A Vietoris functor for bispaces and d-frames

Tomáš Jakl

(joint work with Aleš Pultr and Achim Jung)

Department of Applied Mathematics
Charles University in Prague

AND

School of Computer Science
University of Birmingham

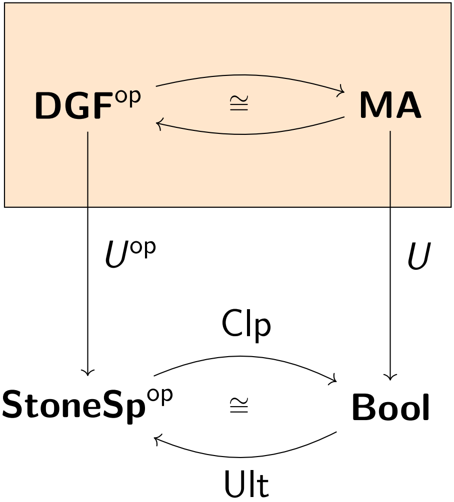
TACL, 30 June 2017

Jónsson-Tarski duality

$$\mathbf{StoneSp}^{\mathrm{op}} \underset{\mathrm{Ult}}{\overset{\mathrm{Clp}}{\rightleftarrows}} \mathbf{Bool} \qquad (\cong)$$

For propositional logic

Jónsson-Tarski duality



For modal logic

DGF are $\mathbb{V}$-coalgebras

Vietoris functor

Let (X, τ) be a Stone space. $\mathbb{V}(X, \tau) \stackrel{\text{def}}{=} (\mathcal{K}X, \mathbb{V}\tau)$ where

1. $\mathcal{K}X$ = compact subsets of X
2. $\mathbb{V}\tau$ is generated by $\boxtimes V$, $\diamondplus V$ (for all $V \in \tau$) where

$$\boxtimes V = \{K \in \mathcal{K}X \mid K \subseteq V\}$$
$$\diamondplus V = \{K \in \mathcal{K}X \mid K \cap V \neq \emptyset\}$$

Theorem (Kupke, Kurz, Venema 2003)

The category of descriptive general frames and the category of $\mathbb{V}$-coalgebras are isomorphic.

Continuous $X \to \mathbb{V}(X)$

Modal algebras are $\mathbb{M}$-algebras

Let B be a Boolean algebra.

$$\mathbb{M}(B) \stackrel{\text{def}}{=} \mathbb{BA}\langle\, \Box a : a \in B \,\rangle \,/\, \approx$$

where $\approx$ is generated by

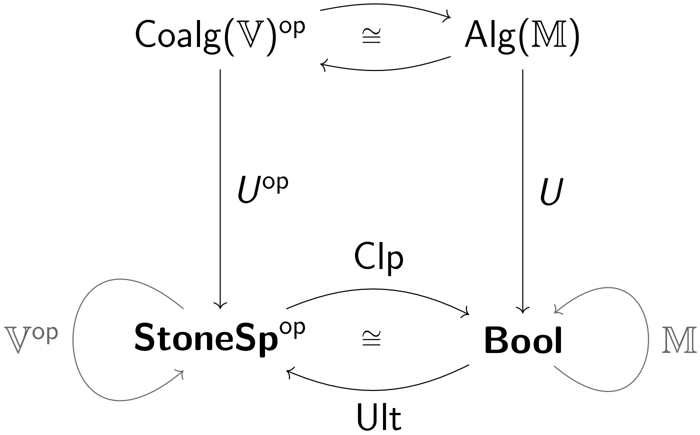
$$\Box(a \wedge b) \approx \Box a \wedge \Box b \quad \text{and} \quad \Box 1 \approx 1$$

Theorem (folklore?)

