Theories of relational lattices

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The natural join and inner union of tables in relational databases can be algebraically modeled as the meet and the join operations in a class of lattices, the class of relational lattices. The connection between these lattices and databases is well illustrated in previous work on the subject, see [8, 3]. We recall here only the mathematical definition of these lattices and discuss some recent advances on their quasiequational and equational theories.

The set of functions from A to D—noted here D^A—can be endowed with the structure of a generalized ultrametric space where the distance takes values in the powerset algebra $P(A)$, see [5, 1]. Namely, define the distance between $f, g \in D^A$ by $\delta(f, g) := \{\, a \in A \mid f(a) \neq f(g) \,\}$. A subset $X \subseteq D^A$ is α-closed if $\delta(f, g) \subseteq \alpha$ and $g \in X$ implies $f \in X$; a pair $(\alpha, X) \in P(A) \times D^A$ is closed if X is α-closed; the closed pairs form a Moore family on $P(A) \times P(D^A)$. The relational lattice $\mathsf{R}(D, A)$ is, up to isomorphism, the lattice of closed pairs of $P(A) \times P(D^A)$.

It was proved in [3] that the quasiequantional theory of relational lattices, over the signature which contains the lattices operations $\wedge, \vee$ as well as an additional constant H (the header constant), is undecidable. We recently refined this result and proved that the quasiequational theory of relational lattices, over the pure lattice signature, is undecidable, [6, 7]. We actually proved there a stronger statement:

Theorem 1. *It is undecidable whether a finite subdirectly irreducible lattice can be embedded into a relational lattice.*

The proof is a reduction from the coverability problem for $S5$ universal product frames, see [2]. It also allows us to find a quasiequation that holds in all the finite $\mathsf{R}(E, A)$, but failing in some infinite $\mathsf{R}(D, A)$, with A finite. A universal product frame is a special dependent product, thus of the form $\prod_{a \in A} D_a$; with the same definition as above, we can give to dependent products the structure of a generalized ultrametric space. The reduction crucially relies on the following statement, whose proof appears in [6].

Theorem 2. *The spaces $(\prod_{a \in A} D_a, \delta)$ are, up to isomorphism, the pairwise-complete and spherically complete generalized ultrametric spaces.*

Using a result from [1] these spaces are, up to isomorphism, the injective objects in the category of generalized ultrametric spaces over $P(A)$.

Coming back to the theories of relational lattices, a natural aim is to move from quasiequations to equations and to relate equational theories of infinite relational lattices to the equational theories of the finite ones. Many informations can be deduced by analysing the functorial properties of the construction $\mathsf{R}(\_, \_)$. For $\psi : E \to D$, $\pi : A \to B$, and $(\alpha, X) \in \mathsf{R}(D, A)$, put

$$\mathsf{R}(D, \pi)(\alpha, X) := (\forall_\pi(\alpha), {\pi^*}^{-1}(X)) \,, \qquad \mathsf{R}(\psi, A)(\alpha, X) := (\alpha, \psi_*^{-1}(X)) \,.$$

Here, for $f \in D^A$, we have $\pi^*(f) = f \circ \psi$, $\psi_*(f) = \psi \circ f$, and $\forall_\pi$ is right adjoint to π^{-1}. Notice that $\mathsf{R}(D, \pi) \circ \mathsf{R}(\psi, A) = \mathsf{R}(\psi, A) \circ \mathsf{R}(D, \pi)$, so we can define $\mathsf{R}(\psi, \pi) := \mathsf{R}(D, \pi) \circ \mathsf{R}(\psi, A)$.

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Proposition 3. *The construction* $\mathsf{R}(\_,\_)$ *is a functor from* $\mathsf{Set}^{op} \times \mathsf{Set}$ *to* $\mathsf{cSL}_\wedge$*, the category of complete meet-semilattices and maps preserving all meets.*

For ψ and π as above, let $\ell_\psi : \mathsf{R}(E,A) \to \mathsf{R}(D,A)$ be left adjoint to $R(\psi,A) : \mathsf{R}(D,A) \to \mathsf{R}(E,A)$; let $\ell_\pi : \mathsf{R}(D,B) \to \mathsf{R}(D,A)$ be left adjoint to $\mathsf{R}(D,\pi) : \mathsf{R}(D,A) \to \mathsf{R}(D,B)$. The following two observations are crucial.

