restriction 2} \setminus F_{j-1}| \leqslant |(B \cap \mathrm{N}_{G_\mathcal{D} \cup G_\nabla}(x_j))^{\restriction 2} \setminus F_{j-1}| ,$$

and B otherwise. Observe that

$$\deg_{G_\mathcal{D}} \Big(x_j, \bigcup \mathcal{R}_j \setminus V(\mathcal{N}_j) \Big) = 0 . \tag{8.56}$$

Also, we have $V(\mathcal{N}^*) \cap \bigcup \mathcal{R}_j \cap V(\mathcal{M}_j) \subseteq V(\mathcal{N}^*) \cap V(\mathcal{M}_j) \subseteq V(\mathcal{M}_j) \setminus F_j$. In particular, (8.54) gives that

$$\Big| V(\mathcal{N}^*) \cap \bigcup \mathcal{R}_j \cap V(\mathcal{M}_j) \Big| \leqslant \frac{2\eta^3 k}{10^9} . \tag{8.57}$$

Let $\mathcal{X}_j$ be the restriction of $\mathcal{R}_j$ on $\mathcal{N}^*$. We then have

$$\begin{aligned}\deg_{G_{\mathcal{D}}}\left(x_j, \bigcup \mathcal{X}_j\right) &= \deg_{G_{\mathcal{D}}}\left(x_j, V(\mathcal{N}^*) \cap \bigcup \mathcal{R}_j\right) \\ \text{(by (8.56))} \quad &\leqslant \deg_{G_{\mathcal{D}}}\left(x_j, V(\mathcal{N}^*) \cap \bigcup \mathcal{R}_j \cap V(\mathcal{M}_j)\right) \\ &\quad + \deg_{G_{\mathcal{D}}}\left(x_j, V(\mathcal{N}_j) \setminus V(\mathcal{M}_j)\right) \\ \text{(by (8.57), Claim 8.21.3)} \quad &\leqslant \frac{3\eta^3 k}{10^9} .\end{aligned}$$

□

For every $j \in [\ell]$ we define $\mathcal{N}_j^* \subseteq \mathcal{N}^*$ as those $\mathcal{N}^*$-edges (A, B) for which we have

$$\left((A \cup B) \setminus \bigcup \mathcal{X}_j\right) \cap \mathfrak{A} = \emptyset .$$

<u>Stage II:</u> We proceed inductively for $j = 1, \ldots, \ell$ with embedding the components of $T_j - r_j$ not included in $\mathcal{F}_j$, which we denote by $\mathcal{K}_j$. There is nothing to do when $\mathcal{K}_j = \emptyset$, so let us assume otherwise.

We write $\mathbf{L} := \{C \in \mathbf{V} : C \subseteq \mathbb{L}_{\eta,k}(G)\}$. Let $K \in \mathcal{K}_j$ be a component that has not been embedded yet. We write U' for the total image of what has been embedded (in Stage I, and Stage II so far), combined with U. We claim that x_j has a substantial degree into one of four specific vertex sets.

Claim 8.21.6. At least one of the following four cases occurs.

(U1) $\deg_{G_{\mathcal{D}}}\left(x_j, V(\mathcal{N}_j^*) \setminus \bigcup \mathcal{X}_j\right) - |U' \cap V(\mathcal{N}_j^*)| \geqslant \frac{\eta^2 k}{10^4}$,

(U2) $\deg_{G_{\mathcal{D}}}(x_j, \mathfrak{A} \setminus U') \geqslant \frac{\eta^2 k}{10^4}$,

(U3) $\deg_{G_{\nabla}}(x_j, V(G_{\mathrm{exp}}) \setminus U') \geqslant \frac{\eta^2 k}{10^4}$,

(U4) $\deg_{G_{\mathcal{D}}}(x_j, \bigcup \mathbf{L} \setminus (L_\# \cup V(G_{\mathrm{exp}}) \cup U')) \geqslant \frac{\eta^2 k}{10^4}$.

Proof. We write $U'' := (U')^{\restriction 2}$ (thus, U'' is what has been embedded in Stage I and

Stage II). We have by (7.36) that

$$
\begin{aligned}
\frac{h}{2} &\leqslant \deg_{G_\nabla}(x_j, V_{\mathrm{good}}^{\restriction 2}) \\
&\leqslant \deg_{G_\mathcal{D}}\Big(x_j, V(\mathcal{N}_j^*) \setminus (V(G_{\mathrm{exp}}) \cup \bigcup \mathcal{X}_j)\Big) \\
&\quad + \deg_{G_\mathcal{D}}\Big(x_j, \mathfrak{A}^{\restriction 2} \setminus (V(\mathcal{N}_j^*) \cup V(G_{\mathrm{exp}}))\Big) \\
&\quad + \deg_{G_\nabla}\Big(x_j, V(G_{\mathrm{exp}})^{\restriction 2}\Big) \\
&\quad + \deg_{G_\mathcal{D}}\Big(x_j, \bigcup \mathbf{L}^{\restriction 2} \setminus (L_\# \cup V(G_{\mathrm{exp}}) \cup V(\mathcal{N}_j^*))\Big) \\
&\quad + \deg_{G_\mathcal{D}}\Big(x_j, V(\mathcal{M}_A \cup \mathcal{M}_B)^{\restriction 2} \setminus (V(\mathcal{N}^*) \cup \mathfrak{A})\Big) + \deg_{G_\mathcal{D}}\Big(x_j, \bigcup \mathcal{X}_j\Big) \\
\text{(by C8.21.4, C8.21.5)} \quad &\leqslant \deg_{G_\mathcal{D}}\Big(x_j, V(\mathcal{N}_j^*) \setminus (V(G_{\mathrm{exp}}) \cup \bigcup \mathcal{X}_j)\Big) \\
&\quad - \Big|U'' \cap \Big(\bigcup \mathcal{X}_j \cup (V(\mathcal{N}_j^*) \setminus V(G_{\mathrm{exp}}))\Big)\Big| \\
&\quad + \deg_{G_\mathcal{D}}\Big(x_j, \mathfrak{A}^{\restriction 2} \setminus (V(\mathcal{N}_j^*) \cup \bigcup \mathcal{X}_j \cup U'')\Big) \\
&\quad + \deg_{G_{\mathrm{exp}}}\Big(x_j, V(G_{\mathrm{exp}})^{\restriction 2} \setminus (\bigcup \mathcal{X}_j \cup U'')\Big) \\
&\quad + \deg_{G_\mathcal{D}}\Big(x_j, \bigcup \mathbf{L}^{\restriction 2} \setminus (L_\# \cup V(G_{\mathrm{exp}}) \cup V(\mathcal{N}_j^*) \cup U'')\Big) \\
&\quad + \frac{4\eta^3 k}{10^9} + |U''| \, .
\end{aligned}
$$

