Jan Hladký
Graphons as weak* limits

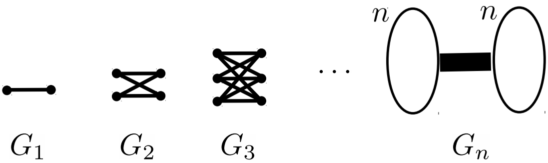
(1) "entropy minimization" with Doležal (arXiv: 1705.09160)
(2) "Vietoris topology" with Doležal, Grebík, Rocha, Rozhoň (arXiv: 1806.07368, 1809.03797)
(3) hypergraphons with Garbe, Noel, Piguet, Rocha, Saumell (????)

Limits of dense graph sequences

Borgs, Chayes, Lovász, Sós, Szegedy, Vesztergombi 2006

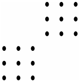
idea: convergence notion for sequences of finite graphs
compactification of the space of finite graphs $\Rightarrow$
$\ldots$ *graphons* symmetric Lebesgue-m. functions $\Omega^2 \to [0,1]$
Ω=separable atomless probability space $\cong [0,1]$

Graphons

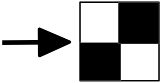


Represent these graphs by their adjacency matrices:

$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix}$$



$\cdots$



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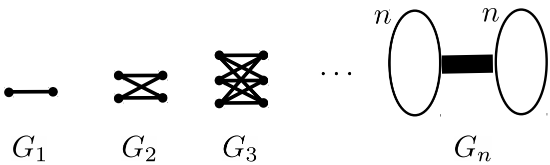

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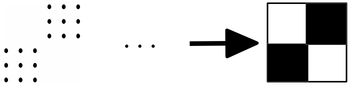

. . . works if you do things the right way. But, . . .

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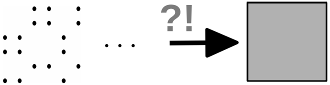

The cut-distance topology

Step 1: "Comparing the number of edges inside any vertex set"

$$d_{\square}(U, W) = \sup_{S \subset \Omega} \left| \int_S \int_S U(x, y) - W(x, y) \right| .$$

Step 2: "Permuting the adjacency matrix"

$$\delta_{\square}(U, W) = \inf_{\pi} d_{\square}(U, W^{\pi}) ,$$

where $\pi : \Omega \to \Omega$ runs through all measure-preserving bijections and $W^{\pi}(x, y) := W(\pi(x), \pi(y))$ version of W

Many important graph parameters still continuous

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$$\mathbf{ACC}_\square(\Gamma_1, \Gamma_2, \ldots) := \{\delta_\square\text{-acc pts of } \Gamma_1, \Gamma_2, \ldots\}$$
$$= \bigcup_{\pi_1, \pi_2, \ldots} \{d_\square\text{-acc pts of } \Gamma_1^{\pi_1}, \Gamma_2^{\pi_2}, \ldots\}$$
$$\mathbf{LIM}_\square(\Gamma_1, \Gamma_2, \ldots) := \bigcup_{\pi_1, \pi_2, \ldots} \{d_\square\text{-limit of } \Gamma_1^{\pi_1}, \Gamma_2^{\pi_2}, \ldots\}$$

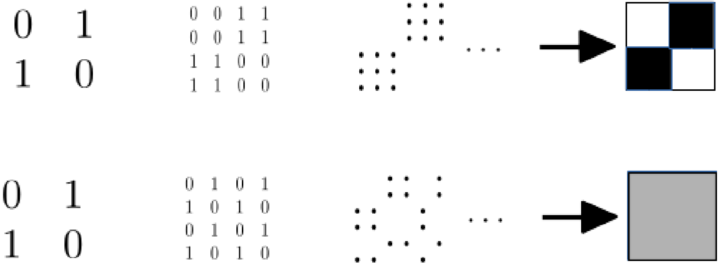
Lovász&Szegedy'06 For any sequence $\Gamma_1, \Gamma_2, \ldots$ we have that $\mathbf{ACC}_\square(\Gamma_1, \Gamma_2, \ldots) \neq \emptyset$.

Proofs of the Lovász–Szegedy Theorem

1. Lovász–Szegedy: Using Szemerédi's Regularity lemma
2. Elek–Szegedy (2012): Ultraproducts
3. Aldous–Hoover theorem on exchangeable arrays (1981)
   Persi Diaconis&Svante Janson and Tim Austin, 2008
4. **our proof(s) based on weak* convergence**

Comparing the weak* and cut-distance topology

Weak* converg.: $\Gamma_1, \Gamma_2, \ldots \xrightarrow{w^*} \Gamma$ iff $\forall X \subset \Omega^2$: $\lim_n \int_X \Gamma_n = \int_X \Gamma$



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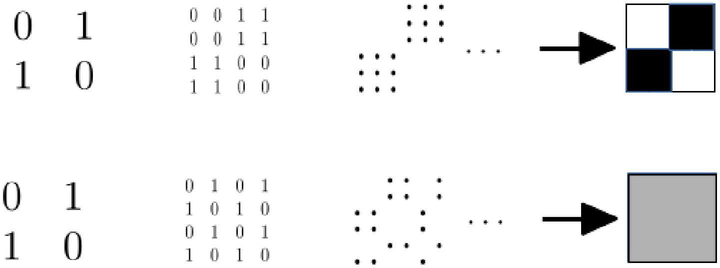
Note: $\mathbf{ACC}_{w^*}(\Gamma_1, \Gamma_2, \ldots)$ is nonempty by Banach–Alaoglu Thm



