

Inhomogeneous random 2SAT

arXiv:2510.17656

&

Digraphons: connectivity and spectral aspects

arXiv:2510.16839

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Slides+this video: <https://www.cs.cas.cz/~hladky>

Highlights: Inhomogeneous random 2SAT

- (The usual/homogeneous) random k -SAT
model of random Boolean formulas
 - $k > 2$: probability theory, statistical mechanics
 - $k = 2$: solved in 1980s, random digraphs
- **Inhomogeneous random 2SAT**
 - Formulated using kernels/graphons
in the spirit of Bollobás-Janson-Riordan
 - Immediately leads to **di**graphons...
 - ...and to functional analysis //
// spectral theory of **non**-self-adjoint operators

Overview

1) Preparations

- a) Sparse Erdős–Rényi random graphs and giant component
- b) Bollobás–Janson–Riordan random graphs
- c) Spectral properties of graphons
- d) Spectral properties of digraphons

2) Random k -SAT

3) Inhomogeneous random 2SAT, our result, proof

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Erdős–Rényi random graphs

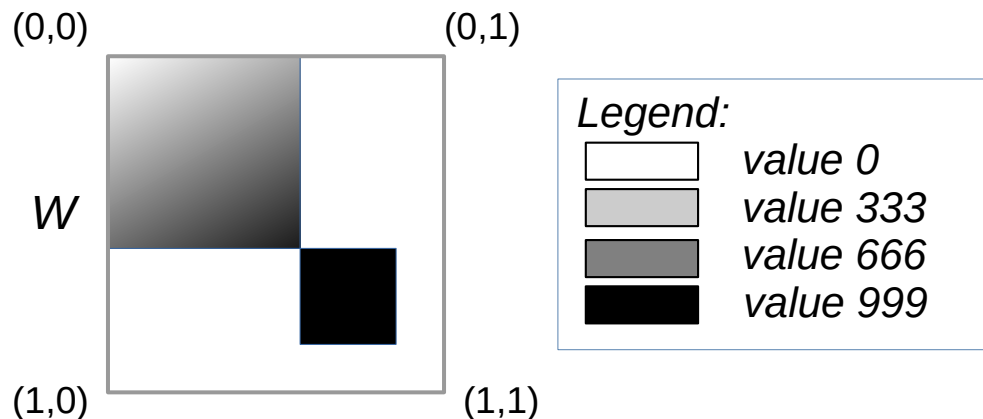
- Erdős–Rényi // Gilbert 1959
- $\mathbf{G}(n,p)$ n fixed vertices $\{1,2,\dots,n\}$
 each pair forms an edge with prob p
- **Phase Transition of the Giant Component (Erdős–Rényi 1959)**
 - For $c < 1$, $\mathbf{G}(n,c/n)$ has largest component of order $\Theta(\log n)$. subcriticality
 - For $c > 1$, $\mathbf{G}(n,c/n)$ has largest component of order $(s(c) \pm o(1))n$. Here, $s(c) \in (0,1)$. supercriticality

Inhomogeneous random graphs

Bollobás-Janson-Riordan 2005

subsequent work by B., Borgs., Chayes, J., R.

- Main idea $\mathbf{G}(n, c/n) \rightsquigarrow \mathbf{G}(n, W / n)$, where $W: [0,1]^2 \rightarrow [0,999]$ is a “**symmetric** kernel”=“graphon”.

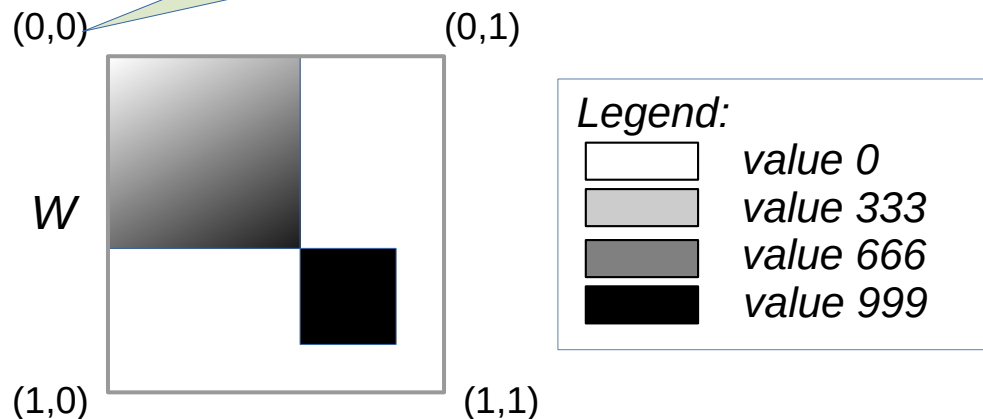


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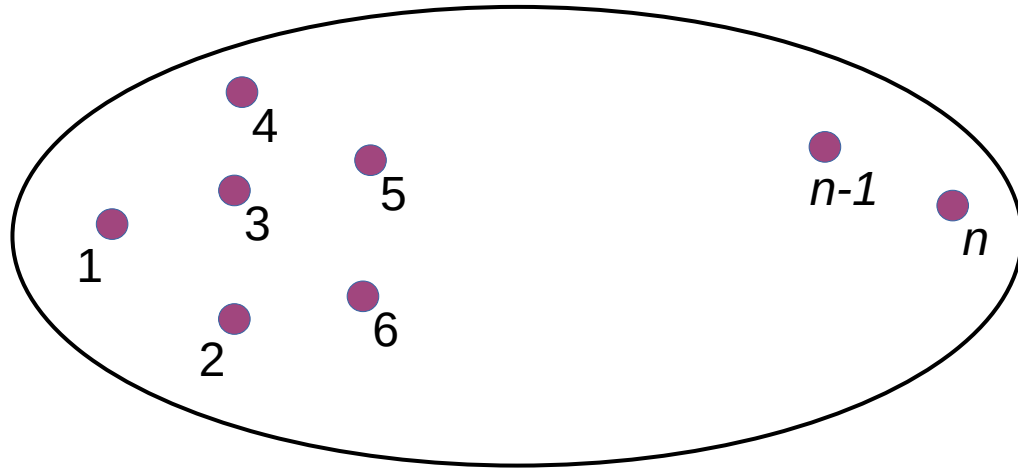
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- Main “matrix orientation of the plane” where $W: [0,1]^2 \rightarrow [0,999]$ is a “symmetric kernel”=“graphon”.

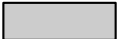


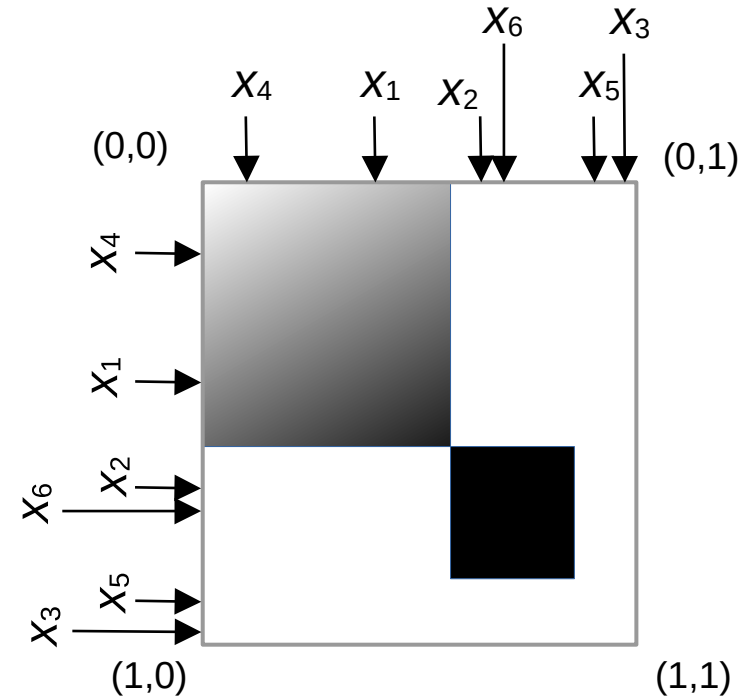
Generating $G(n, W / n)$

- Vertex set $\{1, 2, \dots, n\}$
- Generate $x_1, x_2, \dots, x_n \in [0, 1]$ at random