The claim follows since $|U''| \leqslant \frac{h}{2} - \frac{\eta^2 k}{1000}$. □

We now now briefly describe how to embed K in each of the cases (**U1**)–(**U4**).

- In case (**U1**) recall that each $(\mathcal{M}_A \cup \mathcal{M}_B)$-edge contains at most one $\mathcal{N}_j^*$-edge. Thus by (8.49) we get that there is an $(\mathcal{M}_A \cup \mathcal{M}_B)$-edge (A, B) with

$$
\begin{aligned}
&\deg_{G_\mathcal{D}}\Big(x_j, (V(\mathcal{N}_j^*) \cap (A \cup B)) \setminus \bigcup \mathcal{X}_j\Big) - |V(\mathcal{N}_j^*) \cap U' \cap (A \cup B)| \\
&\qquad \geqslant \frac{\eta^2 k}{10^4} \cdot \frac{\gamma^2}{2(\Omega^*)^2 k} \cdot |A| \, .
\end{aligned} \tag{8.58}
$$

  Let us fix this edge (A, B), and let (A', B') be the corresponding edge in $\mathcal{N}_j^*$. Suppose without loss of generality that $B \in \mathcal{X}_j$. We can now embed K in (A', B') using Lemma 8.5 with the following input: $C_{\triangleright \mathrm{L}8.5} := A', D_{\triangleright \mathrm{L}8.5} := B', X_{\triangleright \mathrm{L}8.5} := A' \setminus U', X^*_{\triangleright \mathrm{L}8.5} := \mathrm{N}_{G_\mathcal{D}}(x_j, A' \setminus U'), Y_{\triangleright \mathrm{L}8.5} := B' \setminus U', \ell_{\triangleright \mathrm{L}8.5} \geqslant |X^*_{\triangleright \mathrm{L}8.5}| \geqslant \frac{\gamma^2 \eta^2 |A|}{4 \cdot 10^4 (\Omega^*)^2}, \varepsilon_{\triangleright \mathrm{L}8.5} := \frac{8 \cdot 10^4 (\Omega^*)^2 \varepsilon)}{\gamma^2 \eta^2}, \beta_{\triangleright \mathrm{L}8.5} := d/6$.

- In Case (**U2**) we embed K by Lemma 8.3 with the following input: $\varepsilon_{\triangleright \mathrm{L}8.3} := \varepsilon', U_{\triangleright \mathrm{L}8.3} := U', U^*_{\triangleright \mathrm{L}8.3} := \mathrm{N}_{G_\mathcal{D}}(x_j, \mathfrak{A} \setminus U'), \ell := 1$.

- In Case (**U3**) we embed K by Lemma 8.4 with the following input: $U_{\triangleright \mathrm{L}8.4} := U', U^*_{\triangleright \mathrm{L}8.4} := \mathrm{N}_{G_{\mathrm{exp}}}(x_j, V(G_{\mathrm{exp}}) \setminus U'), Q_{\triangleright \mathrm{L}8.4} := 1, \zeta_{\triangleright \mathrm{L}8.4} := \rho, \ell_{\triangleright \mathrm{L}8.4} := 1$.

- In Case (**U4**) we proceed as follows. As $\deg_{G_{\mathcal{D}}}(x_j, V_{\not\leadsto\mathbf{\Psi}}) < \frac{\eta^2 k}{10^5}$ (cf. Definition 7.16), we have

$$\deg_{G_{\mathcal{D}}}\left(x_j, \bigcup \mathbf{L} \setminus (L_{\#} \cup V(G_{\exp}) \cup V_{\not\leadsto\mathbf{\Psi}} \cup U')\right) \geqslant \frac{2\eta^2 k}{10^5}\ .$$

We use a similar method as in (8.58) to find a cluster $A \in \mathbf{L}$ such that

$$\deg_{G_{\mathcal{D}}}\left(x_j, A \setminus (L_{\#} \cup V(G_{\exp}) \cup V_{\not\leadsto\mathbf{\Psi}} \cup U')\right) \geqslant \frac{2\eta^2 k}{10^5} \cdot \frac{\gamma^2}{2(\Omega^*)^2 k} \cdot |A|\ .$$

Recall that $\deg^{\min}{}_{G_{\nabla}}(A \setminus (L_{\#} \cup V_{\not\leadsto\mathbf{\Psi}}), V(G) \setminus \mathbf{\Psi}) \geqslant (1 + \frac{4\eta}{5})k$. Thus at least one of the following subcases must occur for the set $A^* := (\mathrm{N}_{G_{\mathcal{D}}}(x_j) \cap A) \setminus (L_{\#} \cup V(G_{\exp}) \cup V_{\not\leadsto\mathbf{\Psi}} \cup U')$:

(**U4a**) For at least $\frac{1}{2}|A^*|$ vertices $v \in A^*$ we have $\deg_{G_{\nabla}}(v, \mathfrak{A} \setminus U') \geqslant \frac{2\eta k}{5}$.

