Prime injective S-acts

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In this article, we define and study prime injectivity which is a generalization of $\mathcal{M}$-injectivity and investigate Skornjakov criterion respect to prime injectivity of acts. We charecterize the behaviour of the property considered under well-known constructions such as product, coproduct and direct sum. Ultimately, among the following results it is proved that an S-act is prime injective if and only if it is a prime-absolute retract if and only if it has no prime-essential extension.

1 Prime injective acts

In this section we define a generalization of injectivity of S-acts and we are going to study some behaviour of it.

Definition 1.1. (1) An act A is said to be *prime injective, if for any prime monomorphism $g : B \to C$, any homomorphism $f : B \to A$ can be lifted to a homomorphism $\bar{f} : C \to A$, such that $\bar{f}g = f$.*
(2) *An act A is said to be weakly prime injective, if it is injective relative to embeddings of all prime ideals into S.*
(3) *An S-act A is called to be f-g prime injective (cyclic prime injective), whenever for each prime homomorhism $g : F \to C$ from finitely generated (cyclic) act F to an act C, and for any homomorphism $f : F \to A$ there exists a homomorphism $h : C \to A$ such that $hg = f$.*

It is clear every injective act is prime injective act and all prime injective acts are weakly prime injective and each $\mathcal{M}$-injective act is a (weakly) prime injective act.

Proposition 1.2. *An act A is weakly prime injective if and only if for any homomorphism $f : I_S \to A$, where $I \subseteq S$ is a prime right ideal. there exists an element $a \in A$ such that $f(s) = as$ for every $s \in I$.*

Lemma 1.3. *The following statements are equivalent for monoid S.*

(1) *Every prime ideal of S is a retract of S.*

(2) *Every prime ideal of S is weakly prime injective.*

Recall that every cofree act is injective. Now we can say every cofree act is prime injective with the similary proof of theorem 3.1.5 of [4]. It is implies that every act can be embedded into a prime injective act. It means that the category of **S-act** has enough prime injective acts.

Lemma 1.4. *Every prime injective act contains a zero.*

Note that the category **S-act** is complete and cocomplete and has all products, coproducts, pushouts, pullbacks.

Proposition 1.5. *Let $\{A_i : i \in I\}$ be a family of S-acts. Then*

(1) $\prod_{i\in I} A_i$ is prime injective (f-g prime injective, cyclic prime injective) if and only if A_i's are prime injective (f-g prime injective, cyclic prime injective) for all $i \in I$.

(2) If the coproduct $\coprod_{i\in I} A_i$ is prime injective (f-g prime injective, cyclic prime injective), then each A_i is prime injective (f-g prime injective, cyclic prime injective) act.

Proposition 1.6. *Each direct sum of f-g prime injective (cyclic prime injective) acts is f-g prime injective (cyclic prime injective) .*

The converse of part (2) of Proposition 1.5, is not necessarily true in general. But we will show in Proposition 1.9, its converse is true for special S.

Theorem 1.7. *Assume an act A contain a zero θ. A is prime injective if and only if it is injective relative to all inclusions prime subact of cyclic acts.*

Definition 1.8. A monoid S is called *left prime reversible if $I \cap J \neq \emptyset$ for any prime right ideals I and J of S.*

Proposition 1.9. *The following statements are equivalent for any monoid S*

(1) *All coproducts of prime injective right acts are prime injective.*

(2) *$\{x, y\}$ is prime injective where x, y are fixed elements.*

(3) *S is left prime reversible.*

Theorem 1.10. *Pushouts transfer prime monomorphisms.*

Next theorem is one of the most interesting theorems about injectivity of S-acts with respect to any subclass of prime monomorphisms. This was proved by P. Berthiaume in [3], for injective acts and B. Banaschewski [2] has proved it for $\mathcal{M}$-injective acts when $\mathcal{M}$ is subclass of monomorhisms.

Theorem 1.11. *Let S be a semigroup. The following are equivalent for an S-act A:*

(1) A is prime injective.

(2) A is a prime-absolute retract.

(3) A has no prime-essential extension.

Theorem 1.11 immediately implies

Corollary 1.12. *For every right S-act there exists an prime injective hull.*

We are going to show that absolutely pure acts are absolutely prime-retract and then by Theorem 1.11, they are prime injective acts.

Theorem 1.13. *Every absolutely pure acts are prime injective.*

References

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