

Hamiltonicity of inhomogeneous random graphs

arXiv:2604.00899

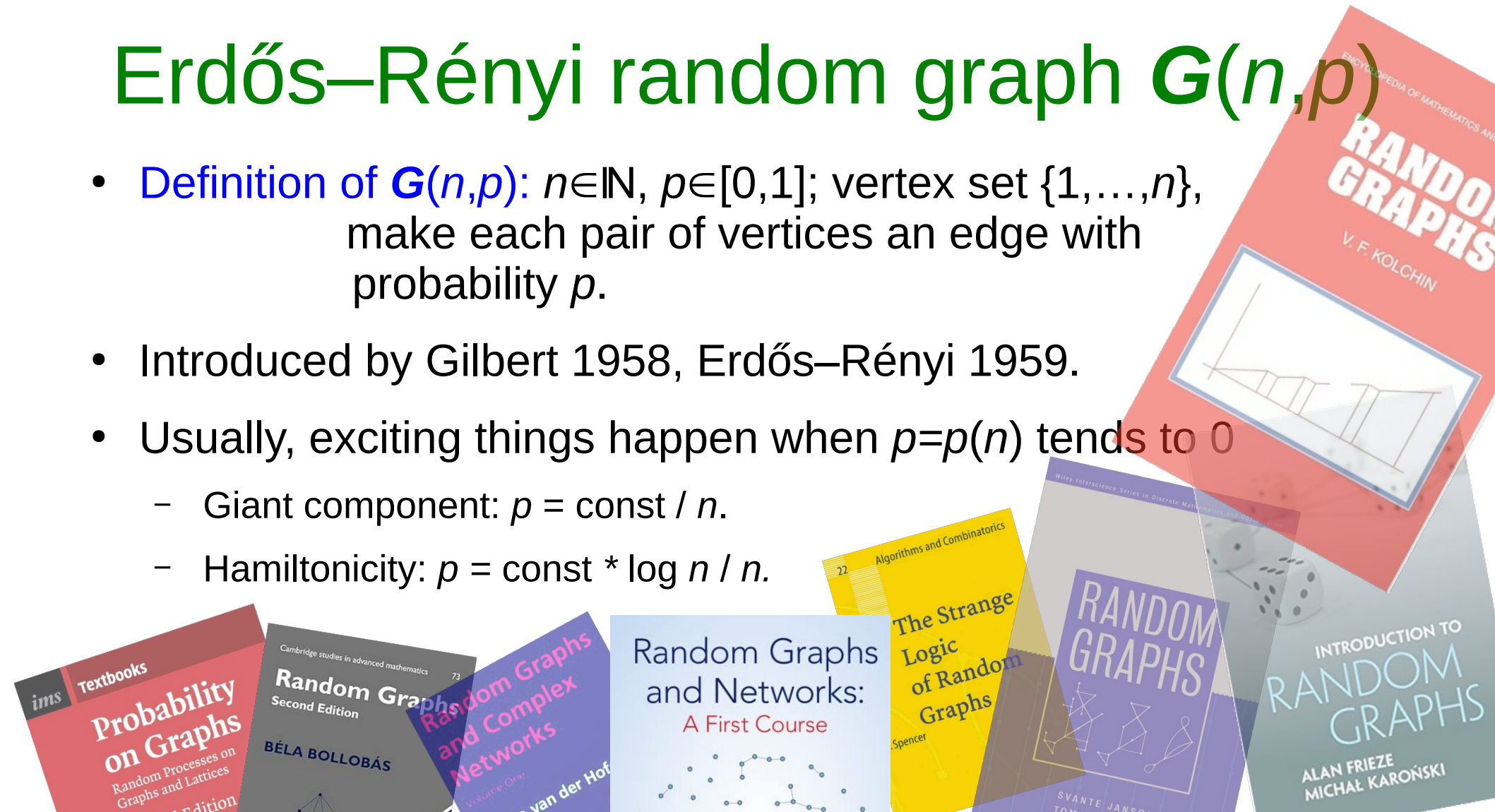
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Erdős–Rényi random graph $G(n, p)$

- **Definition of $G(n, p)$:** $n \in \mathbb{N}$, $p \in [0, 1]$; vertex set $\{1, \dots, n\}$, make each pair of vertices an edge with probability p .
- Introduced by Gilbert 1958, Erdős–Rényi 1959.
- Usually, exciting things happen when $p = p(n)$ tends to 0
 - Giant component: $p = \text{const} / n$.
 - Hamiltonicity: $p = \text{const} * \log n / n$.



Connectedness and Hamiltonicity of $G(n,p)$

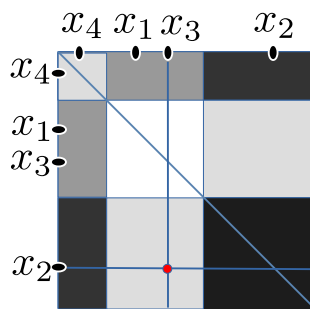
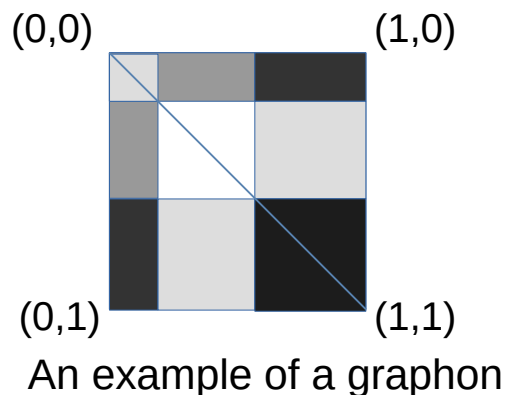
Theorem

Let $\varepsilon > 0$ be fixed.

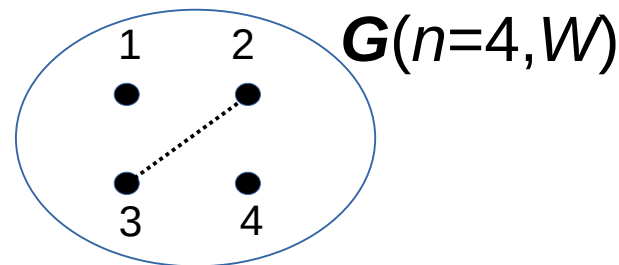
- If $p(n) < (1-\varepsilon) \ln n / n$, then $G(n,p)$ a.a.s. contains an isolated vertex.
- Gilbert 1958, Erdős–Rényi 1959:
If $p(n) > (1+\varepsilon) \ln n / n$, then $G(n,p)$ is a.a.s. connected.
- Pósa 1976, Korshunov 1976:
If $p(n) > (1+\varepsilon) \ln n / n$, then $G(n,p)$ is a.a.s. Hamiltonian.

Graphon-based random graphs

- **Definition of graphon:** $W:[0,1]^2 \rightarrow [0,1]$ measurable, symmetric $W(x,y)=W(y,x)$
- Introduced by Lovász and Szegedy 2006 as “graph limits”
- **Definition of $G(n,W)$:** Vertex set $\{1,\dots,n\}$
 - Generate $x_1, x_2, \dots, x_n \in [0,1]$ at random.
 - For each pair $\{i,j\}$, insert it as an edge with probability $W(x_i, x_j)=W(x_j, x_i)$.

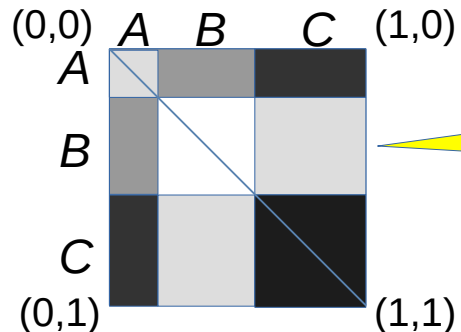


Sampling, $n=4$



Graphon-based random graphs

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This particular example:
stochastic block model
with 3 blocks A, B, C

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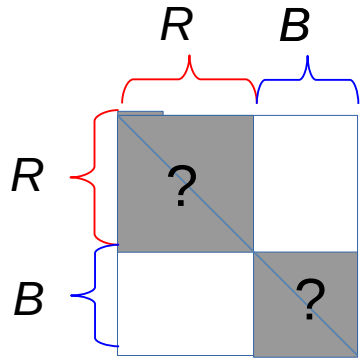
$G(n,W)$ is typically dense (and thus boring???)