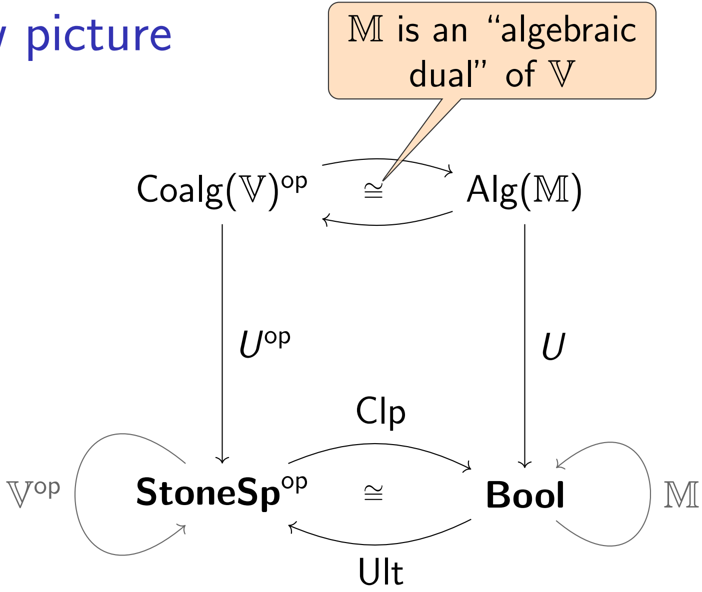
The category of modal algebras and the category of $\mathbb{M}$-algebras are isomorphic.

Homomorphisms $\mathbb{M}(B) \to B$

New picture



New picture



$\mathbb{M}$ is an "algebraic dual" of $\mathbb{V}$
$\mathrm{Coalg}(\mathbb{V})^{\mathrm{op}}$
$\cong$
$\mathrm{Alg}(\mathbb{M})$
U^{op}
U
Clp
$\mathbb{V}^{\mathrm{op}}$
StoneSp$^{\mathrm{op}}$
$\cong$
Bool
$\mathbb{M}$
Ult

Frames

A complete lattice $(L; \bigvee, \wedge, 0, 1)$ is a *frame* iff

$$b \wedge (\bigvee_i a_i) = \bigvee_i (b \wedge a_i)$$

($\implies$ complete Heyting algebra)

Example

$\Omega(X; \tau) = (\tau; \bigcup, \cap, \emptyset, X)$

$\Omega(f \colon X \to Y) \colon U \mapsto f^{-1}[U]$

Frames

A complete lattice $(L; \bigvee, \wedge, 0, 1)$ is a *frame* iff

$$b \wedge (\bigvee_i a_i) = \bigvee_i (b \wedge a_i)$$

$(\implies$ complete Heyting algebra$)$

Example

$\Omega(X; \tau) = (\tau; \bigcup, \cap, \emptyset, X)$

$\Omega(f \colon X \to Y) \colon U \mapsto f^{-1}[U]$

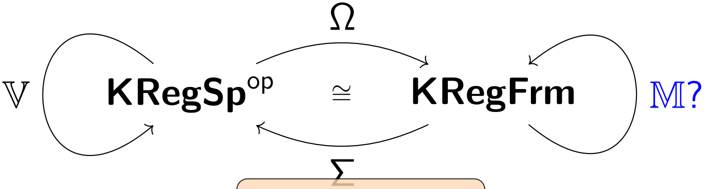
$$\mathbf{Top}^{\mathrm{op}} \underset{\Sigma}{\overset{\Omega}{\underset{\perp}{\rightleftarrows}}} \mathbf{Frm}$$

Frame $\mathbb{M}$

$$\mathbf{KRegSp}^{\mathrm{op}} \underset{\Sigma}{\overset{\Omega}{\rightleftarrows}} \mathbf{KRegFrm} \qquad (\cong)$$

Compact regular spaces

Frame $\mathbb{M}$



$\Diamond a \neq \neg\Box(\neg a)$

$$\mathbb{M}(L) \stackrel{\text{def}}{\equiv} \mathbb{F}\mathrm{r}\langle\, \Box a, \Diamond a : a \in L \,\rangle/_{\approx}$$ where

