Extended Contact Logic

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1 Introduction

Region Connection Calculus is a formalism for reasoning about the relation of contact between regions in topological spaces [5]. Its role in artificial intelligence and computer science stems from the importance of spatial informations in systems for natural language understanding, robotic navigation, etc [2]. After the introduction of Region Connection Calculus, Contact Logic and its different variants have been proposed [1, 4, 6]. Most of them are based on the binary predicate of ordinary contact which holds between regular closed subsets A and B iff $A \cap B \neq \emptyset$ ("regions A and B are in contact"). Recently, a ternary predicate of extended contact has been introduced which holds between regular closed subsets A, B and C iff $A \cap B \subseteq C$ ("regions A and B are jointly bounded by region C"). Remark that two regions are in ordinary contact iff they are not jointly bounded by the empty region. Moreover, the interest to consider the new ternary relation of extended contact lies in the possibility it gives to define the unary predicate of internal connectedness. See [3, Chapter 2] for details. In this note, we introduce the syntax and the semantics of Extended Contact Logic. Then, we give an axiomatization of the set of all valid formulas this semantics gives rise to. Finally, we prove the decidability of the set of all theorems this axiomatization gives rise to.

2 Syntax and semantics

Let VAR be a countable set of variables (p, q, etc). The set TER of all terms (α, β, etc) is defined by

- $\alpha ::= p \mid 0 \mid \alpha^\star \mid (\alpha + \beta)$.

Reading terms as regions, the constructs 0, * and $+$ should be regarded as the empty region, the complement operation and the union operation. The set FOR of all formulas (φ, ψ, etc) is defined by

- $\varphi ::= \alpha \leq \beta \mid (\alpha, \beta) \rhd \gamma \mid \perp \mid \neg \varphi \mid (\varphi \vee \psi)$.

For $\leq$ and $\rhd$, we propose the following readings: $\alpha \leq \beta$ can be read "region α is contained in region β", $(\alpha, \beta) \rhd \gamma$ can be read "regions α and β are jointly bounded by region γ". We will write $\alpha \equiv \beta$ for $(\alpha \leq \beta \wedge \beta \leq \alpha)$. Terms and formulas are interpreted in topological models, i.e. structures of the form (X, τ, V) where (X, τ) is a topological space and V is a valuation on (X, τ), i.e. a map associating with every term α a regular closed subset $V(\alpha)$ of (X, τ) such that

- $V(0) = \emptyset$,

- $V(\alpha^\star) = Cl_\tau(X \setminus V(\alpha))$ where Cl_τ denotes the closure operator in (X, τ),

- $V(\alpha + \beta) = V(\alpha) \cup V(\beta)$.

The connectives $\perp$, $\neg$ and $\vee$ being classically interpreted, the satisfiability of a formula φ in (X, τ, V) (in symbols $(X, \tau, V) \models \varphi$) is defined as follows:

- $(X, \tau, V) \models \alpha \leq \beta$ iff $V(\alpha) \subseteq V(\beta)$,

- $(X, \tau, V) \models (\alpha, \beta) \rhd \gamma$ iff $V(\alpha) \cap V(\beta) \subseteq V(\gamma)$.

We will say that the formula φ is valid (in symbols $\models \varphi$) iff for all topological models (X, τ, V), $(X, \tau, V) \models \varphi$.

3 Axiomatization and decidability

Let $\mathbb{L}_{min}$ be the Hilbert-style axiomatic system consisting of the inference rule of modus ponens and the following axioms:

sentential axioms: instances of tautologies of propositional classical logic,

identity axioms: $\alpha \equiv \alpha$, $\alpha \equiv \beta \to \beta \equiv \alpha$, $\alpha \equiv \beta \wedge \beta \equiv \gamma \to \alpha \equiv \gamma$,

congruence axioms: $\alpha \equiv \beta \to \alpha^\star \equiv \beta^\star$, $\alpha \equiv \beta \wedge \gamma \equiv \delta \to \alpha + \gamma \equiv \beta + \delta$,

Boolean axioms: $(\alpha + \beta) + \gamma \equiv \alpha + (\beta + \gamma)$, $\alpha + \beta = \beta + \alpha$, etc,

nondegenerate axiom: $0 \not\equiv 1$,

extended contact axioms: **(i)** $(\alpha, \beta) \rhd \gamma \to (\beta, \alpha) \rhd \gamma$, **(ii)** $\alpha \leq \gamma \to (\alpha, \beta) \rhd \gamma$, **(iii)** $(\alpha, \beta) \rhd \gamma \wedge (\alpha, \beta) \rhd \delta \wedge (\gamma, \delta) \rhd \epsilon \to (\alpha, \beta) \rhd \epsilon$, **(iv)** $(\alpha, \beta) \rhd \gamma \to \alpha \cdot \beta \leq \gamma$, **(v)** $(\alpha, \gamma) \rhd \delta \wedge (\beta, \gamma) \rhd \delta \to (\alpha + \beta, \gamma) \rhd \delta$.

The notion of proof in $\mathbb{L}_{min}$ is the standard one. All provable formulas will be called theorems of $\mathbb{L}_{min}$.

Proposition 1. *For all formulas φ, $\models \varphi$ iff φ is a theorem of $\mathbb{L}_{min}$.*

Proof. For the soundness, it suffices to check that the inference rule of modus ponens preserve validity and that all axioms of $\mathbb{L}_{min}$ are valid. For the completeness, a detour through a semantical interpretation of terms and formulas in extended contact algebras as the ones studied in [3, Chapter 2] can be done. $\square$

Proposition 2. *The set of all theorems of $\mathbb{L}_{min}$ is decidable.*

Proof. A detour through a semantical interpretation of terms and formulas in relational structures as the ones studied in [1] and an associated finite model property can be done. $\square$

Nevertheless, the exact complexity of the set of all theorems of $\mathbb{L}_{min}$ is not known.

References

[1] Balbiani, P., Tinchev, T., Vakarelov, D.: *Modal logics for region-based theories of space.* Fundamenta Informaticæ **81** (2007) 29–82.

[2] Cohn, A., Renz, J.: *Qualitative spatial representation and reasoning.* In: *Handbook of Knowledge Representation.* Elsevier (2008) 551–596.

[3] Ivanova, T.: *Logics for Relational Geometric Structures: Distributive Mereotopology, Extended Contact Algebras and Related Quantifier-Free Logics.* Sofia University (2016) Doctoral thesis.

[4] Kontchakov, R., Pratt-Hartmann, I., Wolter, F., Zakharyaschev, M.: *Spatial logics with connectedness predicates.* Logical Methods in Computer Science **6** (2010) 1–43.

[5] Randell, D., Cui, Z., Cohn, A.: *A spatial logic based on regions and connection.* In: *Proceedings of the Third International Conference on Principles of Knowledge Representation and Reasoning.* Morgan Kaufman (1992) 165–176.

[6] Vakarelov, D.: *Region-based theory of space: algebras of regions, representation theory, and logics.* In: *Mathematical Problems from Applied Logic. Logics for the XXIst Century. II.* Springer (2007) 267–348.