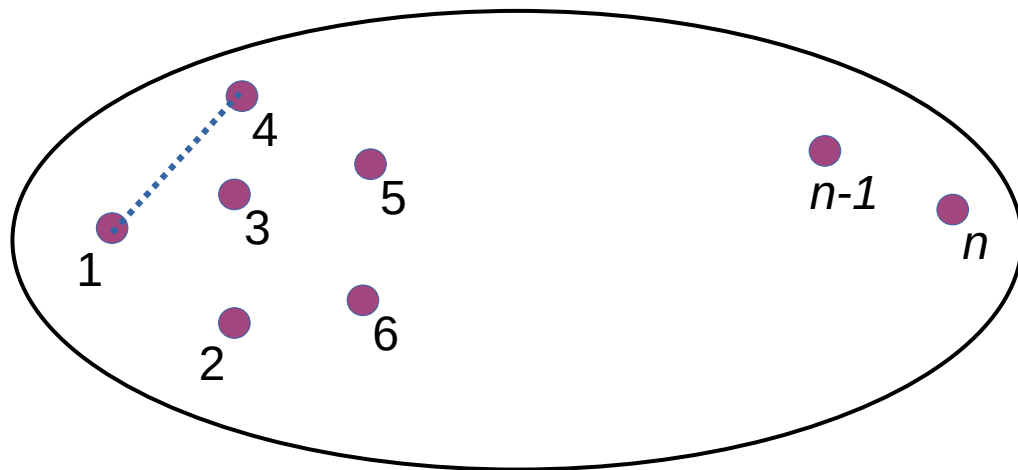
Legend:

	value 0
	value 333
	value 666
	value 999

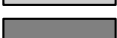


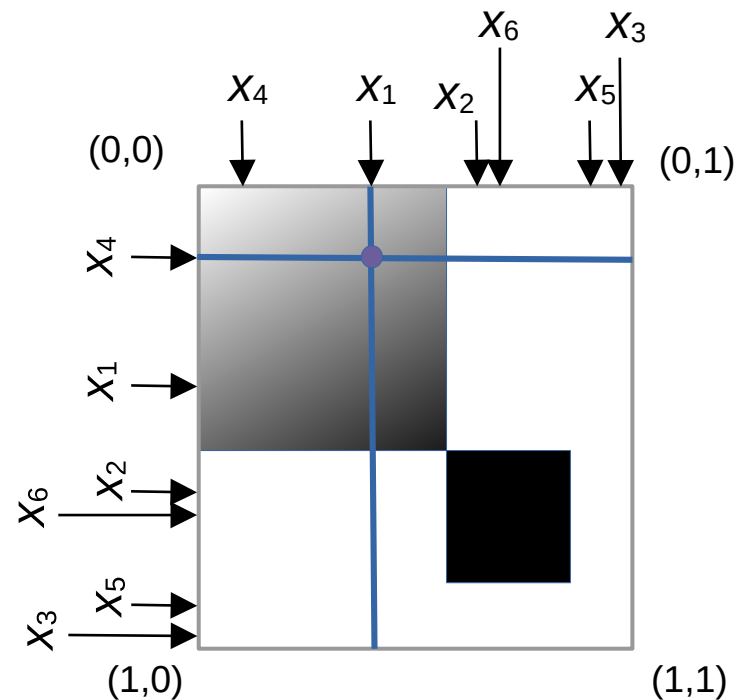
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- Make $\{i, j\}$ an edge with probability $W(x_i, x_j)/n$



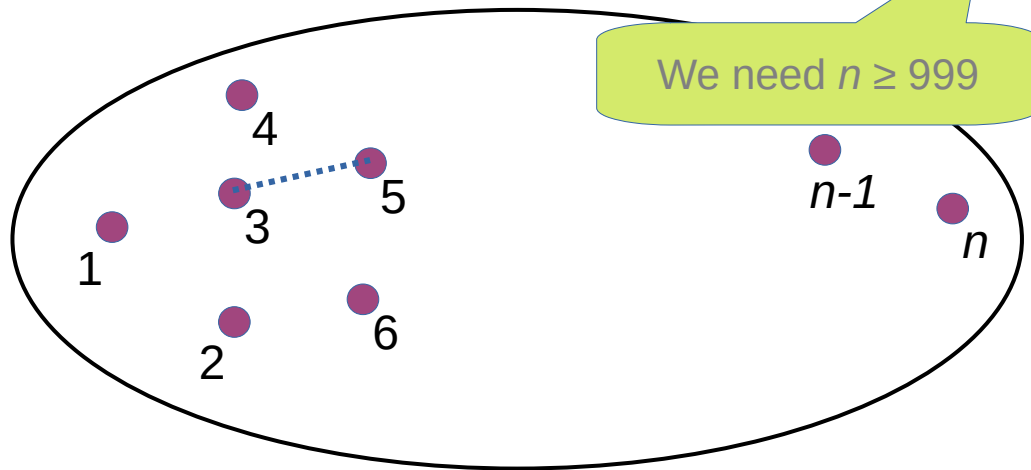
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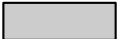


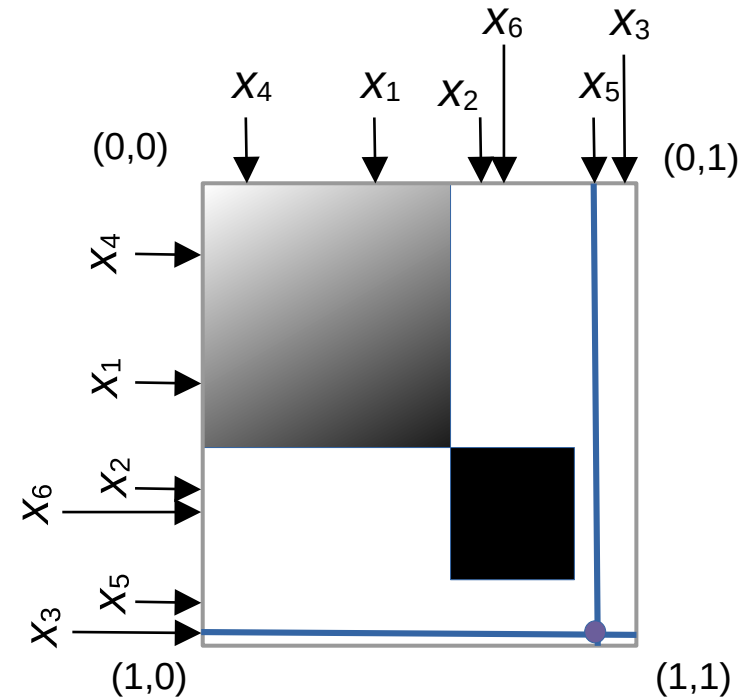
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Giant components in $G(n, c/n)$