(**U4b**) For at least $\frac{1}{2}|A^*|$ vertices $v \in A^*$ we have $\deg_{G_{\mathrm{reg}}}(v, \bigcup \mathbf{V} \setminus U') \geqslant \frac{2\eta k}{5}$.

In case (**U4a**) we embed K using Lemma 8.3. Details are very similar to (**U2**). As for case (**U2b**), let as take an arbitrary vertex $v \in A^*$ with $\deg_{G_{\mathrm{reg}}}(v, \bigcup \mathbf{V} \setminus U') \geqslant \frac{2\eta k}{5}$. In particular, using again the same method as in (8.58), we get that there exists a cluster $B \in \mathbf{V}$ with

$$\deg_{G_{\mathrm{reg}}}(v, B \setminus U') \geqslant \frac{\gamma^2 \eta}{10(\Omega^*)^2}|B|\ .$$

Map the root r_K of K to v and embed $K - r_K$ in (A, B) using Lemma 8.5 with the following input: $C_{\triangleright \mathrm{L}8.5} := B, D_{\triangleright \mathrm{L}8.5} := A, X_{\triangleright \mathrm{L}8.5} := B \setminus U', Y_{\triangleright \mathrm{L}8.5} := A \setminus U', X^*_{\triangleright \mathrm{L}8.5} := \mathrm{N}_{G_{\mathrm{reg}}}(v, B \setminus U'), \ell_{\triangleright \mathrm{L}8.5} := \mathfrak{c}, \beta_{\triangleright \mathrm{L}8.5} := \gamma^2\eta/(5(\Omega^*)^2), \varepsilon_{\triangleright \mathrm{L}8.5} := \varepsilon'$.

<u>Stage III:</u> In this stage we embed the trees $\{T'_j\}_{j=1}^{\ell'}$. The embedding techniques are as in Stage II. The cover $\mathcal{F}'$ from Definition 7.16 plays the same role as the covers $\mathcal{X}_j$ in Stage II. Observe that $\mathcal{F}'$ is universal whereas the covers $\mathcal{X}_j$ are specific for each vertex x_j. In Stage III we use the semiregular matching $\mathcal{M}_A \cup \mathcal{M}_B$ for embedding (in a counterpart of (**U1**)) instead of $\mathcal{N}^*_j$.

Again we proceed inductively for $j = 1, \ldots, \ell$ with embedding the components of $T'_j - r'_j$, which we denote by $\mathcal{K}'_j$. Let $K \in \mathcal{K}'_j$ be a component that has not been embedded yet. We write U' for the total image of what has been embedded (in Stage I, II, and Stage III so far), combined with U. We claim that y_j has a substantial degree into one of four specific vertex sets.

Claim 8.21.7. At least one of the following four cases occurs.

(**U1′**) $\deg_{G_{\mathcal{D}}}(y_j, V(\mathcal{M}_A \cup \mathcal{M}_B) \setminus (\mathfrak{A} \cup \bigcup \mathcal{F}'))$
$- |U' \cap (\bigcup \mathcal{F}' \cup (V(\mathcal{M}_A \cup \mathcal{M}_B) \setminus \mathfrak{A})| \geqslant \frac{\eta k}{80}$,

(**U2′**) $\deg_{G_{\mathcal{D}}}(y_j, \mathfrak{A} \setminus U') \geqslant \frac{\eta k}{80}$,

(**U3′**) $\deg_{G_{\nabla}}(y_j, V(G_{\exp}) \setminus U') \geqslant \frac{\eta k}{80}$,

(**U4′**) $\deg_{G_{\mathcal{D}}}(y_j, \bigcup \mathbf{L} \setminus (L_{\#} \cup V(G_{\exp}) \cup U')) \geqslant \frac{\eta k}{80}$.

Proof. As $y_j \in V_1 \subseteq \mathbb{YA}$, we have that

$$\begin{aligned}
(1+\frac{\eta}{10})k &\leqslant \deg_{G_{\nabla}}(y_j, V_{\mathrm{good}}) \\
&\leqslant \deg_{G_{\mathcal{D}}}\Big(x_j, V(\mathcal{M}_A \cup \mathcal{M}_B) \setminus (\mathfrak{A} \cup V(G_{\exp}) \cup \bigcup \mathcal{F}')\Big) \\
&\quad + \deg_{G_{\mathcal{D}}}\Big(y_j, \mathfrak{A} \setminus (V(G_{\exp}) \cup \bigcup \mathcal{F}'\Big) \\
&\quad + \deg_{G_{\mathcal{D}}}(y_j, \bigcup \mathcal{F}') + \deg_{G_{\mathcal{D}}}\Big(y_j, \bigcup \mathbf{L} \setminus (L_{\#} \cup V(G_{\exp}) \cup V(\mathcal{M}_A \cup \mathcal{M}_B)\Big) \\
&\quad + \deg_{G_{\nabla}}(y_j, V(G_{\exp})) \\
&\leqslant \deg_{G_{\mathcal{D}}}\Big(y_j, V(\mathcal{M}_A \cup \mathcal{M}_B) \setminus (\mathfrak{A} \cup V(G_{\exp}) \cup \bigcup \mathcal{F}')\Big) \\
&\quad - \Big|U' \cap \Big(\bigcup \mathcal{F}' \cup (V(\mathcal{M}_A \cup \mathcal{M}_B) \setminus \mathfrak{A})\Big) \setminus V(G_{\exp})\Big| \\
&\quad + \deg_{G_{\mathcal{D}}}\Big(y_j, \mathfrak{A} \setminus (U' \cup V(G_{\exp}) \cup \bigcup \mathcal{F}')\Big) + \deg_{G_{\nabla}}(y_j, V(G_{\exp}) \setminus U') \\
&\quad + \deg_{G_{\mathcal{D}}}\Big(y_j, \bigcup \mathbf{L} \setminus (L_{\#} \cup V(G_{\exp}) \cup V(\mathcal{M}_A \cup \mathcal{M}_B) \cup U')\Big) + \frac{2\eta^3 k}{10^3} + |U'|\,.
\end{aligned}$$

The claim follows since $|U'| \leqslant k$.