- $W \equiv 0$, or
- $W \neq 0$ and then a.a.s.

$$e(G(n,W)) = (0.5 \pm \varepsilon) \|W\|_1 n^2$$

Connectivity of $G(n, W)$

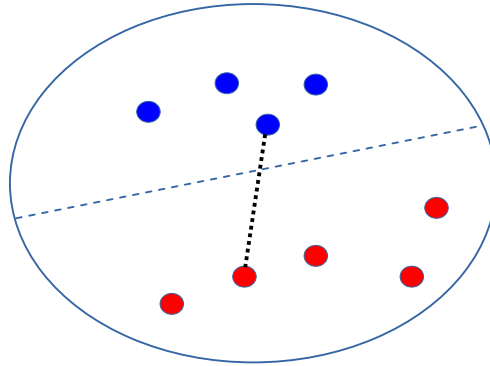
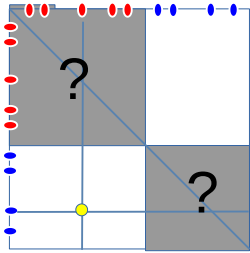
- **First Obstacle:** Disconnectedness of W



Definition: W is *disconnected* if there exists a partition $[0,1]=R\cup B$ into two sets of positive measure such that W is zero on $R\times B$.

Connectivity of $G(n, W)$

- **First Obstacle:** Disconnectedness of W

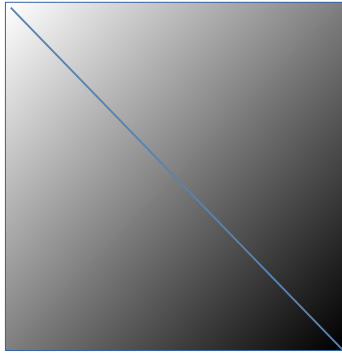


A.a.s., vertices of $G(n, W)$ can be split into two **linear-sized** sets with no edges between.

Connectivity of $G(n, W)$

- **First Obstacle:** Disconnectedness of W
- **Second Obstacle:** Isolated vertices

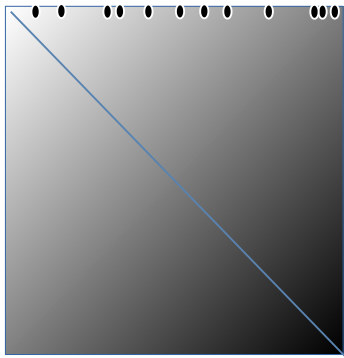
$$W(x, y) = x^6 y^6$$



Connectivity of $G(n, W)$

- **First Obstacle:** Disconnectedness of W
- **Second Obstacle:** Isolated vertices

$$W(x, y) = x^6 y^6$$



The left-most point $\min\{x_i\} \sim 1/n < 1/\sqrt{n}$

Hence, $W(x_i, y) < n^{-3}$ for every y .

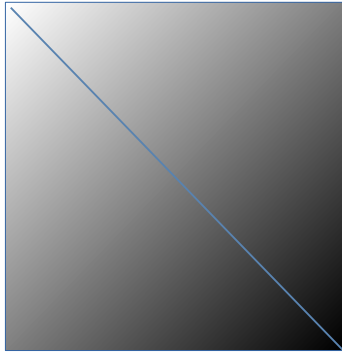
There are $n-1$ people and each of them becomes your friend with probability $< n^{-3}$.

Connectivity of $G(n, W)$

- **First Obstacle:** Disconnectedness of W
- **Second Obstacle:** Isolated vertices

$$W(x, y) = x^6 y^6$$

What about $W(x, y) = x^\alpha y^\alpha$ for $\alpha \in (0, \infty)$?
transition at $\alpha = 1$



The transition is at $\alpha = 1$.

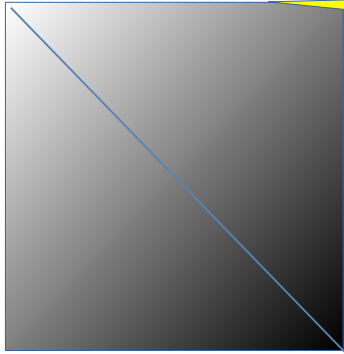
Hence, $W(x_i, y) < n^{-3}$ for every y .

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Connectivity of $G(n, W)$

- **First Obstacle:** Disconnectedness of W
- **Second Obstacle:** Isolated vertices

$$W(x, y) = x^6 y^6$$



THE 1st

Hence, $W(x_i, y) < n^{-3}$ for every y .

There are $n-1$ people and each of them becomes your friend with probability $< n^{-3}$.

For $x \in [0, 1]$, $\deg_w(x) = \int W(x, y) dy$

Vertex $v \in V(G(n, W))$ of type $x \in [0, 1]$ will have degree (almost surely)

$$\deg_{G(n, W)}(v) \approx n \deg_w(x)$$

Connectivity of $\mathbf{G}(n, W)$

Theorem [H.-Viswanathan, [arXiv:2305.03607](#)]

Suppose that W is a graphon.

- If W is disconnected (as a graphon) then $\mathbf{G}(n, W)$ is disconnected (as a graph) a.a.s.
- If W is connected then for $\alpha \in [0, 1]$, write $s(\alpha) \in [0, 1]$ for the measure of elements x with $\deg(x) < \alpha$.
 - If $\lim_{\alpha \rightarrow 0} s(\alpha) / \alpha = 0$ then a.a.s. $\mathbf{G}(n, W)$ is connected.
 - If $\lim_{\alpha \rightarrow 0} s(\alpha) / \alpha = \infty$ then a.a.s. $\mathbf{G}(n, W)$ has an isolated vertex.
 - If $\lim_{\alpha \rightarrow 0} s(\alpha) / \alpha \in (0, \infty)$ then connected AND disconnected with prob > 0 .

Hamiltonicity of $\mathbf{G}(n, W)$

Hamiltonicity = connectivity + ???

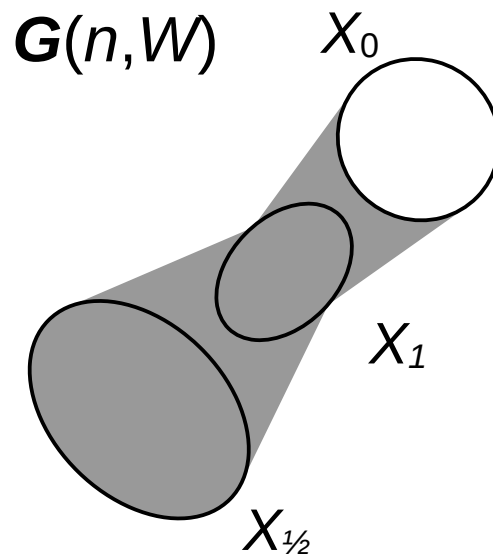
Hamiltonicity of $G(n, W)$

Hamiltonicity = connectivity + ???

W

	X_1	$X_{1/2}$	X_0
X_1	?	?	?
$X_{1/2}$	?	?	
X_0	?		

- W is constant-0 on $X_0 \times (X_0 \cup X_{1/2})$
- $\lambda(X_1) < \lambda(X_0)$



This is a “*half-integral vertex cover of W* ” of total weight less than $\frac{1}{2}$.

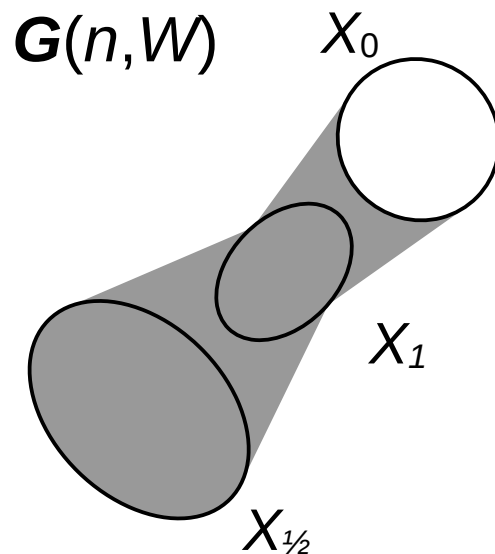
- H.-Hu-Piguet “Tilings in graphons” 2021
- Doležal-H. “Matching polytons” 2019

ty of $G(n, W)$

Half-integrality $\approx$ connectivity + perfect (fractional) matching

	X_1	$X_{\frac{1}{2}}$	X_0
X_1	?	?	?
$X_{\frac{1}{2}}$	?	?	
X_0	?		