Erdős–Rényi 1959

The largest component
in $G(n, c/n)$

$$= \begin{cases} o(n) & \text{if } c < 1 \\ \Theta(n) & \text{if } c > 1 \end{cases}$$

Giant components in $G(n, W/n)$

Bollobás-Janson-Riordan 2005 &
Bollobás-Borgs-Chayes-Riordan 2010

$$W \in L_{\text{symm}}^{\infty}([0,1]^2)$$

The largest component
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$$= \begin{cases} o(n) & \text{if } \rho(W) < 1 \\ \Theta(n) & \text{if } \rho(W) > 1 \end{cases}$$

$\rho(W)$...spectral radius of W

Inhomogeneous random digraphs $G(n, U/n)$

Cao, Olvera-Cravioto 2020

$W \in L_{\text{symm}}^{\infty}([0,1]^2)$ + **continuous**

The largest component
in $\mathbf{G}(n, W/n)$ $= \begin{cases} o(n) & \text{if } \rho(W) < 1 \\ \Theta(n) & \text{if } \rho(W) > 1 \end{cases}$

$\rho(W)$...spectral radius of W

Spectral properties of graphons

Setting $W \in L^\infty_{\text{symm}}([0,1]^2)$ “integral kernel operator”

For $f \in L^2([0,1])$, define $g:=Wf$, where $g \in L^2([0,1])$

$$g(x)=\int_z W(x,z)f(z) \, d(z)$$

Number $\lambda \in \mathbb{R}$ is an **eigenvalue** for eigenfunction $f \neq 0$ if

$$\lambda f=Wf$$

Spectrum is the set of all eigenvalues

Spectral radius $\rho(W)=\max$ eigenvalue (in absolute value)

Spectral properties of graphons

Setting $W \in \mathcal{L}_{\text{symm}}^{\infty}([0,1]^2)$ “integral kernel operator”

For $f \in L^2([0,1])$, $df \in L^2([0,1])$

Symmetric (self-adjoint) integral operators on a Hilbert space have real ($\mathbb{R}$) eigenvalues and a complete orthonormal set of eigenfunctions.

Number $\lambda \in \mathbb{R}$ is eigenvalue if $f \neq 0$ if

Spectrum is the set of all eigenvalues

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Spectral properties of graphons

Setting $W \in L_{\text{symm}}^{\infty}([0,1]^2)$

Spectral theorem (compact + self-adjoint)

For a given graphon W , there exist an at most countable set $\{\lambda_i\}$ of (real) eigenvalues and corresponding orthonormal eigenfunctions $\{f_i \in L^2([0,1])\}$ so that

$$W(x,y) = \sum_i \lambda_i f_i(x) f_i(y)$$

For powers: $W^n(x,y) = \sum_i \lambda_i^n f_i(x) f_i(y)$

Consequently, homomorphism density of cycles: $t(C_k, W) = \sum_i \lambda_i^k$

Spectral properties of digraphons

Setting $W \in \mathbb{L}^\infty([0,1]^2)$

Functional analysis of non-selfadjoint operators:

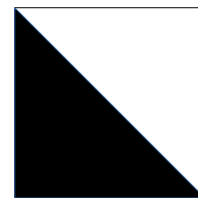
Number $\lambda \in \mathbb{C}$

-**eigenvalue**: $W - \lambda \cdot \text{Id}$ is noninjective on $L^2([0,1])$

-**point of spectrum**: $W - \lambda \cdot \text{Id}$ is not invertible as bounded operator on $L^2([0,1])$

Spectral radius $\rho(W)$ = supremum of moduli over spectrum

No spectral decomposition ~~$W(x,y) = \sum \lambda_i f_i(x) f_i(y)$~~



Volterra operator
 $W(x,y) = \mathbf{1}_{x < y}$

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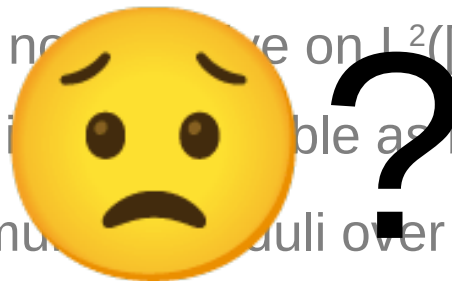
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Spectral properties of digraphons

To the rescue: theory of Banach lattices

digraphons are **nonnegative operators** (if $f \geq g$, then $Wf \geq Wg$)

Spectral radius $\rho(W)$ = supremum of moduli over spectrum

[by **Krein-Rutman**] = maximum eigenvalue

“**Gelfand’s formula**”, **H.-Savický**:

as $n \rightarrow \infty$, $W^n(x,y) = \rho(W)^n f_{\text{left}}(x) f_{\text{right}}(y) + o(\rho(W)^n)$,

where f_{left} is the principal eigenvalue,
and f_{right} is the principal eigenvalue
of the adjoint.

Finite/matrix counterpart to **Krein-Rutman** is **Perron-Frobenius**:

*The adjacency matrix of directed graph...
or*

The transition matrix of a Markov chain...

...has an eigenvector with positive entries (irreducibility, aperiodicity), and the corresponding eigenvalue is positive and larger than any other eigenvalue (even in absolute values).

Spectral properties of digraphons

To the rescue: theory of Banach lattices

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Grebík-Kráľ-Liu-Pikhurko-Slipantschuk: $t(C_k, W) = \sum_i m(\lambda_i) \lambda_i^k$

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Random k -SAT, history

- k -SAT (satisfiability):
 $k=2,3,\dots$
Boolean formula in conjunctive normal form, clauses consist k literals
$$(\neg X_1 \vee X_2 \vee X_3) \wedge (X_1 \vee \neg X_3 \vee X_4) \wedge (\neg X_2 \vee \neg X_4 \vee X_5)$$
- Cook 1971 “The Complexity of Theorem-Proving Procedures”
SAT as the first NP complete problem
- Karp 1972 “Reducibility Among Combinatorial Problems”
3SAT

Random k -SAT, history

- Random k -SAT(n, m); $k=2,3,4,\dots$
 n = number of variables (+ n negations)
 m = number of clauses, each selected at random
- Usually, linear scaling $m = \alpha n$
- **The satisfiability conjecture**
For every k there exists $\alpha_k \in (0, \infty)$:
 - If $\alpha < \alpha_k$ then Random- k -SAT($n, \alpha n$) is a.a.s. satisfiable.
 - If $\alpha > \alpha_k$ then Random- k -SAT($n, \alpha n$) is a.a.s. unsatisfiable.

out of $\approx \binom{2n}{k}$

Random k -SAT

- Random k -SAT(n, m); $k=2, 3, 4, \dots$
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- Chvátal, Reed // Goerdts 1992:
True for $k=2$, $\alpha_2=1$.

- Friedgut 1995: sharp thresholds

- Mertens, Mezard, Zecchina 2006
Coja-Oghlan, Panagiotou 2016

...

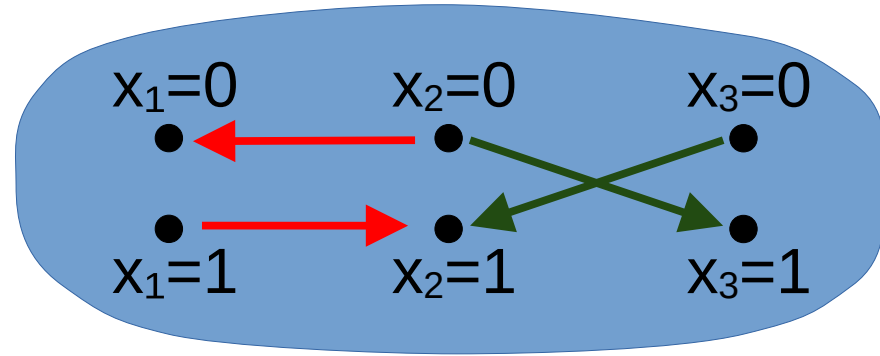
statistical physics//cavity method
 $\alpha_k \approx 2^k \ln(2) - 0.5 - 0.5 \ln(2) + o_k(1)$

- Ding, Sly, Sun 2022, Ann Math, ≈ 400 pages
Proof for k large enough.