Comparing the weak* and cut-distance topology

$$W_n \xrightarrow{d_\square} W \iff \limsup_n \left\{ \sup_{S \subset \Omega} \left| \int_{x \in S} \int_{y \in S} W_n(x,y) - W(x,y) \right| \right\} = 0$$

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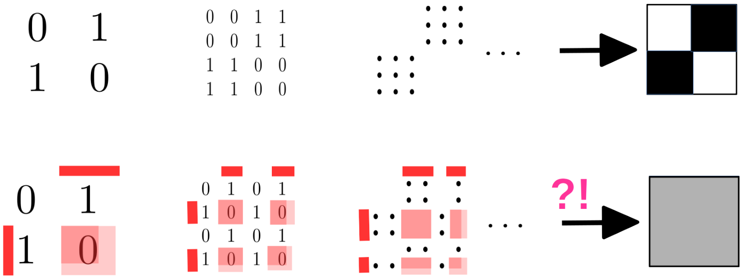
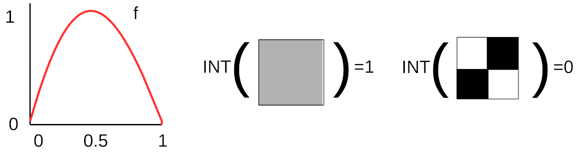


Lovász&Szegedy'06 $\delta_\square$ is a compact topology.

Proof Suppose that $W_1, W_2, \ldots : \Omega^2 \to [0,1]$.

- We need to find an accumulation point w.r.t. cut-distance.
- Lets search only in $\mathbf{ACC}_{w^*}(W_1, W_2, \ldots)$
- From $\mathbf{ACC}_{w^*}(W_1, W_2, \ldots)$ take a most structured graphon a prove that it is also a cut-distance accumulation point:

Fix concave function $f : [0,1] \to \mathbb{R}$. Define $INT(W) := \int_{x,y} f(W(x,y))$



Take $\Gamma \in \mathbf{ACC}_{w^*}(W_1, W_2, \ldots)$ that minimizes $INT(\Gamma)$

Lemma If $U_1, U_2, U_3, \ldots$ converges weak* but not in $d_\square$ to K. Then there exists a subsequence of versions $U_{n_1}^{\pi_{n_1}}, U_{n_2}^{\pi_{n_2}}, U_{n_3}^{\pi_{n_3}}, \ldots$ that weak* converges to some L, $INT(L) < INT(K)$

Graphons and the Vietoris topology

Theorem A For every sequence $W_1, W_2, \ldots$ there exists a subsequence so that

$$\mathbf{ACC}_{w^*}(W_{n_1}, W_{n_2}, \ldots) = \mathbf{LIM}_{w^*}(W_{n_1}, W_{n_2}, \ldots) \ .$$

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Theorem B Any sequence of graphons $U_1, U_2, \ldots$ is cut-distance Cauchy if and only if $\mathbf{ACC}_{w^*}(U_1, U_2, \ldots) = \mathbf{LIM}_{w^*}(U_1, U_2, \ldots)$

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Envelopes: $\langle W \rangle := \mathbf{LIM}_{w^*}(W, W, \ldots)$
Structurdness order: $U \preceq W \quad \text{iff} \quad \langle U \rangle \subseteq \langle W \rangle$

Range frequencies and degree frequencies

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Theorem If $U \prec W$ then Ψ_U is strictly flatter than Ψ_W, and Φ_U is at least as flat than Φ_W.

Λ_1 is at least as flat as Λ_2 if there exists a finite measure Δ on $[0,1]^2$ such that Λ_1 is the marginal of Δ on the first coordinate, Λ_2 is the marginal of Δ on the second coordinate, and for each $D \subset [0,1]$ we have $\int_{D\times[0,1]} x \, \mathrm{d}\Delta(x,y) = \int_{D\times[0,1]} y \, \mathrm{d}\Delta(x,y)$.

Cut-distance identifying graphon parameters

Motivation: The Chung-Graham-Wilson Theorem:
Among all graphons with edge density p, the constant-p graphon is the only graphon U satisfying any of the following:

- $t(C_4, U) \leq p^4$
- $|\lambda_1(U)| \leq p$ and $|\lambda_2(U)| \leq 0$.

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Definition $F : \mathcal{W} \to \mathbb{R}$ is a cut-distance identifying graphon parameter (CDIGP) if for each $U \prec W$ we have $F(U) < F(W)$.
(cut-distance compatible if for each $U \prec W$ we have $F(U) \leq F(W)$.)

Results:

- $t(C_4, \cdot)$ is a CDIGP
- each kth eigenvalue is CDIGP (not precise)

The (Erdos-Simonovits-)Sidorenko conjecture (1984, 1993)

Density of a graph H in a graphon W:

$$t(H,W) = \int_{x_1} \dots \int_{x_k} \prod_{ij \in E(H)} W(x_i, x_j) \ .$$

Sidorenko's conjecture: For any bipartite graph H, and any graphon W of density p, $t(H,W) \geq p^{e(H)}$.
Forcing conjecture: ...strict, unless H is a forest or W is constant

Definition (Kral-Martins-Pach-Wrochna): H has step Sidorenko property if for every W and every partition $\mathcal{P}$ of Ω we have $t(H,W) \geq t(H,W_{\mathcal{P}})$ (step forcing property analogously)
(Lets work with connected graphs)

Theorem

H step Sidorenko $\Leftrightarrow$ $t(H,\cdot)$ is CDCGP $\Leftrightarrow$ H is weakly norming
H step forcing $\Leftarrow$ $t(H,\cdot)$ is CDCGP $\Leftarrow$ H is norming