- W is constant-0 on $X_0 \times (X_0 \cup X_{\frac{1}{2}})$
- $\lambda(X_1) < \lambda(X_0) \iff \frac{1}{2}\lambda(X_{\frac{1}{2}}) + \lambda(X_1) < \frac{1}{2}$



Hamiltonicity of $\mathbf{G}(n, W)$

Theorem [Garbe-H.-Piga], imprecise

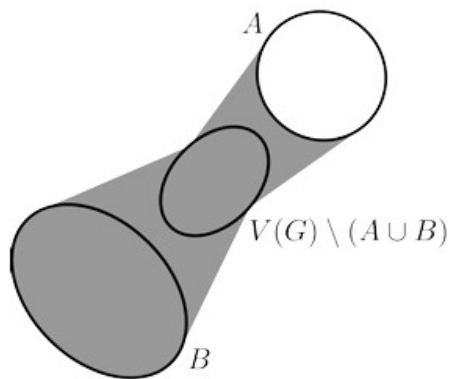
Suppose that W is a graphon. Then $\mathbf{G}(n, W)$ is Hamiltonian a.a.s if and only if

- If W is connected (as a graphon),
- $\lim_{\alpha \rightarrow 0} s(\alpha) / \alpha = 0$
- There is no partition $[0, 1] = X_0 \cup X_{1/2} \cup X_1$ such that
 - $\frac{1}{2}\lambda(X_{1/2}) + \lambda(X_1) < \frac{1}{2}$,
 - and W is constant-0 on $X_0 \times (X_0 \cup X_{1/2})$

Theorem

Suppose that
only if

- If W is
- $\lim_{\alpha \rightarrow 0} s(\alpha) / \alpha = 0$
- There is no partition $[0,1] = X_0 \cup X_{1/2} \cup X_1$ such that
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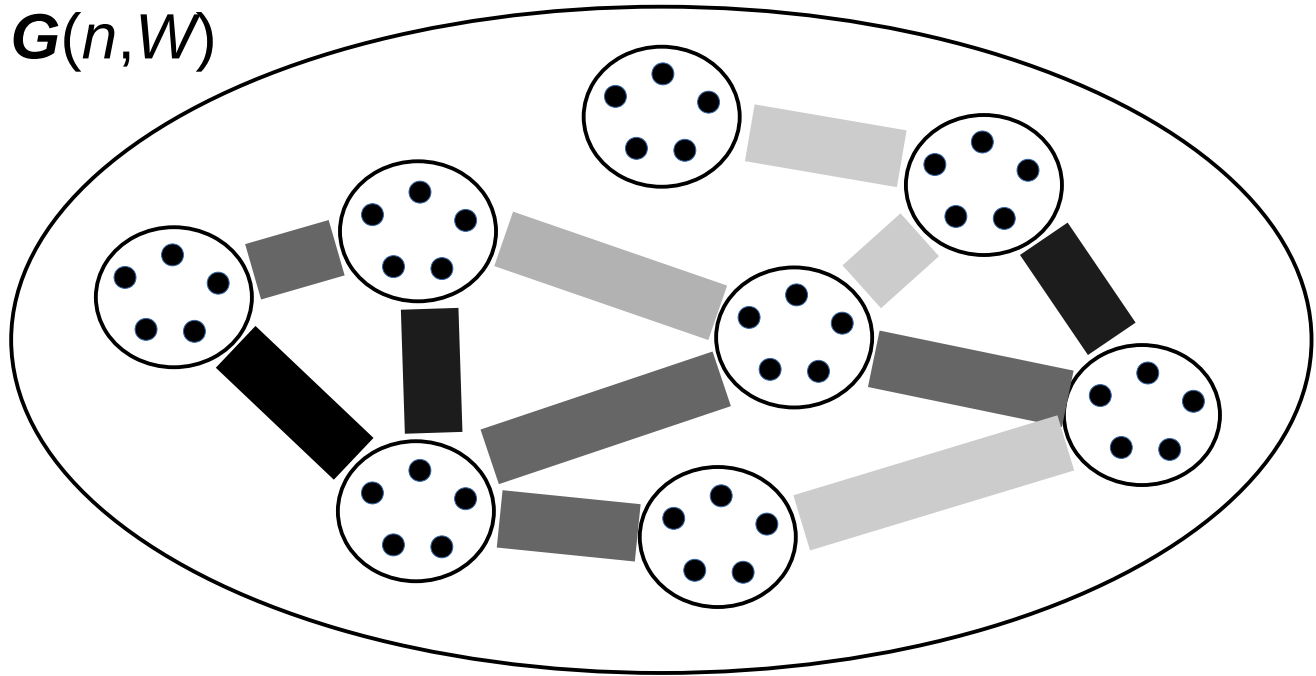
} no peninsula

Szemerédi's Regularity Lemma & Łuczak's connected matchings

Apply the SzRL

- “clusters” of vertices
- regular pairs of different densities

$G(n, W)$



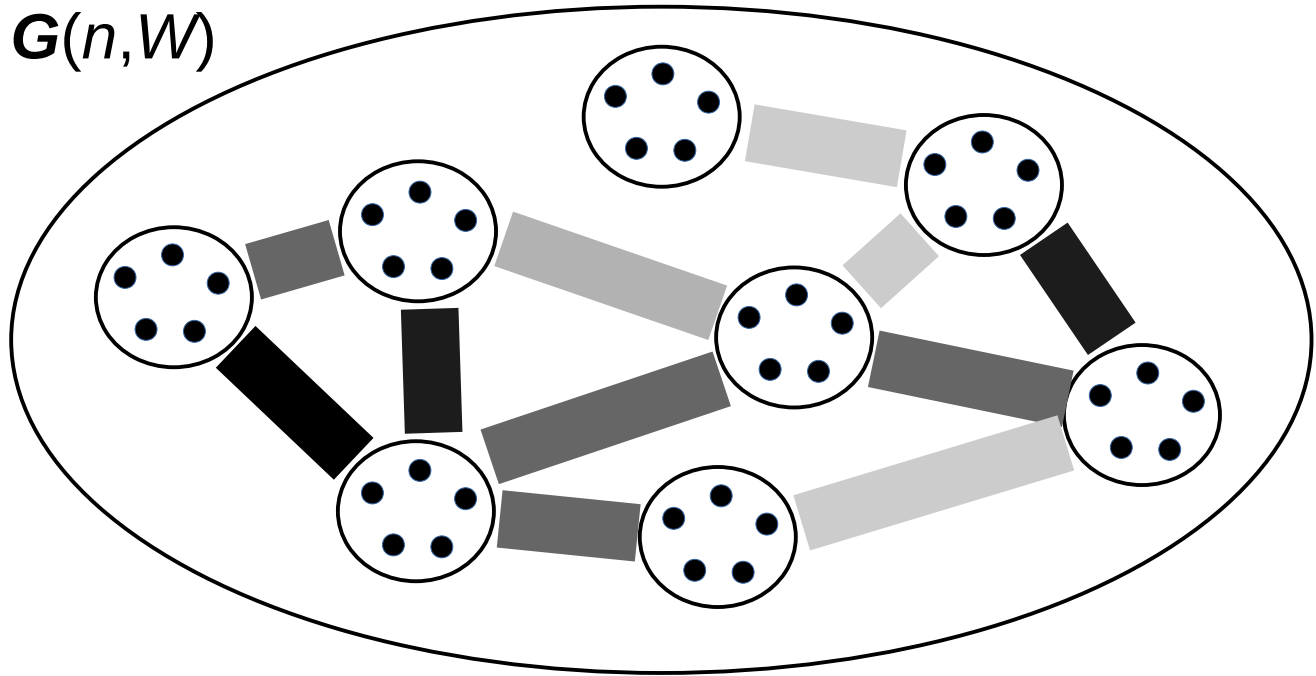
Szemerédi's Regularity Lemma & Łuczak's connected matchings