Random 2-SAT

- Chvátal, Reed // Goerdt 1992:
 - If $\alpha < 1$ then Random-2SAT($n, \alpha n$) is a.a.s. satisfiable.
 - If $\alpha > 1$ then Random-2SAT($n, \alpha n$) is a.a.s. unsatisfiable.
- Proof: Implication digraphs

$$(\neg x_1 \vee x_2) \wedge (x_2 \vee x_3)$$



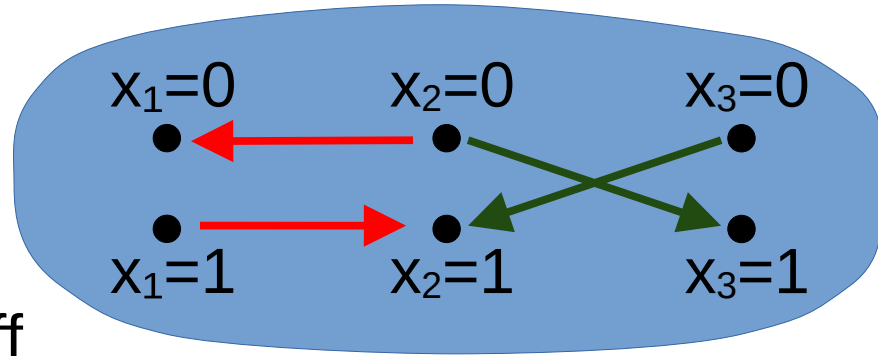
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Lemma:

A 2SAT formula is satisfiable iff
its implication digraph contains no “contradictory cycle”
(directed cycle through $x_i=0$ and $x_i=1$)



Random 2-SAT

- Chvátal, Reed // Goerdts 1992:
 - If $\alpha < 1$ then $2SAT(n, \alpha n)$ is a.a.s. satisfiable.
 - If $\alpha > 1$ then $2SAT(n, \alpha n)$ is a.a.s. unsatisfiable.
- Proof:

So, does the implication digraph of $2SAT(n, \alpha n)$ contain a contradictory cycle?

$2n$ vertices

up to $(2n)^2$ directed edges

αn clauses $\rightarrow 2\alpha n$ directed edges

edge density $\frac{2\alpha n}{(2n)^2} = \frac{\alpha}{2n} = \frac{\alpha}{\text{vertices}}$

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Approximately a random
directed graph with edge
probability $\alpha/\text{vertices}$

- Proof:

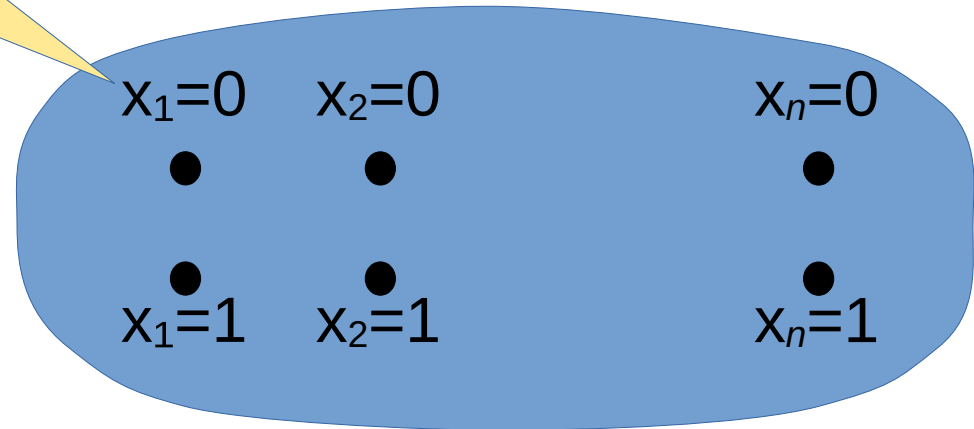
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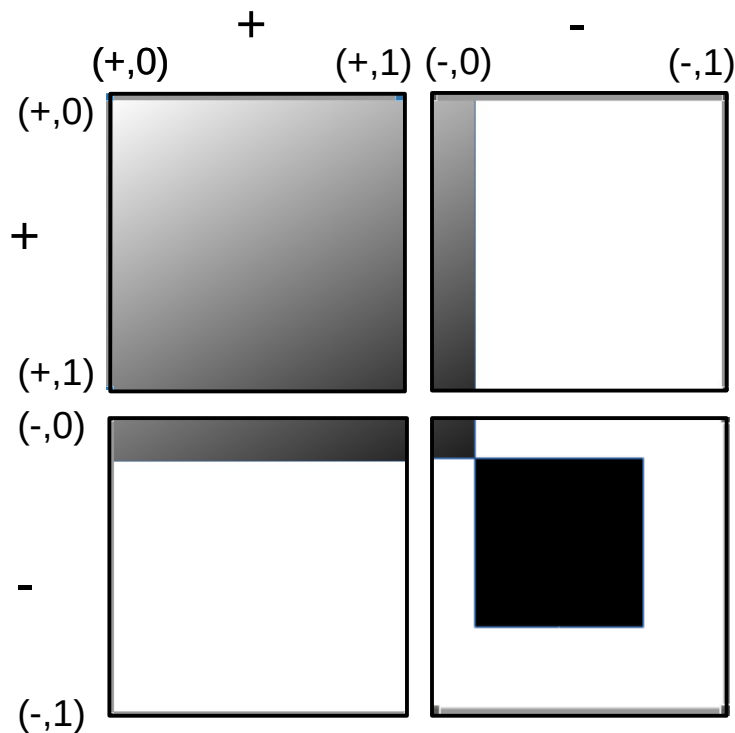
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Inhomogeneous 2SAT(n, W)

- Setting $W \in \mathbb{L}_{\text{symm}}^{\infty}(\{\{+, -\} \times [0, 1]\}^2)$