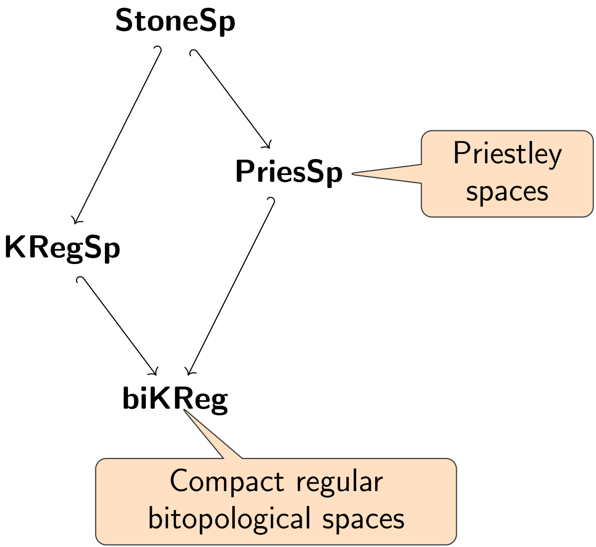
$$\Box a \wedge \Box b \approx \Box(a \wedge b) \quad \Box 1 \approx 1$$

$$\Diamond a \vee \Diamond b \approx \Diamond(a \vee b) \quad \Diamond 0 \approx 0$$

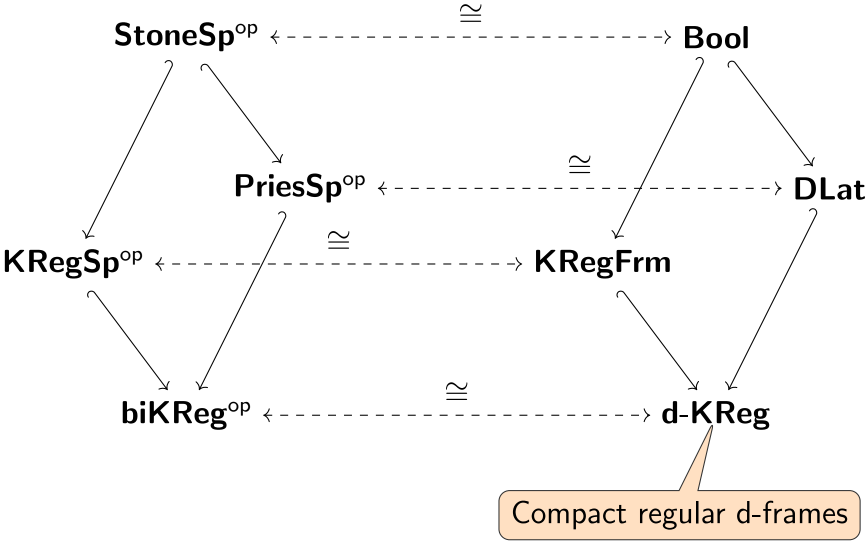
$$\Box a \wedge \Diamond b \preceq \Diamond(a \wedge b) \quad \Box(a \vee b) \preceq \Box a \vee \Diamond b$$

$$\bigvee\nolimits^{\uparrow} \Box a_i \approx \Box(\bigvee\nolimits^{\uparrow} a_i) \quad \bigvee\nolimits^{\uparrow} \Diamond a_i \approx \Diamond(\bigvee\nolimits^{\uparrow} a_i)$$