Apply the SzRL

- “clusters” of vertices
- regular pairs of different densities

1. The structure of the cluster graph is similar (?) to the graphon W .

$G(n, W)$

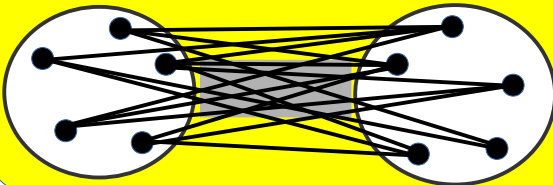


Szemerédi's Regularity Lemma & Łuczak's connected matchings

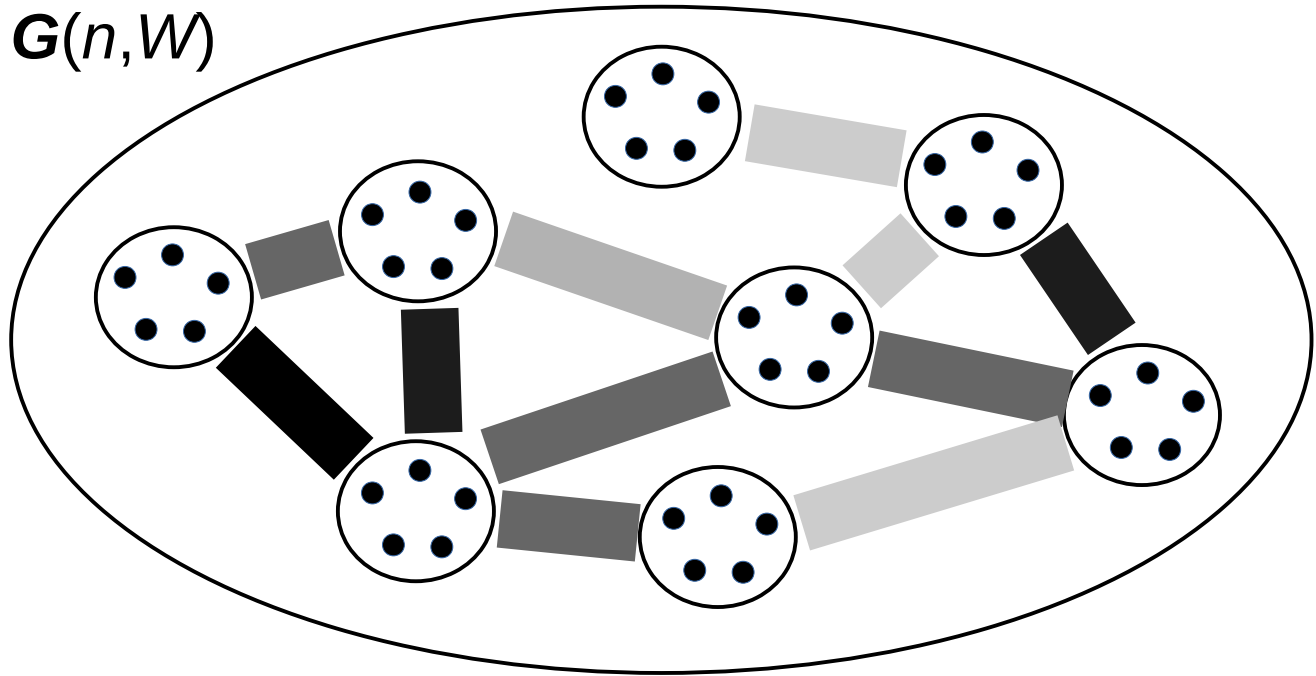
Apply the SzRL

- “clusters” of vertices
- regular pairs of different densities

2. Properties of a regular pair (Blow-up lemma, ...):
Regular pair of positive density contains (almost) spanning path.



$G(n, W)$



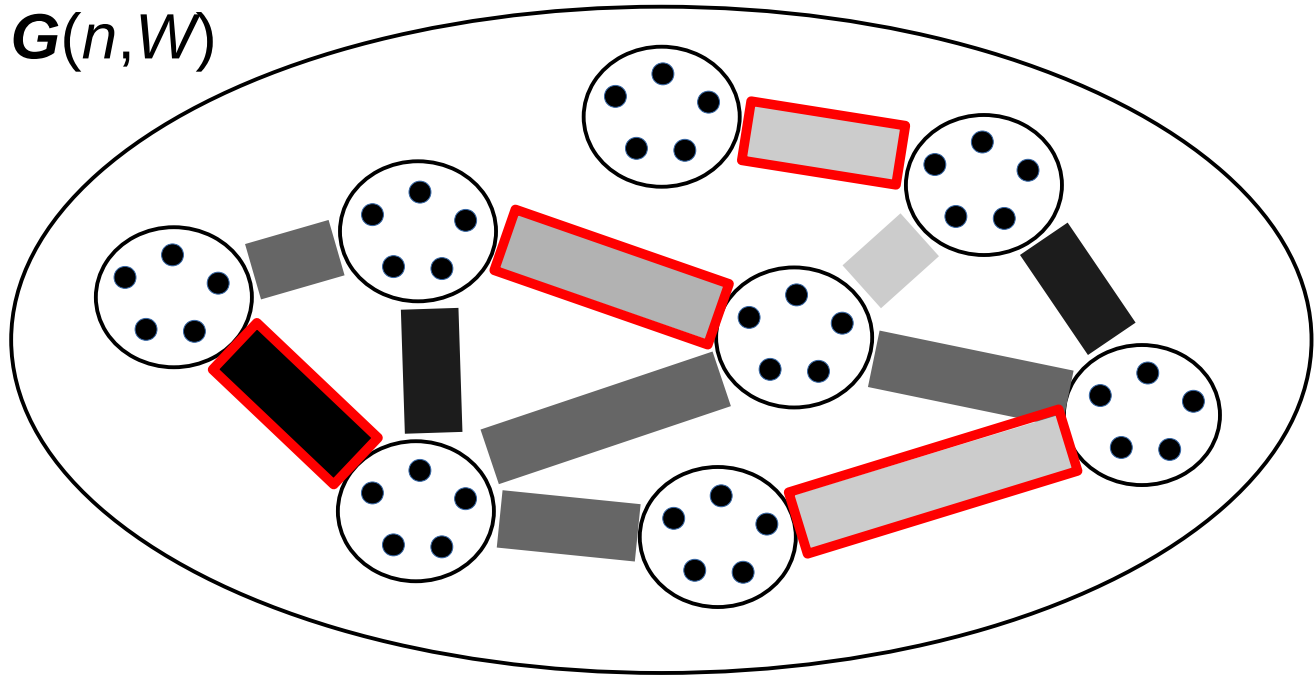
Szemerédi's Regularity Lemma & Łuczak's connected matchings

Łuczak's connected matchings:

- Find a perfect matching in the cluster graph
- +connectedness

Can we do that?
Not quite...

$G(n, W)$



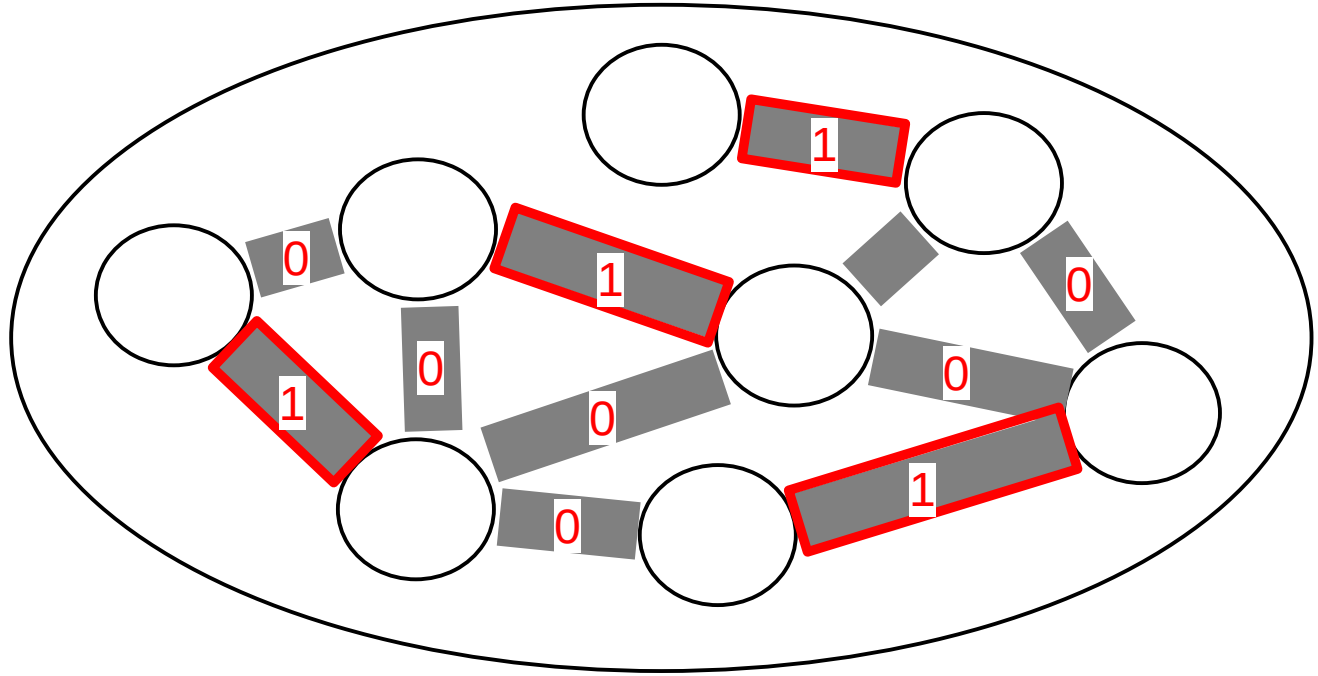
(Fractional) matchings in graphs

A **perfect matching**:

$$m: E \rightarrow \{0,1\}$$

so that for every vertex v :

$$\sum_{u \sim v} m(uv) = 1.$$



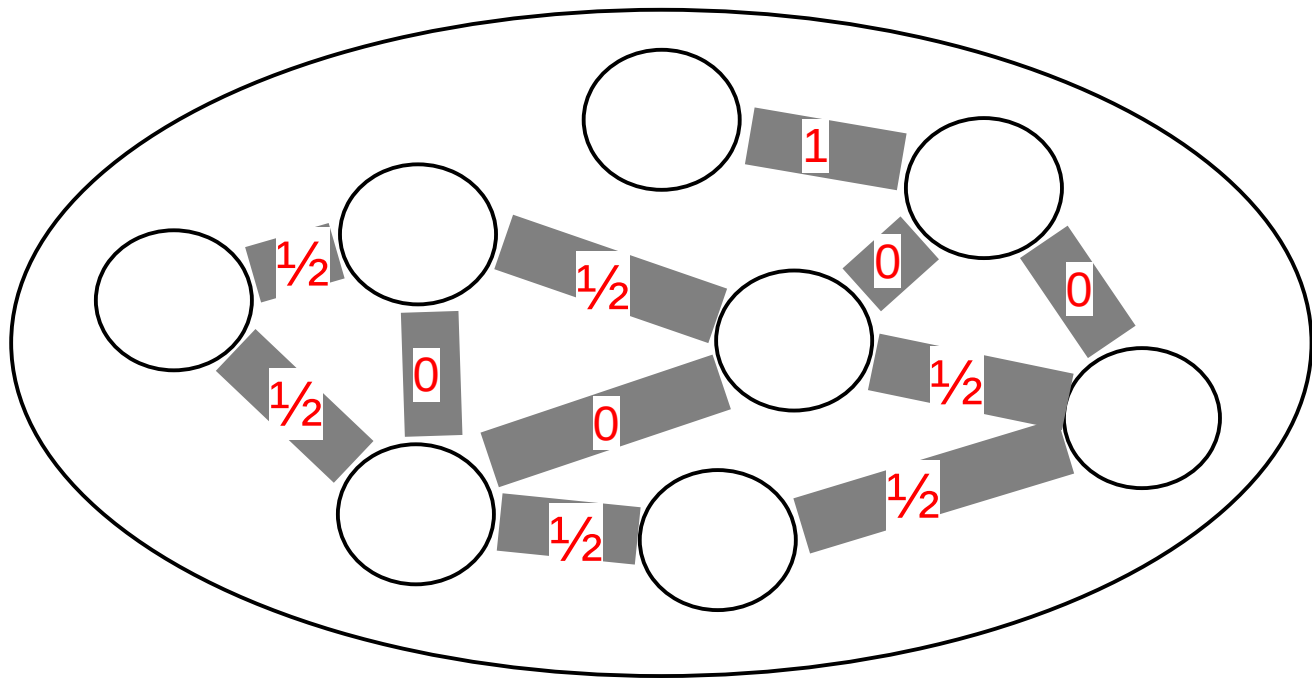
(Fractional) matchings in graphs

A **perfect fractional matching**:

$$m: E \rightarrow [0,1]$$

so that for every vertex v :

$$\sum_{u \sim v} m(uv) = 1.$$



(Fractional) matchings in graphs

A **perfect fractional matching**:

$$m: E \rightarrow [0,1]$$

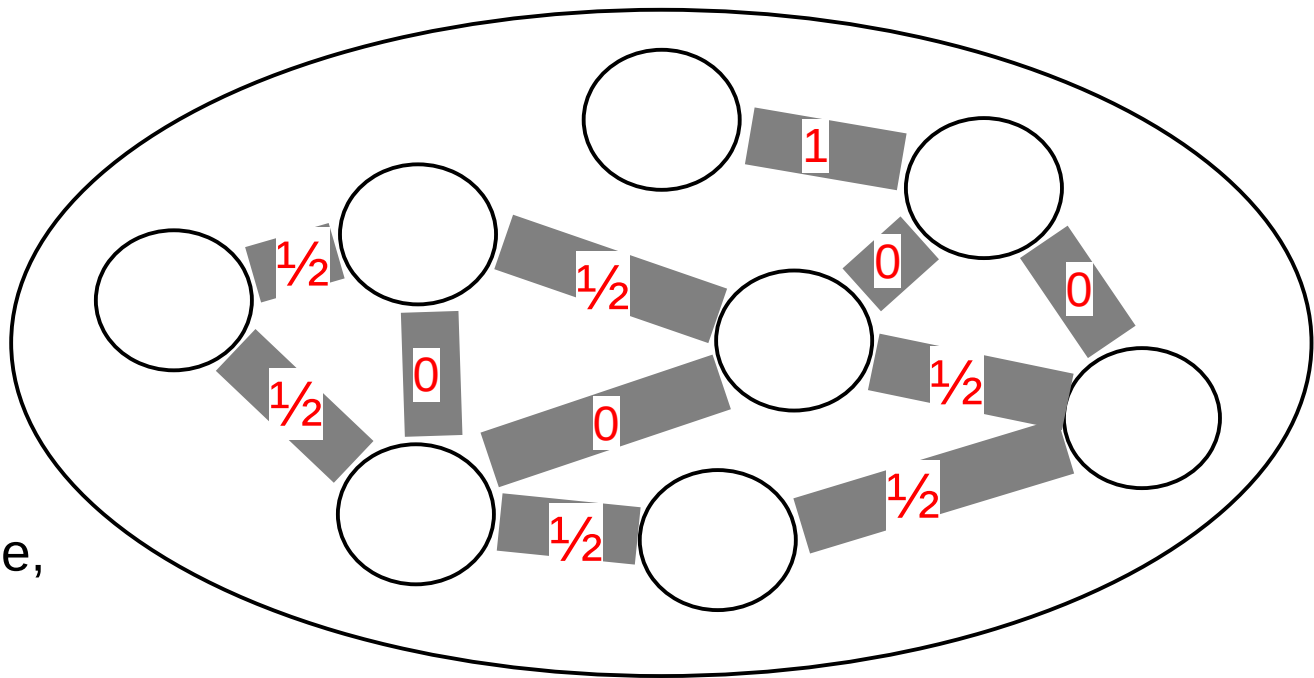
so that for every vertex v :

$$\sum_{u \sim v} m(uv) = 1.$$

Half-integrality of the matching polytope:

If a perfect fractional matching exists, then there also exists a half-integral one,