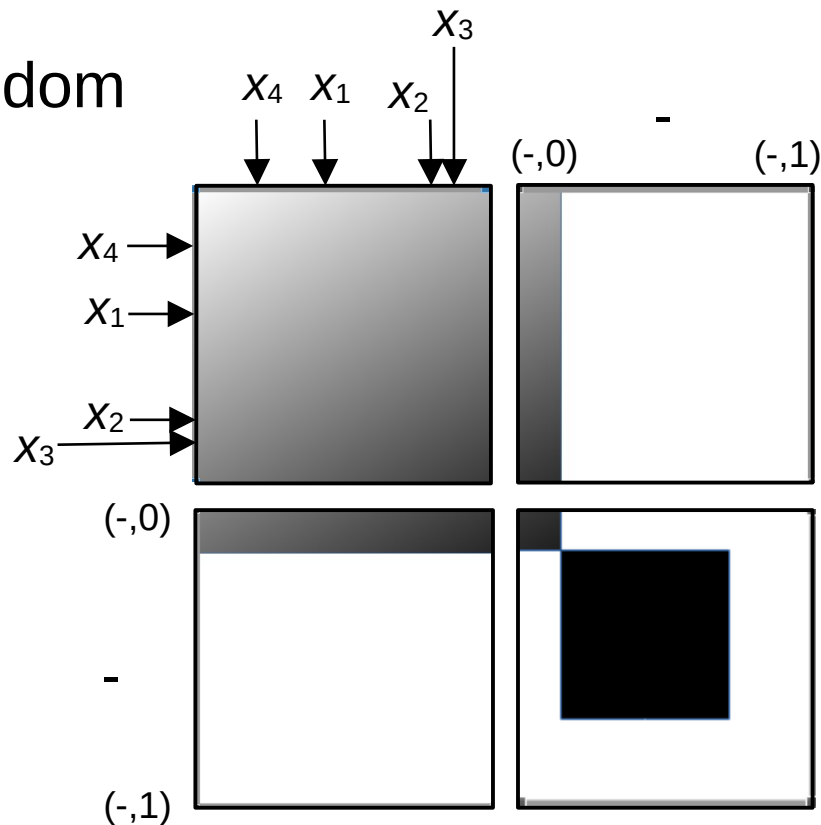
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	value 2
	value 3



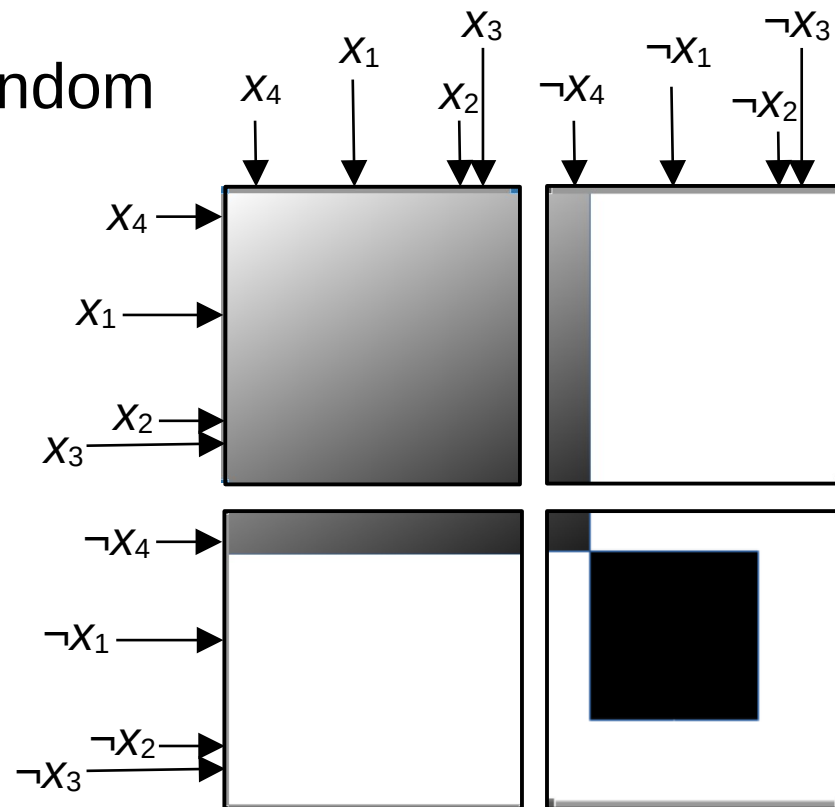
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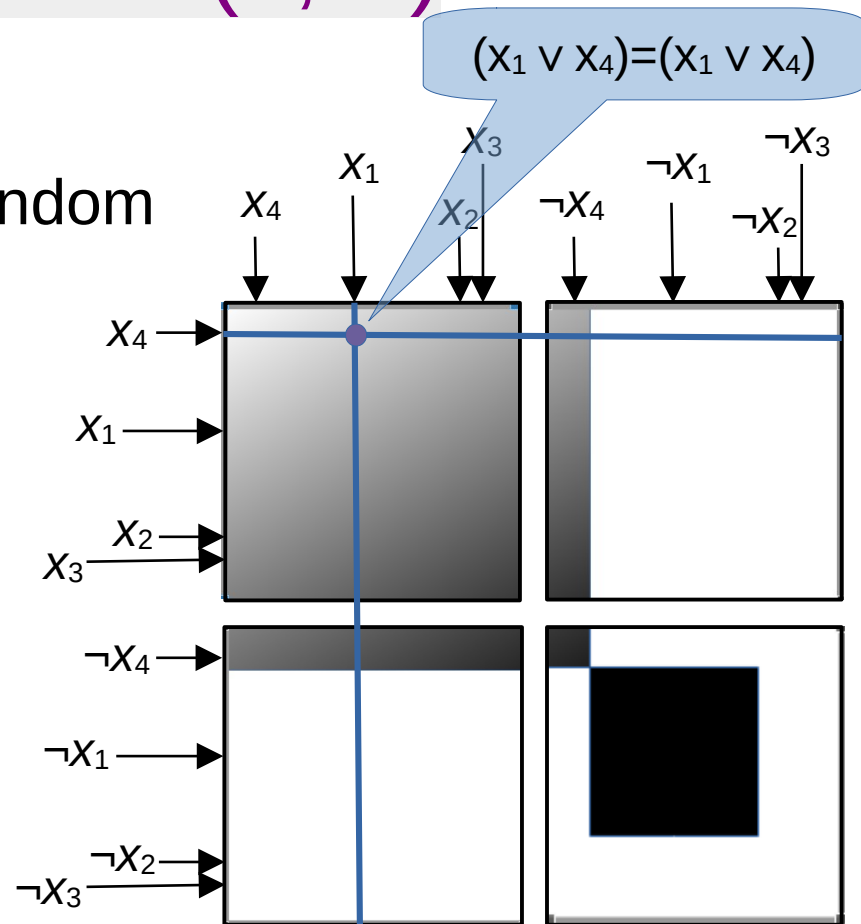
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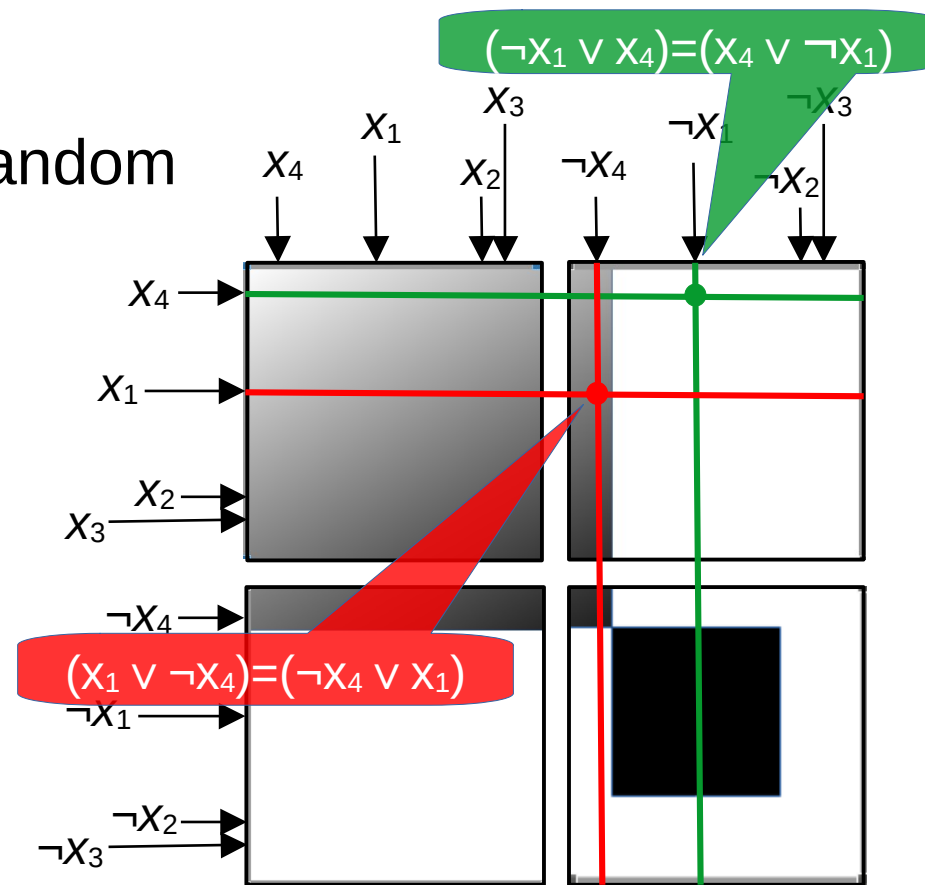
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- The random 2SAT formula:
 - Insert each clause $(x_i \vee x_j)$ with prob = $\frac{W(x_i, x_j)}{2n}$