Cases (**U1′**)–(**U4′**) are treated analogously as Cases (**U1**)–(**U4**). □

□

Lemma 8.22. *Suppose we are in Setting 7.4 and 7.7, and that the sets V_0 and V_1 witness Preconfiguration $(\heartsuit\mathbf{2})(h)$. Suppose that $U \subseteq \mathfrak{P}_0 \cup \mathfrak{P}_1$, such that $|U| \leqslant k$. Suppose that $\{x_j\}_{j=1}^{\ell} \subseteq V_0 \cup V_1$ are distinct vertices. Let $\{(T_j, r_j)\}_{j=1}^{\ell}$ be a family of rooted trees such that each component of $T_j - r_j$ has order at most τk.*

If $\sum_j v(T_j) \leqslant h - \eta^2 k/1000$ and $|U| + \sum_j v(T_j) \leqslant k$ then there exist disjoint $(r_j \hookrightarrow x_j, V(T_j) \setminus \{r_j\} \hookrightarrow V(G) \setminus U)$-embeddings of T_j in G.

Proof. The proof is contained in the proof of Lemma 8.21. It just suffices to repeat the first two stages of the embedding process in the proof. In that setting, we use $h_{\triangleright \mathrm{L}8.21} = 2h$. It now suffices to realize that the condition $\{x_j\} \subseteq V_0$ in the setting of Lemma 8.21 gives us the same possibilities for embedding as the condition $\{x_j\} \subseteq V_0 \cup V_1$ in the current setting (cf. (7.36) and (7.39)). □

Lemma 8.23. *Suppose we are in Setting 7.4 and 7.7, at least one of the following configurations occurs:*

- *Configuration* $(\diamond\mathbf{6})(\frac{\eta^3\rho^4}{10^{14}(\Omega^*)^4}, 4\varepsilon_{\odot}, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2\cdot 10^3}, h)$,

- *Configuration* $(\diamond\mathbf{7})(\frac{\eta^3\gamma^3\rho}{10^{12}(\Omega^*)^4)}, \frac{\eta\gamma}{400}, 4\varepsilon_{\odot}, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\eta^2\nu}{2\cdot 10^4}, \frac{3\eta^3}{2\cdot 10^3}, h)$*, or*
- *Configuration* $(\diamond\mathbf{8})(\frac{\eta^4\gamma^4\rho}{10^{15}(\Omega^*)^5)}, \frac{\eta\gamma}{400}, \frac{4\varepsilon}{\mathfrak{p}_1}, 4\varepsilon_{\odot}, \frac{d}{2}, \frac{\gamma^3\rho}{32\Omega^*}, \frac{\mathfrak{p}_1\pi\mathfrak{c}}{2k}, \frac{\eta^2\nu}{2\cdot 10^4}, h_1, h)$.

Suppose that $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ *is a* τk*-fine partition of a rooted tree* (T, r) *of order* k*. If the total order of the end shrubs is at most* $h - \frac{\eta^2 k}{10^3}$ *and the total order of the internal shrubs is at most* $h_1 - \frac{2\eta^2}{10^5}$*, then* $T \subseteq G$*.*

Proof. Let T' be the tree induced by all the cut-vertices $W_A \cup W_B$ and all the internal shrubs. Fix an embedding of T' as in Lemma 8.19 (in configurations $(\diamond\mathbf{6})$ and $(\diamond\mathbf{7})$), or in in Lemma 8.20 (in configuration $(\diamond\mathbf{8})$). This embedding now extends to external shrubs by Lemma 8.21 (in Preconfiguration $(\heartsuit\mathbf{1})$, which can only occur in Configuration $(\diamond\mathbf{6})$ and $(\diamond\mathbf{7})$), or by Lemma 8.22 (in Preconfiguration $(\heartsuit\mathbf{2})$). ☐

The next lemma completely resolves Theorem 1.3 in the presence of Configuration $(\diamond\mathbf{9})$.

Lemma 8.24. *Suppose we are in Setting 7.4 and 7.7, and we have Configuration*

$$(\diamond\mathbf{9})(\delta, \frac{2\eta^3}{10^3}, h_1, h_2, \varepsilon_1, d_1, \mu_1, \varepsilon_2, d_2, \mu_2)$$

with $d_2 > 10\varepsilon_2 > 0$, $4\cdot 10^3 \leqslant d_2\mu_2\tau k$, $\max\{d_1, \tau/\mu_1\} \leqslant \gamma^2\eta^2/(4\cdot 10^7(\Omega^*)^2)$, $d_1^2/6 > \varepsilon_1 \geqslant \tau/\mu_1$ *and* $\delta k > 10^3/\tau$*. Suppose that* $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ *is a* τk*-fine partition of a rooted tree* (T, r) *of order* k*. If the total order of the internal shrubs of* $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ *is at most* $h_1 - \frac{\eta^2 k}{10^5}$*, and the total order of the end shrubs is at most* $h_2 - \frac{\eta^2 k}{10^5}$ *then* $T \subseteq G$*.*

Proof. Let $V_0, V_1, V_2, \mathcal{N}, \{Q_0^{(j)}, Q_1^{(j)}\}_{j\in\mathcal{Y}}$ and $\mathcal{F}'$ witness $(\diamond\mathbf{9})$. The embedding process has two stages. In the first stage we embed the knags and the internal shrubs of T. In the second stage we embed the end shrubs. The knags will be embedded in $V_0 \cup V_1$, and the internal shrubs will be embedded in $V(\mathcal{N})$. Lemma 8.21 will be used to embed the end shrubs.