$$m: E \rightarrow \{0, \frac{1}{2}, 1\}$$



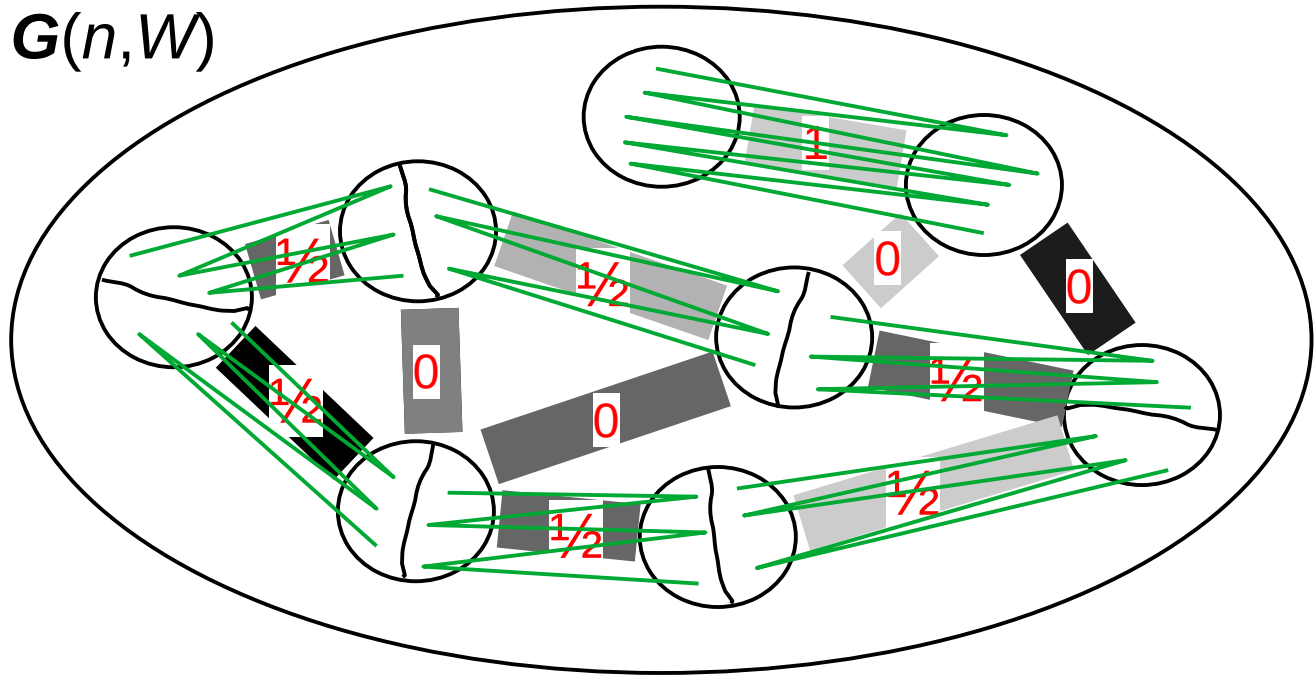
Łuczak's connected matchings, fractionally

Łuczak's connected matchings:

- Find a perfect **half-integral** matching in the cluster graph
- +connectedness

Can we do that?

$G(n, W)$



LP duality: frac matching VS frac vertex cover

A **fractional vertex cover** of a graph H :

$$c: V \rightarrow [0,1]$$

so that for every edge uv :

$$c(u) + c(v) \geq 1.$$

Every graph has a fractional vertex cover constant- $\frac{1}{2}$.

LP duality: frac matching VS frac vertex cover

A **fractional vertex cover** of a graph H :

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Every graph has a fractional vertex cover constant $\leq 1/2$.

LP duality: If a graph does not contain a perfect fractional (half-integral) matching then it contains a fractional (half-integral) vertex cover of average less than $1/2$.

LP duality: frac matching VS frac vertex cover

A **half-integral vertex cover** of a graph H :

$$c: V \rightarrow \{0, \frac{1}{2}, 1\}$$

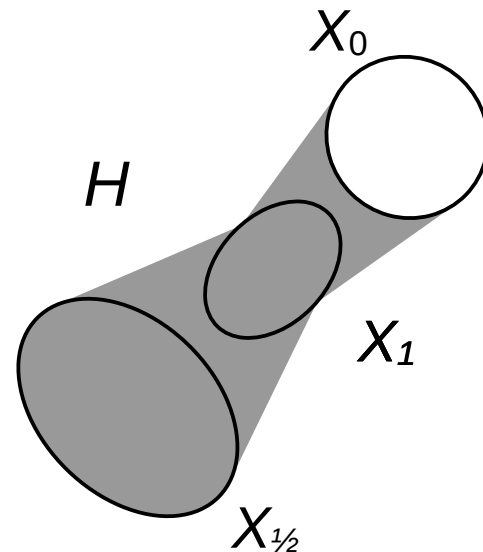
so that for every edge uv :

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$$X_0 := c^{-1}(0)$$

$$X_{\frac{1}{2}} := c^{-1}(\frac{1}{2})$$

$$X_1 := c^{-1}(1)$$



LP duality: If a graph does not contain a perfect fractional (half-integral) matching then it contains a fractional (half-integral) vertex cover of average less than $\frac{1}{2}$.

Failure??? of Łuczak's fractional connected matchings

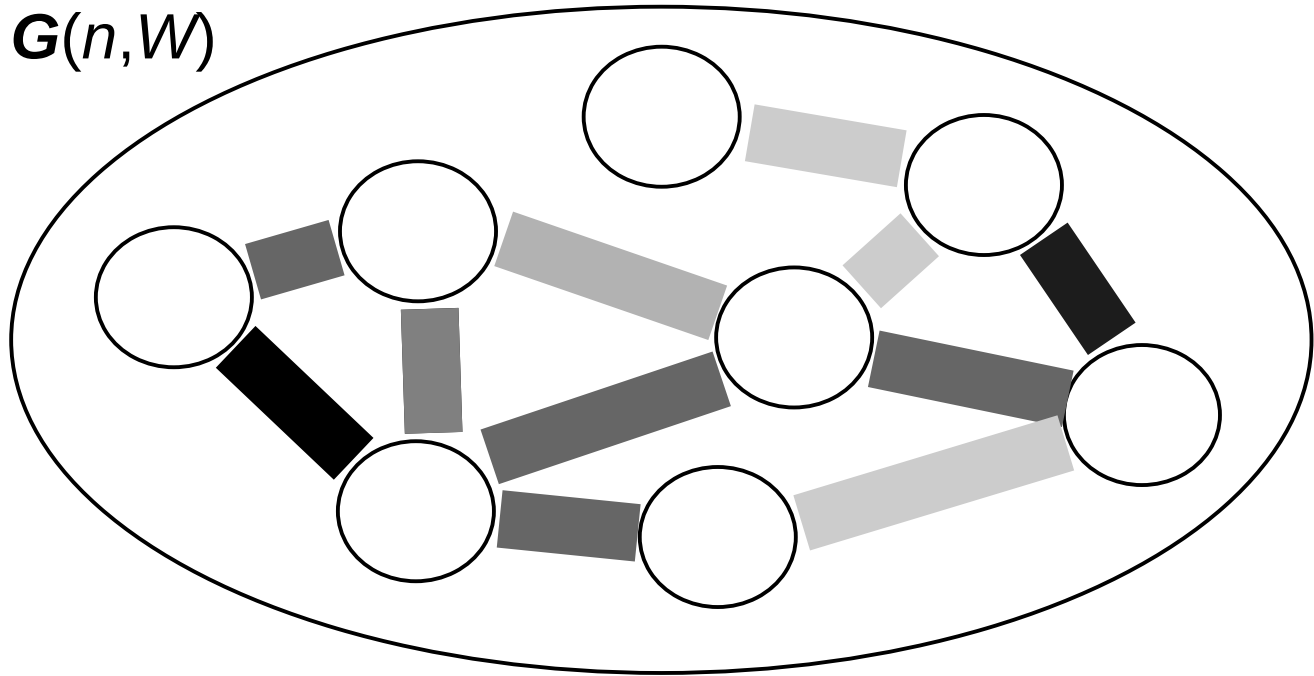
Łuczak's connected matchings:

- Find a perfect half-integral matching in the cluster graph
- (+connectedness)

Can we do that?

For contradiction, assume NOT. Then there exist half-integral vertex cover of average $< \frac{1}{2}$.

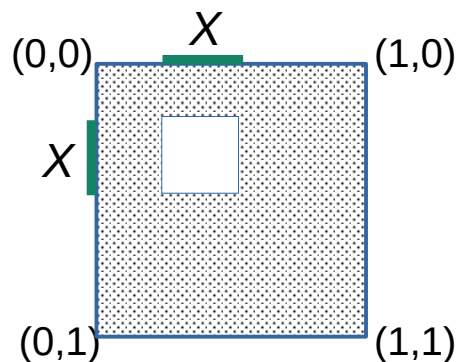
$G(n, W)$



The method of weak* limits

Toy example: Independent sets in a graphon

[H.-Rocha: “Independent sets, cliques, and colorings in graphons”, 2006]



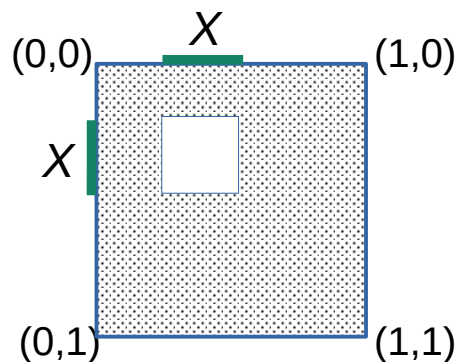
A set $X \subseteq [0,1]$ is an **independent set** in W if W is constant-0 almost everywhere on $X \times X$.