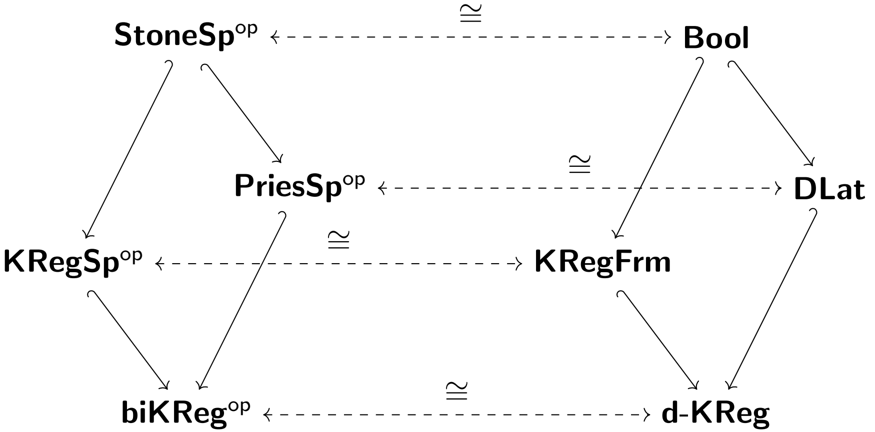
The whole perspective



The whole perspective



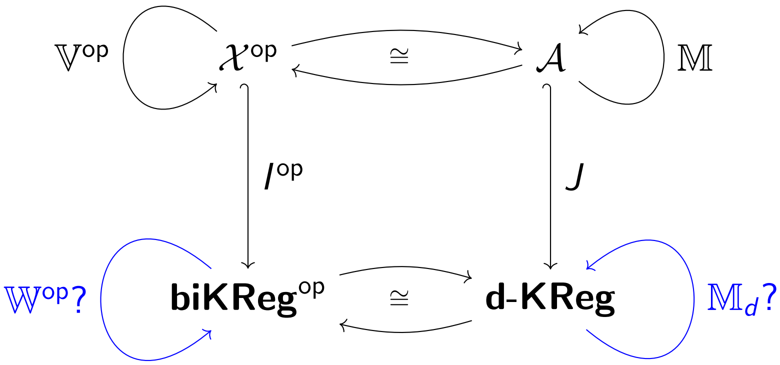
The whole perspective



$$\forall \mathcal{X} \in \{\mathbf{StoneSp}, \mathbf{KRegSp}, \mathbf{PriesSp}\} \quad \exists \mathbb{V} \colon \mathcal{X} \to \mathcal{X}$$
$$\forall \mathcal{A} \in \{\mathbf{Bool}, \mathbf{KRegFrm}, \mathbf{DLat}\} \quad \exists \mathbb{M} \colon \mathcal{A} \to \mathcal{A}$$

and $\quad \mathsf{Coalg}(\mathbb{V})^{op} \cong \mathsf{Alg}(\mathbb{M})$ $\quad$ (whenever $\mathcal{X}^{op} \cong \mathcal{A}$)

The task: generalise $\mathbb{V}$'s and $\mathbb{M}$'s



such that $I \circ \mathbb{V} \cong \mathbb{W} \circ I$ and $J \circ \mathbb{M} \cong \mathbb{M}_d \circ J$

D-frames (Jung & Moshier, 2006)

D-frame is a structure $\mathcal{L} = (L_+, L_-; \mathrm{con}, \mathrm{tot})$ where

- L_+ and L_- are frames
- $\mathrm{con}, \mathrm{tot} \subseteq L_+ \times L_-$

(+ axioms, e.g. $\dfrac{(x_+, x_-) \in \mathrm{con},\ \ x'_+ \leq x_+,\ \ x'_- \leq x_-}{(x'_+, x'_-) \in con}$)

Example

$\Omega_d(X, \tau_+, \tau_-) = (\tau_+, \tau_-, \mathrm{con}_X, \mathrm{tot}_X)$

$$(U, V) \in \mathrm{con}_X \stackrel{\text{def}}{\equiv} U \cap V = \emptyset$$

$$(U, V) \in \mathrm{tot}_X \stackrel{\text{def}}{\equiv} U \cup V = X$$

D-frames (Jung & Moshier, 2006)

D-frame is a structure $\mathcal{L} = (L_+, L_-; \text{con}, \text{tot})$ where

- L_+ and L_- are frames
- $\text{con}, \text{tot} \subseteq L_+ \times L_-$

$$(+ \text{ axioms, e.g. } \frac{(x_+, x_-) \in \text{con},\ x'_+ \leq x_+,\ x'_- \leq x_-}{(x'_+, x'_-) \in con})$$

$$\mathbf{biTop}^{\mathrm{op}} \underset{\Sigma_d}{\overset{\Omega_d}{\rightleftarrows}} \mathbf{d\text{-}Frm} \quad (\bot)$$