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When $\{+, -\} \times [0, 1]$ is viewed as a probability space

Observation 1: Number of clauses $\approx \|W\|_1 n$

Observation 2: When $W = \text{constant-}\alpha$ then $2\text{SAT}(n, W) = 2\text{SAT}(n, \alpha)$

Main Theorem

Let $W \in L_{\text{symm}}^{\infty}(\{\{+, -\} \times [0, 1]^2\})$

- If $\rho^*(W) < 1$ then $2\text{SAT}(n, W)$ is a.a.s. satisfiable.
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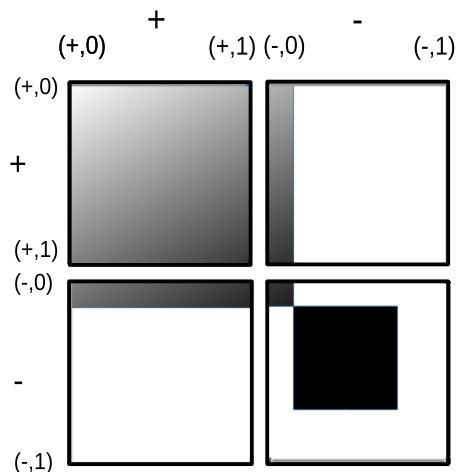
$\rho^*(W)$...maximum over all strong components that are contradictory of spectral radius of the “implication digraphon corresponding to W ”

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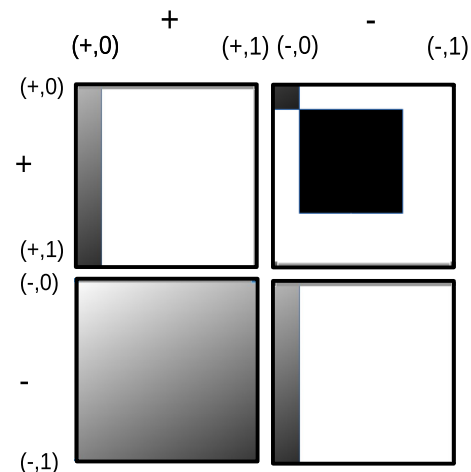
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W



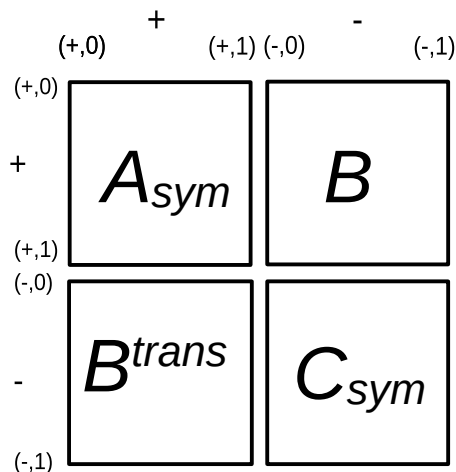
Implication digraphon of W

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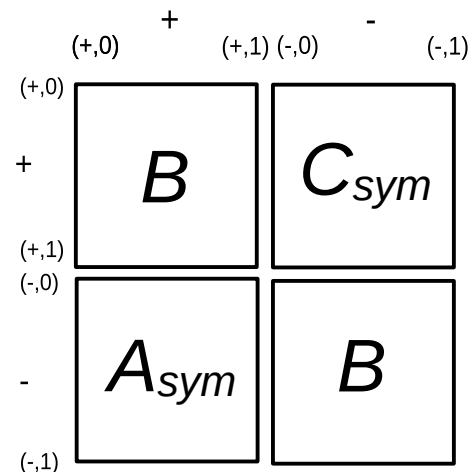
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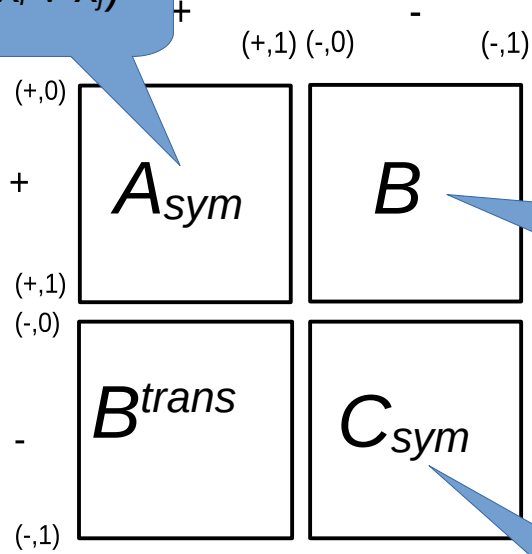


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Implication digraphon

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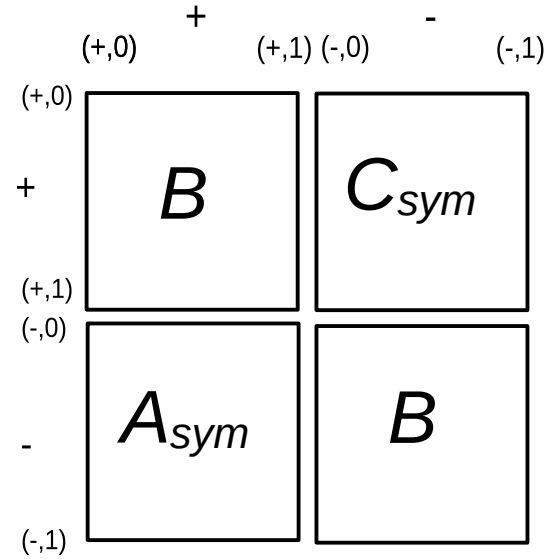
$(x_i \vee x_j)$



$(x_i \vee \neg x_j)$

$(\neg x_i \vee \neg x_j)$

Implication digraphon of W



Implication digraphon

W

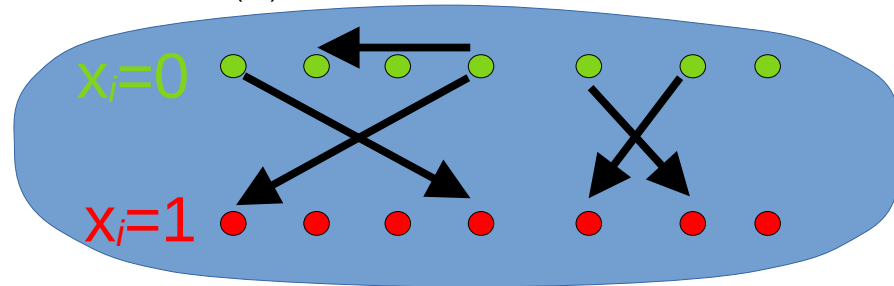
			+			-	
	(+,0)	(+,1)	(-,0)		(-,1)		
+	999	999					
-	999	0					