The **independence number** of W , denoted $\alpha(W)$ is the supremum of Lebesgue measures over independent sets.

The method of weak* limits

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Claim: There exists an independent set of measure $\alpha(W)$.

Proof: Weak* limits.

The method of weak* limits

Toy example: Independent sets in a graphon

Set $X \subseteq [0,1]$ is an independent set in W if $\int_{X \times X} W = 0$.

Function $f: [0,1] \rightarrow \{0,1\}$ is an independent set in W if $\int_{[0,1] \times [0,1]} W(x,y) f(x) f(y) = 0$.

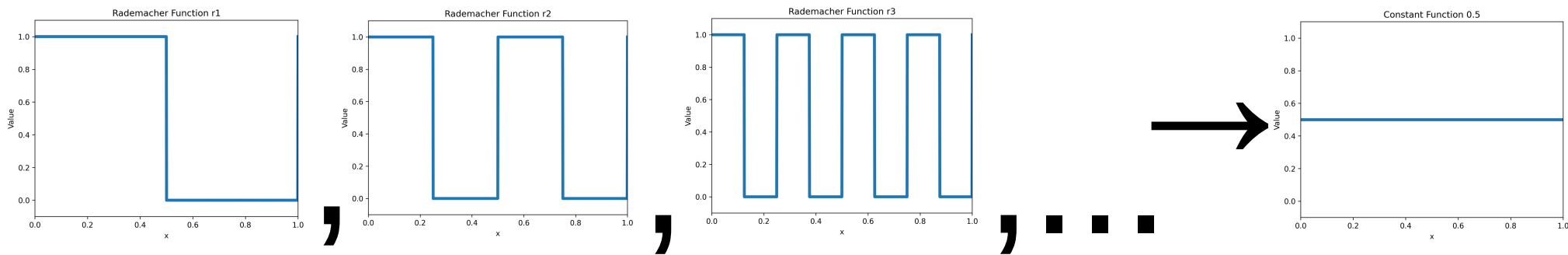
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Weak* limits of functions $f_n:[0,1] \rightarrow \{0,1\}$



The method of weak* limits

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Weak* limits of functions $f_n:[0,1] \rightarrow [0,1]$

A sequence $(f_n:[0,1] \rightarrow [0,1])_n$ **weak* converges** to a function $F:[0,1] \rightarrow [0,1]$ if

for every measurable $A \subseteq [0,1]$, we have $\lim_n \int_A f_n = \int_A F$.

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Weak* limits of functions: $f_n \xrightarrow{W^*} F \iff \lim_n \int_A f_n = \int_A F$ for every $A \subseteq [0,1]$

Banach–Alaoglu theorem:

The space of functions $f:[0,1] \rightarrow [0,1]$ is compact in the weak* topology.

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Claim: There exists an independent set of measure $\alpha(W)$.

Proof: Take independent sets $f_1, f_2, \dots$ whose measures converge to $\alpha(W)$,

$$\lim_n \int_{[0,1]} f_n = \alpha(W).$$

The method of weak* limits

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$$\lim_n \int_{[0,1]} f_n = \alpha(W).$$

Let $F: [0,1] \rightarrow [0,1]$ be a weak* accumulation point of $f_1, f_2, \dots$

We have $\int_{[0,1]} F = \alpha(W)$ and $\int_{[0,1] \times [0,1]} W(x,y) F(x) F(y) = 0$.

The method of weak* limits

Toy example: Independent sets in a graphon

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We have $\int_{[0,1]} F = \alpha(W)$ and $\int_{[0,1] \times [0,1]} W(x,y)F(x)F(y) = 0$.

“Round F up”: define $G: [0,1] \rightarrow \{0,1\}$, $G := \text{sgn}(F)$.

$$\int_{[0,1]} G \geq \alpha(W) \text{ and } \int_{[0,1] \times [0,1]} W(x,y)G(x)G(y) = 0.$$

The method of weak* limits

Toy example: Independent sets in a graphon

Takeaway idea: Weak* limits of independent set in a graphon, after “rounding up”, are an independent set in that graphon.

Strengthening: $W_1, W_2, W_3, \dots$ a sequence of graphons converging in the cut-norm to a graphon W .

f_1 an independent set in W_1 , f_2 an independent set in W_2 , ...

G is a “rounded up” weak* limit of $f_1, f_2, \dots$

Then G is an independent set in W .

The method of weak* limits

Half-integral vertex covers

Toy example: Independent

Takeaway idea: Weak* limit
“rounding

Strengthening: $W_1, W_2, \dots$
cut-norm

f_1 an inde

G is a “rounded up” weak* limit of $f_1, f_2, \dots$

Then G is an independent set in W .

	X_1	$X_{1/2}$	X_0
X_1	?	?	?
$X_{1/2}$	?	?	
X_0	?		

after
graphon.

erging in the

set in $W_2, \dots$

The method of weak* limits

Half-integral vertex covers

Toy example: Independent

Takeaway idea: Weak* limit
“rounding”

Strengthening: $W_1, W_2, \dots$
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G is a “ro

Then G is a

	X_1	$X_{1/2}$	X_0
X_1	?	?	?
$X_{1/2}$	?	?	
X_0	?		

Takeaway: Properties of half-integral
vertex covers are (lower semi-)continuous
with respect to the cut norm metric.

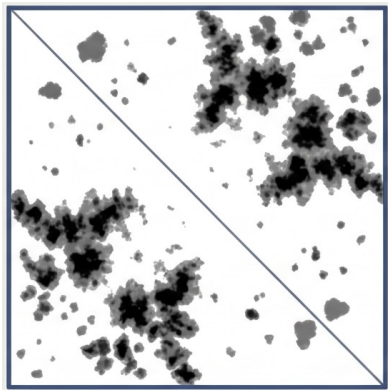
after
graphon.