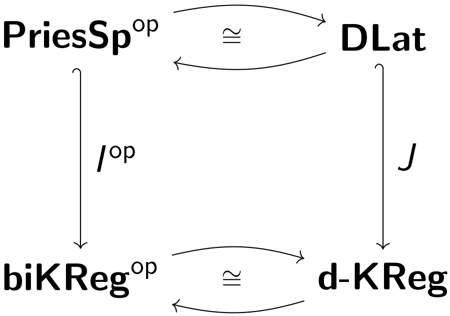
Example: embedding the frame duality

$$\begin{array}{ccc} \mathbf{KRegSp}^{\mathrm{op}} & \overset{\longrightarrow}{\underset{\longleftarrow}{\cong}} & \mathbf{KRegFrm} \\ \Big\downarrow{\scriptstyle I^{\mathrm{op}}} & & \Big\downarrow{\scriptstyle J} \\ \mathbf{biKReg}^{\mathrm{op}} & \overset{\longrightarrow}{\underset{\longleftarrow}{\cong}} & \mathbf{d\text{-}KReg} \end{array}$$

- $I\colon (X, \tau) \mapsto (X, \tau, \tau)$
- $J\colon L \mapsto (L, L, \mathrm{con}_L, \mathrm{tot}_L)$ where

$$\begin{aligned} (a, b) \in \mathrm{con}_L \quad &\text{iff} \quad a \wedge b = 0 \\ (a, b) \in \mathrm{tot}_L \quad &\text{iff} \quad a \vee b = 1 \end{aligned}$$

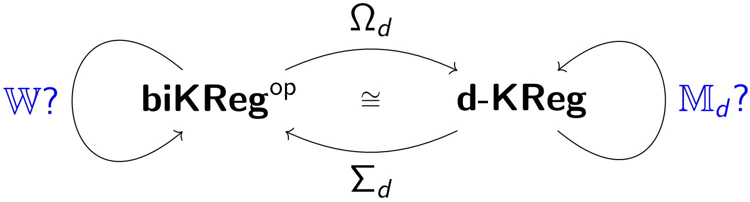
Example: embedding the Priestley duality

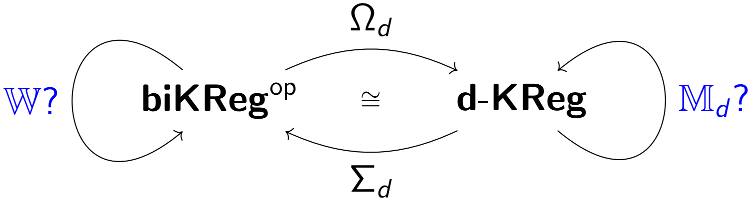


- $I \colon (X, \tau, \leq) \mapsto (X, \tau_+, \tau_-)$ where
  $\tau_+ = \mathrm{Up}(X, \leq) \cap \tau \qquad \tau_- = \mathrm{Down}(X, \leq) \cap \tau$

- $J \colon D \mapsto (\mathrm{Idl}(D), \mathrm{Filt}(D), \mathrm{con}_D, \mathrm{tot}_D)$ where

$$
\begin{aligned}
(I, F) \in \mathrm{con}_D \quad &\text{iff} \quad \forall i \in I, f \in F : i \leq f \\
(I, F) \in \mathrm{tot}_D \quad &\text{iff} \quad I \cap F \neq \emptyset
\end{aligned}
$$


𝕎?
biKReg^op
Ω_d
≅
Σ_d
d-KReg
𝕄_d?