$(X_1 \vee X_2) \wedge (X_1 \vee X_3) \wedge (X_2 \vee X_3) \wedge \dots$
 $\dots \wedge (\neg X_1 \vee X_2) \wedge (X_1 \vee \neg X_2) \wedge (\neg X_2 \vee X_3) \wedge$
 $\dots$

Satisfiable! $x_1=x_2=\dots=1$

Implication digraphon of W

			+			-	
	(+,0)	(+,1)	(-,0)		(-,1)		
+	999	0					
-	999	999					



Implication digraphon

W

	+	-
(+,0)	999	999
(+,1)	999	999
(-,0)	999	0
(-,1)	999	0

$\rho^*(W)$...maximum over all strong components that are contradictory of spectral radius of the "implication digraphon corresponding to W "

Implication digraphon of W

	+	-
(+,0)	999	0
(+,1)	999	999
(-,0)	999	999
(-,1)	999	999

$(X_1 \vee X_2) \wedge (X_1 \vee X_3) \wedge (X_2 \vee X_3) \wedge \dots$
 $\dots \wedge (\neg X_1 \vee X_2) \wedge (X_1 \vee \neg X_2) \wedge (\neg X_2 \vee X_3) \wedge$
 $\dots$

Satisfiable! $x_1=x_2=\dots=1$

Implication digraphon

W

	+	-
(+,0)	999	999
(-,0)	999	0
(+,1)		
(-,1)		

$\rho^*(W)$...maximum over all strong components that are contradictory of spectral radius of the "implication digraphon corresponding to W "
 $\rho^*(W)=0$

Implication digraphon of W

	+	-
(+,0)	999	0
(-,0)	999	999
(+,1)		
(-,1)		

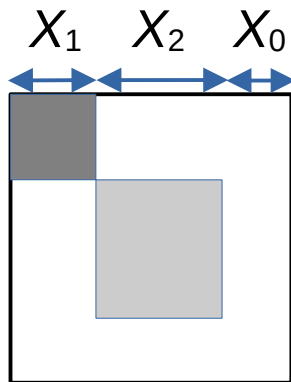
$(X_1 \vee X_2) \wedge (X_1 \vee X_3) \wedge (X_2 \vee X_3) \wedge \dots$
 $\dots \wedge (\neg X_1 \vee X_2) \wedge (X_1 \vee \neg X_2) \wedge (\neg X_2 \vee X_3) \wedge$
 $\dots$

Satisfiable! $x_1=x_2=\dots=1$

Components in graphons and digraphons

Graphons (Janson 2008)

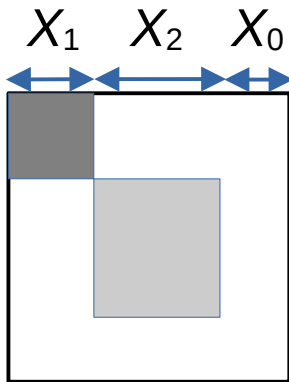
- Set $X \subseteq [0,1]$ in a graphon W is **disconnected** if $\exists Y \subseteq X$ “proper” so that $\int_{Y \times (X \setminus Y)} W = 0$.
- **Connected components**
- **Thm:** Every graphon admits unique decomposition into components $X_1, X_2, \dots$ and zero set X_0 .



Components in graphons and digraphons

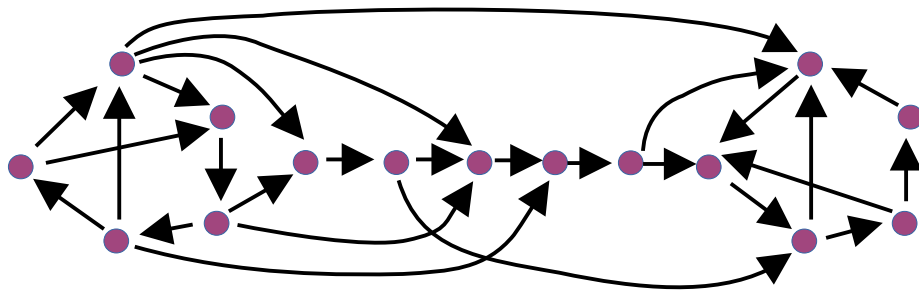
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Digraphs

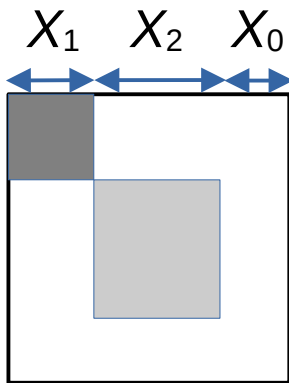
- Strong components**



Components in graphons and digraphons

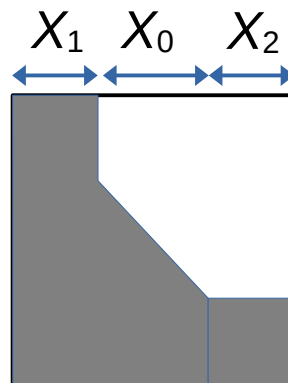
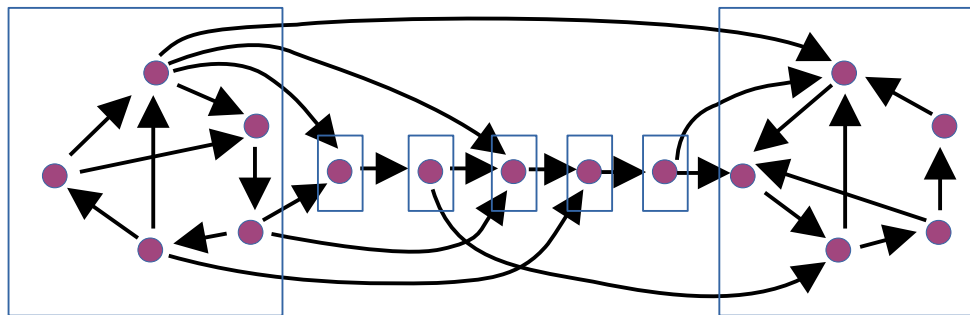
Graphons (Janson 2008)

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Digraphs

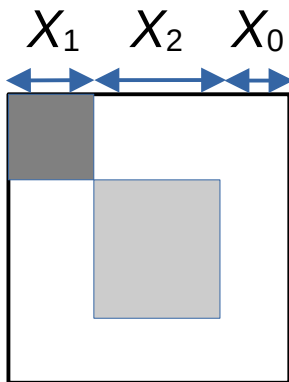
- Strong components**



Components in graphons and digraphons

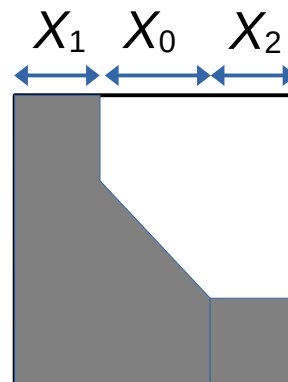
Graphons (Janson 2008)

- Set $X \subseteq [0,1]$ in a graphon W is **disconnected** if $\exists Y \subseteq X$ “proper” so that $\int_{Y \times (X \setminus Y)} W = 0$.
- Connected components**
- Thm:** Every graphon admits unique decomposition into components $X_1, X_2, \dots$ and zero set X_0 .