The knags of $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ are embedded in such a way that W_A is embedded in V_1 and W_B is embedded in V_0. Since no other part of T is embedded in $V_0 \cup V_1$ in the first stage, each knag can be embedded greedily using the minimum degree condition arising from the super-regularity of the pairs $\{(Q_0^{(j)}, Q_1^{(j)})\}_{j\in\mathcal{Y}}$ using the bound on the total order of knags coming from Definition 3.1(c) and Lemma 8.7 wiht the following input: $\varepsilon_{\triangleright\text{L8.7}} := \varepsilon_2$, $d_{\triangleright\text{L8.7}} := d_2$, $\ell_{\triangleright\text{L8.7}} := \mu_2 k$, $U_A \cup U_B$ be the image of $W_A \cup W_B$ embedded so far and $\{A_{\triangleright\text{L8.7}}, B_{\triangleright\text{L8.7}}\} := \{Q_0^{(j)}, Q_1^{(j)}\}$, where $j \in \mathcal{Y}$ is arbitrary for the first knag, and is such that $\mathrm{N}_{G_{\mathcal{D}}}(\phi(f)) \cap Q_1^{(j)} \setminus U_A \neq \emptyset$, for $f = \mathrm{Par}(P)$, when we want to embed the knag P.

We now describe how to embed an internal shrub $T^* \in \mathcal{S}_A$ whose parent $u \in W_A$ is embedded on a vertex $x \in V_1$. Let $w \in V(T^*)$ be the unique neighbor of a vertex from

$W_A \setminus \{u\}$ (cf. Definition 3.1(h)). Let U be the image of the part of T embedded so far. The next claim finds a suitable $\mathcal{N}$-edge for accommodating T^*.

Claim 8.24.1. There exists an $\mathcal{N}$-edge (A,B), or an $\mathcal{N}$-edge (B,A) such that

$$\min\left\{|\mathrm{N}_{G_\mathcal{D}}(x) \cap V_2 \cap (A \setminus U)|, |B \setminus U|\right\} \geqslant 100d_1|A| + \tau k \, .$$

Proof of Claim 8.24.1. For the purpose of this claim we reorient $\mathcal{N}$ so that $V_2(\mathcal{N}) \subseteq \bigcup \mathcal{F}'$.

Suppose the claim fails. Then for each $(A,B) \in \mathcal{N}$ we have $|\mathrm{N}_{G_\mathcal{D}}(x) \cap V_2 \cap (A \setminus U)| < 100d_1|A| + \tau k$ or $|B \setminus U| < 100d_1|A| + \tau k$. In either case we get

$$|\mathrm{N}_{G_\mathcal{D}}(x) \cap V_2 \cap A| - |U \cap (A \cup B)| < 100d_1|A| + \tau k \, . \tag{8.59}$$

We write $S := \bigcup\{V(D) : D \in \mathcal{D}, x \in V(D)\}$. Combining Fact 4.3 and Fact 4.4 we get that

$$|S| \leqslant \frac{2(\Omega^*)^2 k}{\gamma^2} \, . \tag{8.60}$$

Let us look at the number

$$\lambda := \sum_{(A,B)\in\mathcal{N}} \left(|\mathrm{N}_{G_\mathcal{D}}(x) \cap V_2 \cap A| - |U \cap (A \cup B)|\right) . \tag{8.61}$$

For a lower bound on λ, we write $\lambda = |\mathrm{N}_{G_\mathcal{D}}(x) \cap V_2| - |U \cap V(\mathcal{N})|$. The first term is at least h_1 by (7.58), while the second term is at most $h_1 - \frac{\eta^2 k}{10^5}$ by the assumptions of the lemma. Thus $\lambda \geqslant \frac{\eta^2 k}{10^5}$.

For an upper bound on λ we only consider those $\mathcal{N}$-edges (A,B) for which $\mathrm{N}_{G_\mathcal{D}}(x) \cap A \neq \emptyset$. In that case $A \subseteq S$ (cf. 3 of Setting 7.4). Thus, combining (8.60) with the fact that $\mathcal{N}$ is $(\varepsilon_1, d_1, \mu_1 k)$-semiregular we get that

$$|\{(A,B) \in \mathcal{N} \, : \, \mathrm{N}_{G_\mathcal{D}} \cap A \neq \emptyset\}| \leqslant \frac{2(\Omega^*)^2}{\gamma^2 \mu_1} \, . \tag{8.62}$$

Thus,

$$\begin{aligned} \lambda &\leqslant \sum_{(A,B)\in\mathcal{N}, \mathrm{N}_{G_\mathcal{D}}(x)\cap A\neq\emptyset} \left(|\mathrm{N}_{G_\mathcal{D}}(x) \cap V_2 \cap A| - |U \cap (A \cup B)|\right) \\ \text{(by (8.59), (8.62))} \quad &\leqslant 100d_1|S| + \frac{2(\Omega^*)^2}{\gamma^2\mu_1}\tau k \\ \text{(by (8.60))} \quad &< \frac{\eta^2 k}{10^5} \, , \end{aligned}$$

a contradiction. This finishes the proof of the claim. □

By symmetry we suppose that Claim 8.24.1 gives an $\mathcal{N}$-edge (A,B) such that $\min\left\{|\mathrm{N}_{G_\mathcal{D}}(x) \cap V_2 \cap (A \setminus U)|, |B \setminus U|\right\} \geqslant 100d_1|A| + \tau k$. We apply Lemma 8.5 with input $C_{\triangleright\mathrm{L}8.5} := A$, $D_{\triangleright\mathrm{L}8.5} := B$ $X_{\triangleright\mathrm{L}8.5} = X^*_{\triangleright\mathrm{L}8.5} := \mathrm{N}_{G_\mathcal{D}}(x) \cap V_2 \cap (A \setminus U)$, $Y_{\triangleright\mathrm{L}8.5} := B \setminus U$