erging in the

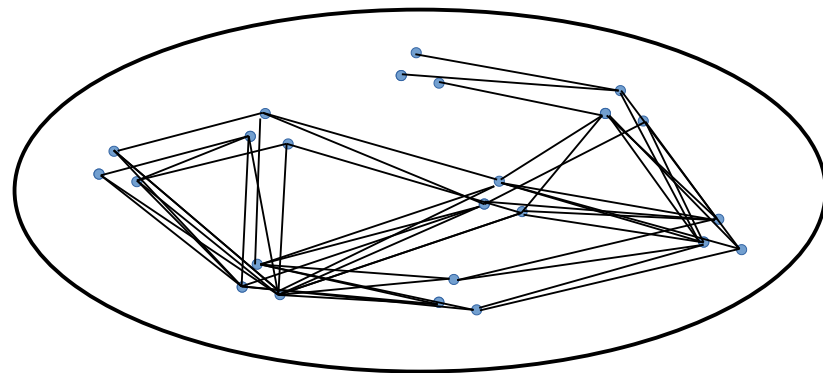
set in $W_2, \dots$

The proof

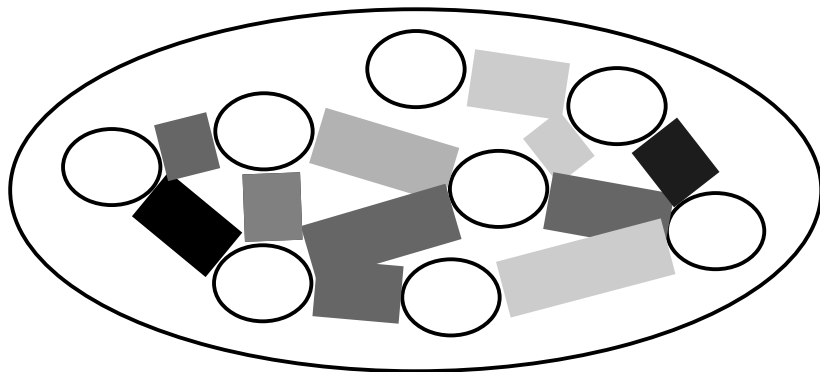
I.
graphon W
with no
peninsula



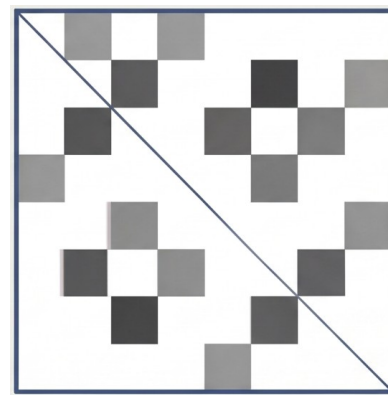
II.
 $\mathbf{G}(n, W)$



III.
regularization
 R of $\mathbf{G}(n, W)$

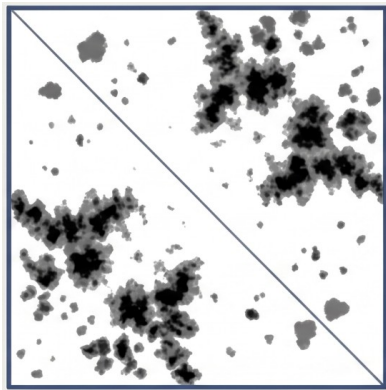


IV.
graphon
representation
of R

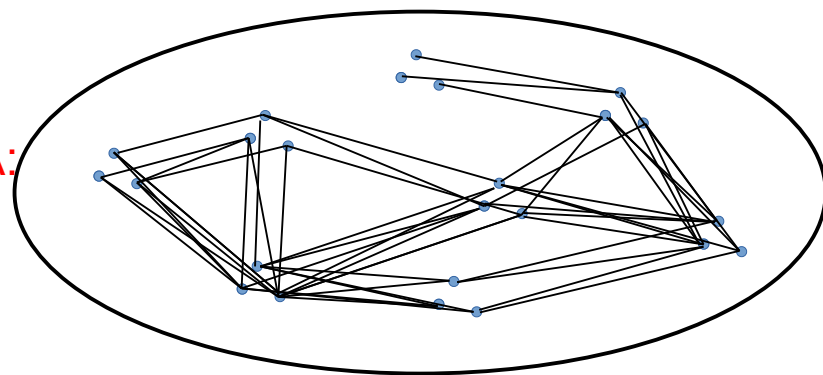


The proof

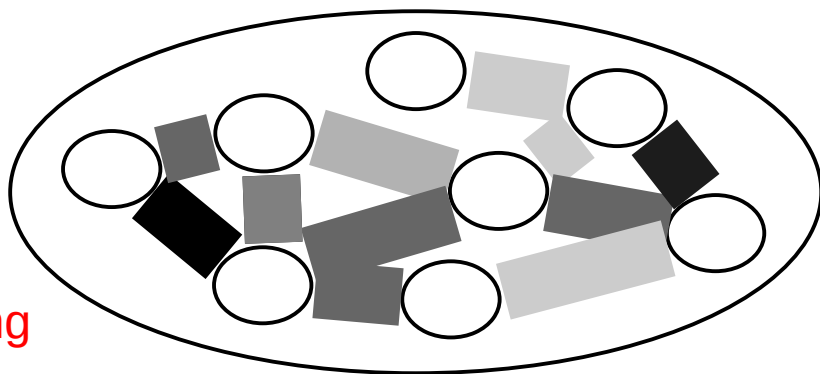
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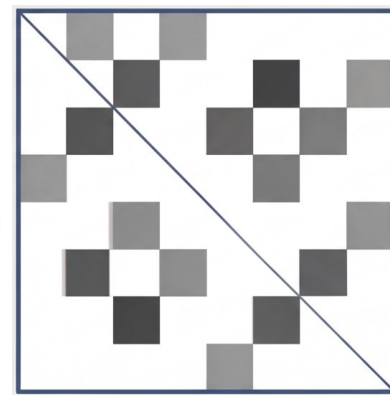
II.
 $\mathbf{G}(n, W)$
Łuczak's CMA:
almost perf.
cover by
paths.



III.
regularization
 R of $\mathbf{G}(n, W)$
almost perfect
half-int. matching

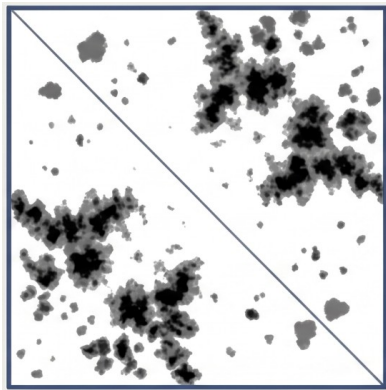


IV.
graphon
representation
of R
 $\approx$ no peninsula

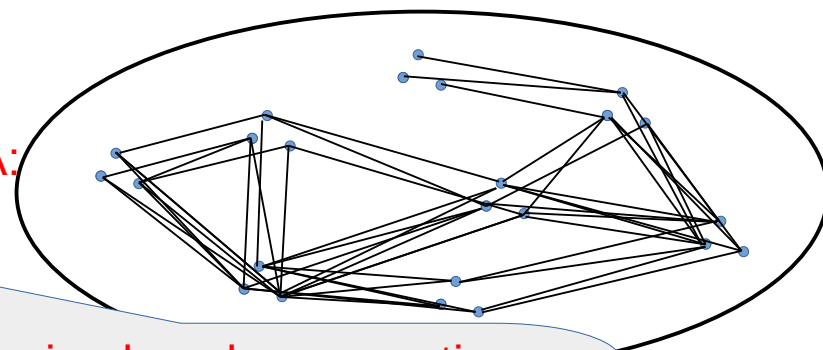


The proof

I.
graphon W
with no
peninsula

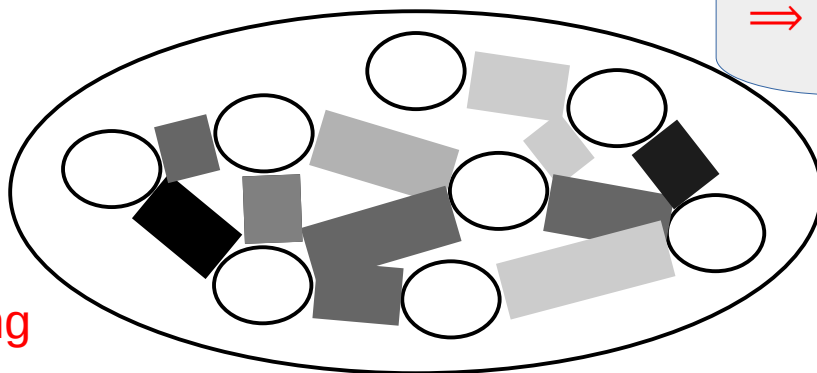


II.
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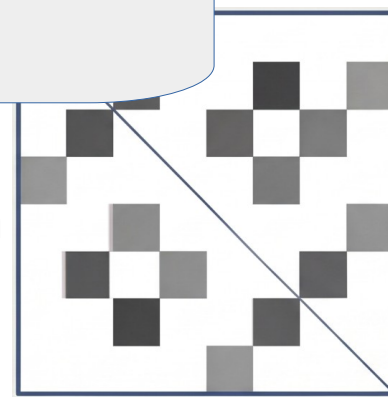


+covering low-degree vertices
+absorbers
 $\Rightarrow$ **Hamilton cycle**

III.
regularization
 R of $\mathbf{G}(n, W)$
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half-int. matching



IV.
graphon
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 $\approx$ no peninsula



Hamiltonicity of $\mathbf{G}(n, W)$

Theorem [Garbe-H.-Piga], imprecise

Suppose that W is a graphon. Then $\mathbf{G}(n, W)$ is Hamiltonian a.a.s if and only if

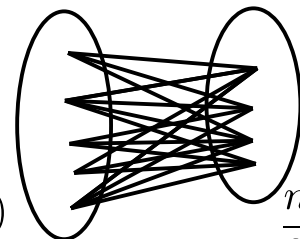
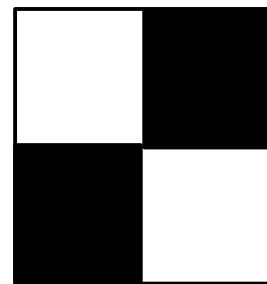
- If W is connected (as a graphon),
- $\lim_{\alpha \rightarrow 0} s(\alpha) / \alpha = 0$
- There is no partition $[0, 1] = X_0 \cup X_{1/2} \cup X_1$ such that
 - $\frac{1}{2}\lambda(X_{1/2}) + \lambda(X_1) < \frac{1}{2}$,
 - and W is constant-0 on $X_0 \times (X_0 \cup X_{1/2})$

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 - $\frac{1}{2}\lambda(X_{1/2}) + \lambda(X_1) \leq \frac{1}{2}$ and $\lambda(X_{1/2}) < 1$,
 - and W is constant-0 on $X_0 \times (X_0 \cup X_{1/2})$



$$\frac{n}{2} \pm \Theta(\sqrt{n})$$

$$\frac{n}{2} \mp \Theta(\sqrt{n})$$