$\mathbb{W}\colon (X;\, \tau_+, \tau_-) \mapsto (\mathcal{K}_c X;\, \mathbb{V}\tau_+, \mathbb{V}\tau_-)$ where

1. $\mathcal{K}_c X$ = compact convex subsets of X

   (Note: $(\leq_{\tau_+}) = (\geq_{\tau_-})$)

2. $\mathbb{V}\tau_+$ is generated by $\boxtimes U_+$, $\diamondplus U_+$ (for all $U_+ \in \tau_+$)
3. $\mathbb{V}\tau_-$ is generated by $\boxtimes U_-$, $\diamondplus U_-$ (for all $U_- \in \tau_-$)

Free d-frame construction

For $(B_+, \approx_+, B_-, \approx_-, \mathsf{con}_1, \mathsf{tot}_1)$ where

1. $L_+ = \mathbb{F}\mathrm{r}\langle\, B_+ \,\rangle/_{\approx_+}$
2. L_- defined similarly
3. $\mathsf{con}_1, \mathsf{tot}_1 \subseteq B_+ \times B_-$

Theorem

$(L_+, L_-; \mathtt{CON}\langle \mathsf{con}_1 \rangle, \mathtt{TOT}\langle \mathsf{tot}_1 \rangle)$ *is a d-frame if*

Closure under con-*operations in* $L_+ \times L_-$

Closure under tot-*operations in* $L_+ \times L_-$

Free d-frame construction

For $(B_+, \approx_+, B_-, \approx_-, \mathsf{con}_1, \mathsf{tot}_1)$ where

1. $L_+ = \mathbb{F}\mathrm{r}\langle\, B_+ \,\rangle/_{\approx_+}$
2. L_- defined similarly
3. $\mathsf{con}_1, \mathsf{tot}_1 \subseteq B_+ \times B_-$

Theorem

$(L_+, L_-; \mathtt{CON}\langle \mathsf{con}_1 \rangle, \mathtt{TOT}\langle \mathsf{tot}_1 \rangle)$ *is a d-frame if*

$\uparrow \mathbb{DL}_{\vee,\wedge} \langle \mathsf{tot}_1 \rangle$

$\mathbb{DCPO}_{\bigsqcup^{\uparrow}} \langle \downarrow \mathbb{DL}_{\vee,\wedge} \langle \mathsf{con}_1 \rangle \rangle$

Free d-frame construction

For $(B_+, \approx_+, B_-, \approx_-, \mathsf{con}_1, \mathsf{tot}_1)$ where

1. $L_+ = \mathbb{F}\mathrm{r}\langle\, B_+ \,\rangle/_{\approx_+}$
2. L_- defined similarly
3. $\mathsf{con}_1, \mathsf{tot}_1 \subseteq B_+ \times B_-$

$\forall \alpha \in \mathsf{con}, \beta \in \mathsf{tot}$:
$\alpha_+ = \beta_+$ or $\alpha_- = \beta_-$ $\implies \alpha \sqsubseteq \beta$

Theorem

$(L_+, L_-; \mathtt{CON}\langle \mathsf{con}_1 \rangle, \mathtt{TOT}\langle \mathsf{tot}_1 \rangle)$ *is a d-frame if*

- $\alpha \in \mathsf{con}_\vee$, $\beta \in \mathsf{tot}_\wedge$, $\beta_+ \leq \alpha_+ \implies \alpha_- \leq \beta_-$
- $(L_+ \times B_-) \cap {\downarrow}\mathsf{con}_{\wedge,\bigvee} \subseteq {\downarrow}\mathsf{con}_\vee$

(+ symmetric variants)

Vietoris functor for d-frames

$$\mathbb{M}_d \colon (L_+, L_-; \mathsf{con}, \mathsf{tot}) \mapsto (\mathbb{M}L_+, \mathbb{M}L_-;\ \mathtt{CON}\langle \mathsf{con}_1 \rangle, \mathtt{TOT}\langle \mathsf{tot}_1 \rangle)$$

where

$$\mathsf{tot}_1 = \{(\Box a, \Diamond b), (\Diamond a, \Box b) : (a, b) \in \mathsf{tot}\}$$
$$\mathsf{con}_1 = \{(\Box a, \Diamond b), (\Diamond a, \Box b) : (a, b) \in \mathsf{con}\}$$