, $\varepsilon_{\triangleright L8.5} := \varepsilon_1$, $\beta_{\triangleright L8.5} := d_1/3$, $\ell_{\triangleright L8.5} := \mu_1 k$. Then there exists an embedding of T^* in $V(\mathcal{N}) \setminus U$ such that w is embedded in V_2. Condition (7.59) then guarantees that the cut-vertices W_A neighboring w can be embedded in V_1.

We remark that there may be several internal shrubs extending from $u \in W_A$. However Claim 8.24.1 and the subsequent application of Lemma 8.5 allows a sequential embedding of these shrubs. This finishes the first stage of the embedding process.

For the second stage, i.e., the embedding the end shrubs of $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$, we first recall that the total order of end shrubs in $\mathcal{S}_A$ is at most $h_1 - \frac{\eta^2 k}{10^5}$, and the total order of end shrubs in $\mathcal{S}_A$ is at most $\frac{1}{2}\big(h_1 - \frac{\eta^2 k}{10^5}\big)$ by Definition 3.1(l). The embedding is a straightforward application of Lemma 8.21. □

The next lemma completely resolves Theorem 1.3 in the presence of Configuration ($\diamond\mathbf{10}$).

Lemma 8.25. *For every $\eta', d', \Omega > 0$ there exists $\tilde{\varepsilon} > 0$ such that for every $\nu' > 0$ there exists k_0 such that the following holds for each $k > k_0$. If G is a graph with Configuration ($\diamond\mathbf{10}$)$(\tilde{\varepsilon}, d', \nu' k, \Omega k, \eta')$ then $\mathbf{trees}(k) \subseteq G$.*

Lemma 8.25 was basically resolved in [PS12] and we do not give a proof here. See Section 8.1.5 for discussion.

9 Proof of Theorem 1.3

Suppose that $\alpha > 0$ is arbitrary. Let $\eta := \min\{\frac{1}{30}, \frac{\alpha}{2}\}$. We wish to fix further constants as in (7.3). A trouble is that we do not know the right choice of Ω^* and Ω^{**} yet. Therefore we take $g := \lfloor \frac{100}{\eta^2} \rfloor + 1$ and fix suitable constants

$$\eta \gg \frac{1}{\Omega_1} \gg \frac{1}{\Omega_2} \gg \ldots \gg \frac{1}{\Omega_{g+1}} \gg \rho \gg \gamma \gg d \geqslant \frac{1}{\Lambda}$$
$$\geqslant \varepsilon \geqslant \pi \geqslant \varepsilon_\odot \geqslant \alpha_\odot \geqslant \varepsilon' \geqslant \nu \gg \tau \gg \frac{1}{k_0} > 0\,,$$

where the "$\gg$" relation is dictated by the use of the lemmas below. In particular, this gives us a relation between between α and k_0.

Suppose now that $k > k_0$, and $G \in \mathbf{LKS}(n, k, \alpha)$ is a graph, and $T \in \mathbf{trees}(k)$ is a tree. It is our goal to show that $T \subseteq G$.

We follow the plan outlined in Figure 1.3. First, we process the tree T by considering any (τk)-fine partition $(W_A, W_B, \mathcal{S}_A, \mathcal{S}_B)$ of T rooted at an arbitrary root r. Such a partition exists by Lemma 3.4. Let m_1 and m_2 be the total order of internal and end shrubs, respectively. For $i = 1, 2$ set $\mathfrak{p}_i := \frac{\eta}{50} + \frac{m_i}{(1+\frac{\eta}{10})k}$, and $\mathfrak{p}_0 := 1 - (\mathfrak{p}_1 + \mathfrak{p}_2)$. We have $\mathfrak{p}_i \in (\frac{\eta}{100}, 1)$ for $i = 1, 2, 3$.

To find a suitable structure in the graph G we proceed as follows. We apply Lemma 4.14 with input graph $G_{\triangleright L4.14} := G$ and parameters $\eta_{\triangleright L4.14} := \alpha$, $\Lambda_{\triangleright L4.14} := \Lambda$, $\gamma_{\triangleright L4.14} := \gamma$, $\varepsilon_{\triangleright L4.14} := \varepsilon'$, $\rho_{\triangleright L4.14} := \rho$, and sequence $(\Omega_j)_{j=1}^{g+1}$. The lemma outputs a graph $G'_{\triangleright L4.14} \in \mathbf{LKSsmall}(n,k,\eta)$, and two numbers $\Omega^* = \Omega_i$ and $\Omega^{**} = \Omega_{i+1}$ for some $i \in [g]$. By abusing the of notation slightly, we call this graph still G. Taking $b_{\triangleright L4.14} := \frac{\rho k}{100\Omega^*}$, the second part of the lemma outputs a $(k,\Omega^{**},\Omega^*,\Lambda,\gamma,\varepsilon',\nu,\rho)$-sparse decomposition $\nabla = (\mathbf{\Psi},\mathbf{V},\mathcal{D},G_{\mathrm{reg}},G_{\mathrm{exp}},\mathfrak{A})$. Let $\mathfrak{c}$ be the size of an arbitrary cluster in $\mathbf{V}$.