Digraphons (H.-Savický)

- Set $X \subseteq [0,1]$ in a digraphon D is **not strongly connected** if $\exists Y \subseteq X$ “proper” so that $\int_{Y \times (X \setminus Y)} D = 0$.
- Strong components**
- Thm:** Every digraphon admits unique decomposition into strong components $X_1, X_2, \dots$ and zero set X_0 .



Implication digraphon

W

			+				-		
	(+,0)		(+,1)	(-,0)			(-,1)		
+	999		999						
	(+,1)								
	(-,0)								
-	999		0						
	(-,1)								

Implication digraphon of W

			+				-		
	(+,0)		(+,1)	(-,0)			(-,1)		
+	999		0						
	(+,1)								
	(-,0)								
-	999		999						
	(-,1)								

$$(X_1 \vee X_2) \wedge (X_1 \vee X_3) \wedge (X_2 \vee X_3) \wedge \dots$$

$$\dots \wedge (\neg X_1 \vee X_2) \wedge (X_1 \vee \neg X_2) \wedge (\neg X_2 \vee X_3) \wedge \dots$$

Satisfiable! $x_1=x_2=\dots=1$

Implication digraphon

W

	+		-	
	(+,0)	(+,1)	(-,0)	(-,1)
+	999	999		
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$(X_1 \vee X_2) \wedge (X_1 \vee X_3) \wedge (X_2 \vee X_3) \wedge \dots$
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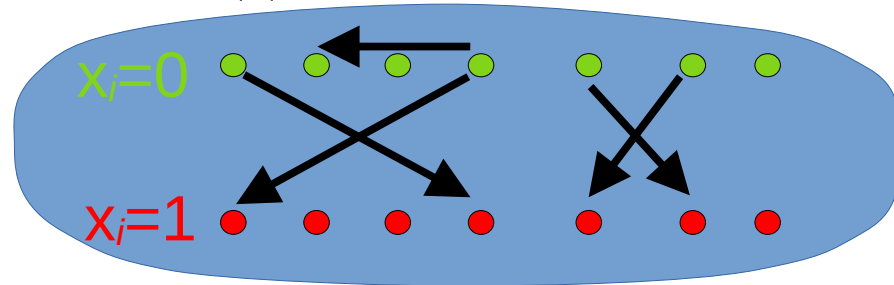
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Implication digraphon of W

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	(+,0)	(+,1)	(-,0)	(-,1)
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Strong component

Strong component

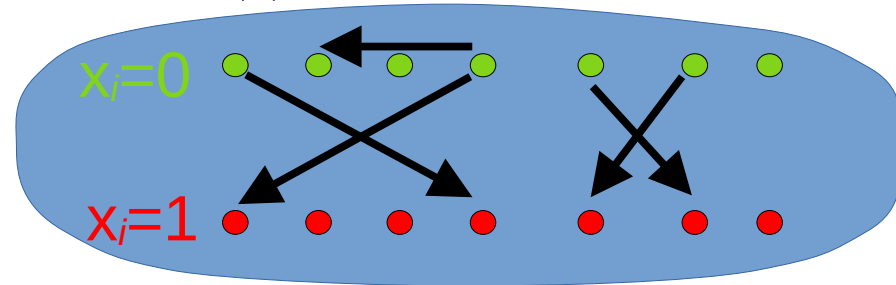
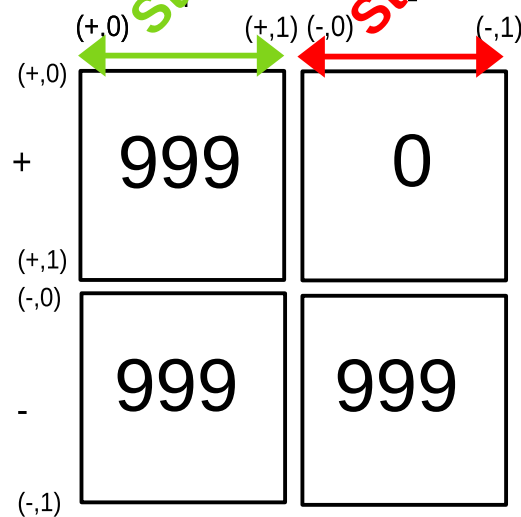


Implication digraphon

Main theorem with details

- $2SAT(n, W)$
- $D := \text{implication digraphon of } W$
- Decompose D into strong components $\{X_i\}_{i \in I}$ and zero set X_0 .
- A strong component X_i is **contradictory** if $\lambda(\{x \in [0, 1] : (+, x) \in X_i \text{ and } (-, x) \in X_i\}) > 0$.
- $J \subseteq I \dots$ indices of contradictory components
- $\rho^*(W) := \max\{\rho(D[X_i]) : i \in J\}$
 - If $\rho^*(W) < 1$ then $2SAT(n, W)$ is a.a.s. satisfiable.
 - If $\rho^*(W) > 1$ then $2SAT(n, W)$ is a.a.s. unsatisfiable.

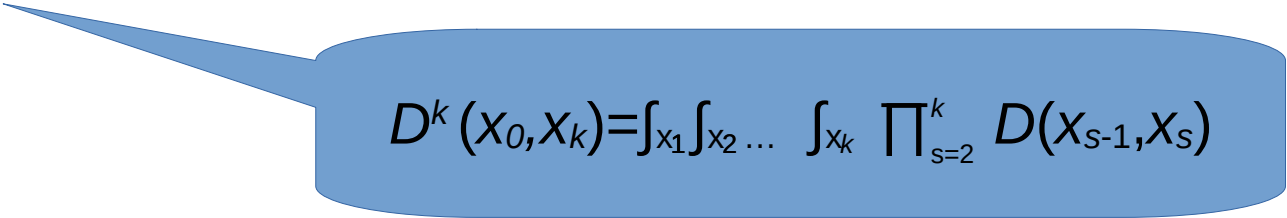
Implication digraphon of W



Spectral properties of digraphons

Our “Gelfand’s formula”,

For k large, $D^k(x, y) \approx \rho(D)^k f_{\text{left}}(x) f_{\text{right}}(y)$.


$$D^k(x_0, x_k) = \int_{x_1} \int_{x_2} \dots \int_{x_k} \prod_{s=2}^k D(x_{s-1}, x_s)$$

Contradictory cycles and spectral radius

Digraphon $D \in L^\infty((\{+,-\} \times [0,1])^2)$

Directed edge from x_i to $\neg x_j$ with prob $\frac{D(x_i, \neg x_j)}{n}$

Generate from D random digraph H on $x_1, \neg x_1, x_2, \neg x_2, \dots, x_n, \neg x_n$

Take $k, h \in \mathbb{N}$ large

Probability that $(y, x_1, \dots, x_k, \neg y, x_{k+1}, x_{k+2}, \dots, x_h, x_1)$ is a (contradictory) cycle:

$$\int_y \left(\frac{D^{k+1}(y, \neg y)}{n^{k+1}} \right) \left(\frac{D^{h+1}(\neg y, y)}{n^{h+1}} \right) = \Theta \left(\rho(D)^{k+h} \cdot n^{-(k+h+2)} \right)$$

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E.g. the unsatisfiable regime, $\rho(D) > 1$. Take $k, h = \Theta(\log n)$:

$\rho(D)^{k+h} \cdot n^{-(k+h+2)} \cdot n^{k+h+1}$ goes to infinity!