Vietoris functor for d-frames

$$\mathbb{M}_d \colon (L_+, L_-; \mathsf{con}, \mathsf{tot}) \mapsto (\mathbb{M}L_+, \mathbb{M}L_-;\ \mathtt{CON}\langle \mathsf{con}_1 \rangle, \mathtt{TOT}\langle \mathsf{tot}_1 \rangle)$$

where

$$\mathsf{tot}_1 = \{(\Box a, \Diamond b), (\Diamond a, \Box b) : (a, b) \in \mathsf{tot}\}$$
$$\mathsf{con}_1 = \{(\Box a, \Diamond b), (\Diamond a, \Box b) : (a, b) \in \mathsf{con}\}$$

$\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$ and $\Diamond 0 = 0$

because $(a, b) \in \mathsf{con}$ mimics $a \wedge b = 0$

Facts

1. $\mathbb{M}_d$ is comonadic
2. If $\mathcal{L}$ is regular, zero-dimensional or compact regular then also $\mathbb{M}_d\mathcal{L}$ is.
3. Points of $\mathbb{M}_d\mathcal{L}$ (i.e. $\Sigma_d(\mathbb{M}_d\mathcal{L})$), are in bijection with $\alpha \in L_+ \times L_-$ such that
   (A+) $(\alpha_+ \vee u_+, \alpha_-) \in \mathrm{tot} \implies (u_+, \alpha_-) \in \mathrm{tot}$
   (A−) $(\alpha_+, \alpha_- \vee u_-) \in \mathrm{tot} \implies (\alpha_+, u_-) \in \mathrm{tot}$
4. $\mathbb{W} \circ \Sigma_d \cong \Sigma_d \circ \mathbb{M}_d$
5. $\implies \mathrm{Coalg}(\mathbb{W})^{\mathrm{op}} \cong \mathrm{Alg}(\mathbb{M}_d)$

Conclusion

- Free constructions of d-frames.
- $\mathbb{W}$ and $\mathbb{M}_d$ are generalisations of $\mathbb{V}$ and $\mathbb{M}$ for all our $\mathcal{X}^{\mathsf{op}} \cong \mathcal{A}$
- Domain Theory: because **biKReg** is equivalent to the category of stably compact spaces, we obtained an algebraic dual of $\mathbb{V}$ for stably compact spaces (open problem)

Conclusion

- Free constructions of d-frames.
- $\mathbb{W}$ and $\mathbb{M}_d$ are generalisations of $\mathbb{V}$ and $\mathbb{M}$ for all our $\mathcal{X}^{\mathrm{op}} \cong \mathcal{A}$
- Domain Theory: because **biKReg** is equivalent to the category of stably compact spaces, we obtained an algebraic dual of $\mathbb{V}$ for stably compact spaces (open problem)

Thank you for your attention!

Axioms of d-frames

- con $= \uparrow$con and tot $= \downarrow$tot
- con and tot are $(\wedge, \vee)$-subalgebras of $L_+ \times L_-^{\mathrm{op}}$
- con is DCPO-closed
- $\forall \alpha \in$ con, $\beta \in$ tot:
  $\alpha_+ = \beta_+$ or $\alpha_- = \beta_- \implies \alpha \sqsubseteq \beta$

Topological properties

	frames	d-frames
a^* :	$\bigvee\{x \mid x \wedge a = 0\}$	$\bigvee\{x \in L_- \mid (a, x) \in \mathrm{con}\}$
$a \lhd b$:	$b \vee a^* = 1$	$(b, a^*) \in \mathrm{tot}$

Regularity: $a = \bigvee\{x \mid x \lhd a\}$

Zero-dimensionality: $a = \bigvee\{x \mid x \lhd x \leq a\}$

Compactness:

For all $U \subseteq L$:

$$\bigvee U = 1 \implies \exists F \subseteq_{\mathrm{fin}} U \text{ s.t. } \bigvee F = 1$$

For all $\mathcal{U} \subseteq L_+ \times L_-$:

$$\bigsqcup \mathcal{U} \in \mathrm{tot} \implies \exists \mathcal{F} \subseteq_{\mathrm{fin}} \mathcal{U} \text{ s.t. } \bigsqcup \mathcal{F} \in \mathrm{tot}$$