We now apply Lemma 6.1 with parameters $\eta_{\triangleright L6.1} := \eta$, $\Omega_{\triangleright L6.1} := \Omega_{g+1}$, $\gamma_{\triangleright L6.1} := \gamma$, $\beta_{\triangleright L6.1} := d$, $\varepsilon_{\triangleright L6.1} := \varepsilon$, $\varepsilon'_{\triangleright L6.1} := \varepsilon'$, $\pi_{\triangleright L6.1} := \pi$, $\nu_{\triangleright L6.1} := \nu$ and $\Omega^*_{\triangleright L6.1} := \Omega^*$. Given the graph G with its sparse decomposition ∇ the lemma outputs three $(\varepsilon,d,\pi\mathfrak{c})$-semiregular matchings $\mathcal{M}_A$, $\mathcal{M}_B$, and $\mathcal{M}_{\mathrm{good}} \subseteq \mathcal{M}_A$ which fulfill the assertion either of case **(K1)**, or of **(K2)**. The matchings $\mathcal{M}_A$ and $\mathcal{M}_B$ also define the sets $\mathbb{XA}$ and $\mathbb{XB}$.

The additional features provided by Lemma 4.14 and Lemma 6.1 guarantee that we are in the situation described in Setting 7.4. We apply Lemma 7.3 as described in Definition 7.6; recall that the numbers $\mathfrak{p}_0,\mathfrak{p}_1,\mathfrak{p}_2$ are given by the ratios of types of shrubs in $(W_A,W_B,\mathcal{S}_A,\mathcal{S}_B)$. This puts us in the setting described in Setting 7.7. We now use Lemma 7.31 to obtain one of the following configurations.

- $(\diamond\mathbf{1})$,
- $(\diamond\mathbf{2})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}},\frac{\sqrt[4]{\Omega^{**}}}{2},\frac{\eta^9\rho^2}{128\cdot 10^{22}\cdot(\Omega^*)^5}\right)$,
- $(\diamond\mathbf{3})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}},\frac{\sqrt[4]{\Omega^{**}}}{2},\frac{\gamma}{2},\frac{\eta^9\gamma^2}{128\cdot 10^{22}\cdot(\Omega^*)^5}\right)$,
- $(\diamond\mathbf{4})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}},\frac{\sqrt[4]{\Omega^{**}}}{2},\frac{\gamma}{2},\frac{\eta^9\gamma^3}{384\cdot 10^{22}(\Omega^*)^5}\right)$,
- $(\diamond\mathbf{5})\left(\frac{\eta^{27}\Omega^{**}}{4\cdot 10^{66}(\Omega^*)^{11}},\frac{\sqrt[4]{\Omega^{**}}}{2},\frac{\eta^9}{128\cdot 10^{22}\cdot(\Omega^*)^3},\frac{\eta}{2},\frac{\eta^9}{128\cdot 10^{22}\cdot(\Omega^*)^4}\right)$,
- $(\diamond\mathbf{6})\big(\frac{\eta^3\rho^4}{10^{14}(\Omega^*)^4},4\varepsilon_{\odot},\frac{\gamma^3\rho}{32\Omega^*},\frac{\eta^2\nu}{2\cdot 10^4},\frac{3\eta^3}{2000},\mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{7})\big(\frac{\eta^3\gamma^3\rho}{10^{12}(\Omega^*)^4},\frac{\eta\gamma}{400},4\varepsilon_{\odot},\frac{\gamma^3\rho}{32\Omega^*},\frac{\eta^2\nu}{2\cdot 10^4},\frac{3\eta^3}{2\cdot 10^3},\mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{8})\big(\frac{\eta^4\gamma^4\rho}{10^{15}(\Omega^*)^5},\frac{\eta\gamma}{400},\frac{400\varepsilon}{\eta},4\varepsilon_{\odot},\frac{d}{2},\frac{\gamma^3\rho}{32\Omega^*},\frac{\eta\pi\mathfrak{c}}{200k},\frac{\eta^2\nu}{2\cdot 10^4},\mathfrak{p}_1(1+\frac{\eta}{20})k,\mathfrak{p}_2(1+\frac{\eta}{20})k\big)$,
- $(\diamond\mathbf{9})\big(\frac{\rho\eta^8}{10^{27}(\Omega^*)^3},\frac{2\eta^3}{10^3},\mathfrak{p}_1(1+\frac{\eta}{40})k,\mathfrak{p}_2(1+\frac{\eta}{20})k,\frac{400\varepsilon}{\eta},\frac{d}{2},\frac{\eta\pi\mathfrak{c}}{200k},4\varepsilon_{\odot},\frac{\gamma^3\rho}{32\Omega^*},\frac{\eta^2\nu}{2\cdot 10^4}\big)$,
- $(\diamond\mathbf{10})\big(\varepsilon,\frac{\gamma^2 d}{2},\pi\sqrt{\varepsilon'}\nu k,\frac{2(\Omega^*)^2 k}{\gamma^2},\frac{\eta}{40}\big)$

Depending on the actual configuration Lemma 8.15, Lemma 8.18, Lemma 8.23, Lemma 8.24, or Lemma 8.25 guarantee that $T \subseteq G$. This finishes the proof of the theorem.

10 Concluding remarks

10.1 Theorem 1.3 algorithmically

We now discuss the algorithmic aspects of our proof. That is, we would like to find an algorithm which finds a copy of a given tree $T \in \mathbf{trees}(k)$ in any given graph $G \in \mathbf{LKS}(n,k,\alpha)$ in time $O(n^C)$. Here the degree C of the polynomial is allowed to depend on α, but not on k. It can be verified that each of the steps of our proof — except the extraction of dense spots (cf. Section 4.7) — can be turned into a polynomial time algorithm. The two randomized steps — random splitting in Section 7.2 and the use of the stochastic process Duplicate in Section 8 — can be also efficiently derandomized using a standard technique for derandomizing the Chernoff bound. Let us sketch how to deal with extracting dense spots.

The idea is as follows. Initially, we pretend that G_{exp} consists of the entire bounded-degree part $G - \mathbf{\Psi}$ (cleaned for minimum degree ρk as in (4.8)). With such a supposed sparse classification ∇_1 we go through Lemma 6.1 and Lemma 7.31 (which builds on Lemmas 7.32, 7.33, and 7.34) to obtain a configuration. We now start embedding T as in Section 8. Note that G_{reg} and $\mathfrak{A}$ are absent, and so, the only embedding techniques are those involving $\mathbf{\Psi}$ and G_{exp}. Now, either we embed T, or we fail. The only possible reason for the failure is that we were unable to perform the one-step look-ahead strategy described in Section 4.5 because G_{exp} was not really nowhere-dense. But then we actually localized a dense spot D_1. We get an updated supposed sparse classification ∇_2 in which D_1 is removed from G_{exp} and put in $\mathcal{D}$ (which of course can give rise to G_{reg} or $\mathfrak{A}$). We keep iterating. Since in each step we extract at least $O(k^2)$ edges we iterate the above at most $e(G)/\Theta(k^2) = O(\frac{n}{k})$ times. We are certain to succeed eventually, since after $\Theta(\frac{n}{k})$ iterations we get an honest sparse classification.

It seems that this iterative method is generally applicable for problems which employ a sparse classification.

10.2 Strengthenings of Theorem 1.3

It would be possible to strengthen Theorem 1.3 with not too much extra effort (say 3 additional pages) by removing the approximation concerning the number of large vertices. Actually, having approximation on the degrees, one can even prove the theorem with negative approximation on the number of large vertices, in the following form.

Theorem 10.1. *For every $\alpha > 0$ there exists k_0 such that for any $k > k_0$ we have the following. Each n-vertex graph with at least $(\frac{1}{2} - \frac{\alpha}{100})n$ vertices of degree at least $(1+\alpha)k$ contains each tree of order k.*

Of course, the term $\frac{\alpha}{100}$ is not optimal, but this is not the point.

To prove Theorem 10.1 the only thing which has to be done — apart from obvious notational changes to the classes $\mathbf{LKS}(n,k,\eta)$, $\mathbf{LKSmin}(n,k,\eta)$, $\mathbf{LKSsmall}(n,k,\eta)$ — is to strengthen Lemma 6.1. An appropriately changed Lemma 6.1 can still provide one of the structures **(K1)** or **(K2)** under the weakened hypothesis. The subsequent steps of the proof then do not have to be modified at all. More importantly, it seems that stability type arguments — even though quite subtle in the sparse setting — will lead to a full resolution of Conjecture 1.2 for all k bigger than an absolute constant.

Chapter III

Conclusion

We presented a solution of a weaker version of the Loebl-Komlós-Sós Conjecture. The central tool was a certain graph decomposition which extends the Szemerédi Regularity Lemma. Much of this method was developed by Ajtai, Komlós, Simonovits and Szemerédi during their work on the Erdős-Sós Conjecture, a work which started in the early 1990's but is unpublished as of now. We enhanced this method by several further steps and our resolution of the weak Loebl-Komlós-Sós Conjecture indicates that this method can be used in some generality. As indicated in Section II.10 it seems that the ideas of the Stability Method of Simonovits [Sim68] are compatible with the sparse decomposition,[i] and can for example lead to an exact solution of the Loebl-Komlós-Sós Conjecture for all k sufficiently large. It would be of most interest to see what other tree-embedding problems (and perhaps other) can be attacked using this approach. A most tantalizing prospect is the applicability of a similar decomposition to other combinatorial structures such as directed graphs or uniform hypergraphs.

It seems that the method we used brings certain tedious details to be taken care of. Indeed the resulting 140+ pages to prove Theorem II.1.3 is quite a jump from some 20 pages Piguet and Stein [PS12] needed to prove the dense counterpart, Theorem II.1.5. Of course, this is given by the complexity of the sparse decomposition. However, there is always a hope when a new method appears that further tools will be developed and the method will eventually be simplified.[ii] In the context of the original Regularity Lemma, an example of such a tool certainly was the Blow-up lemma.

[i] even though an application of the Stability Method seems highly non-trivial

[ii] Very recent results about graph minors may serve as an encouragement. Kawarabayashi, Thomas, and Wollan [KTW12] reproved one of the core results of the Graph Minor Project mentioned in Chapter I as an example of a work of extraordinary length and depth, the "Weak Structure Theorem", in a much shorter way. Further, they write that they believe they can substantially simplify even the proof of its extension, the "Excluded Clique Minor Theorem", which is one of the most applicable outcomes of the project.

The decomposition lemma, Lemma II.4.13 as given here is general enough, and keenly awaiting to be recycled in other applications. Another example which may be readily used elsewhere (and feels to be needed) is the technique of augmenting a matching in Section II.5. The real challenge however would be to formulate a general statement that would incorporate the cleaning lemmas (in Section II.7.6) and various tree-embedding techniques (in Section II.8). Such a statement would in the ideal situation allow one to conclude containment of certain trees already after a "rough structural result", such as that given in Lemma II.6.1. Such a result would reduce problems employing the sparse decomposition really to the essence, which is finding a suitable rough structure in the sparse decomposition. However it is too early to call for such a metastatement as we need to see other applications of the method. Only then will we be able to capture the needs for a wide range of settings.

Another big question is whether there is no alternative approach which would avoid the notion of sparse decomposition, and indeed the notion of regular pairs. Such a programme has been being developed in the dense setting by Szemerédi and his collaborators, see [LSS10] for a particular instance of "deregularizing" a result originally resolved [KSS96] using the Regularity Method. However this programme has not given a general alternative view on such problems, as of yet.

Symbol index

General index

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