Proposition 4. *If* $E \neq \emptyset$ *and* $\psi : E \to D$ *is injective, then* ℓ_ψ *is injective and preserves all the meets. If* $\psi : A \to B$ *is surjective, then* ℓ_π *is injective and preserves all the meets.*

In particular, $\mathsf{R}(E,B)$ belongs to the variety generated by $\mathsf{R}(D,A)$ whenever $E \subseteq D$ and $A \subseteq B$. When all these sets are finite, it is possible to look at the combinatorial proprieties the OD-graphs to assert that the two varieties are not equal, see [4]. Using Proposition 2, we derive the following theorem, showing that, for equations, the situation is quite different from the one of quasiequations.

Theorem 5. *If* A *is finite, then* $\mathsf{R}(D,A)$ *belongs to the variety generated by all the finite* $\mathsf{R}(E,A)$*.*

Indeed, $\mathsf{R}(D,A)$ is an algebraic lattice, thus it is isomorphic to the ideal completion of the join-semilattice of its compact elements. Yet, this join-semilattice is the colimit of the diagram $\ell_{\psi_{E_0,E_1}}$ where $E_0 \subseteq E_1 \subseteq D$, E_0, E_1 are non-empty and finite, and ψ_{E_0,E_1} is the inclusion of E_0 into E_1. In particular this colimit is a lattice in the variety generated by the finite $\mathsf{R}(E,A)$. It is well known that the ideal completion of a lattice and the lattice satisfy same the same identities.

If A is infinite, then $\mathsf{R}(D,A)$ is not an algebraic lattice, yet something can be said when D is finite.

Theorem 6. *If* D *is finite, then* $\mathsf{R}(D,A)$ *lies in the variety generated by all the finite* $\mathsf{R}(D,B)$*.*

Let $\mathrm{Part}_{\mathfrak{f}}(A)$ be the set of finite partitions of A, and consider the canonical maps $\pi_Q : A \to Q$ with $Q \in \mathrm{Part}_{\mathfrak{f}}(A)$, as well as the maps $\pi_{Q,P} : Q \to P$, for $Q, P \in \mathrm{Part}_{\mathfrak{f}}(A)$ such that Q refines P, sending a block of Q to the block of P that contains it. The maps $\mathsf{R}(D,\pi_Q)$ induce a canonical map $\pi : \mathsf{R}(D,A) \to \lim_{Q\in\mathrm{Part}_{\mathfrak{f}}(A)} \mathsf{R}(D,Q)$ in the category $\mathsf{cSL}_\wedge$, where $\lim_{Q\in\mathrm{Part}_{\mathfrak{f}}(A)} \mathsf{R}(D,Q)$ is the inverse limit of the maps $\mathsf{R}(D,\pi_{Q,P})$. We argue that if D is finite, then π is injective and preserves finite joins. Now, $\lim_{Q\in\mathrm{Part}_{\mathfrak{f}}(A)} \mathsf{R}(D,Q)$ is an algebraic lattice, and the poset of its compact element can be identified with the colimit (in the category of join-semilattices) of the diagram $\ell_{\pi_{Q,P}} : \mathsf{R}(D,P) \to \mathsf{R}(D,Q)$, for $Q, P \in \mathrm{Part}_{\mathfrak{f}}(A)$ and Q refines P. As before, $\lim_{Q\in\mathrm{Part}_{\mathfrak{f}}(A)} \mathsf{R}(D,Q)$ belongs to the variety generated by the $\mathsf{R}(D,Q)$, that are finite. As $\mathsf{R}(D,A)$ embeds into $\lim_{Q\in\mathrm{Part}_{\mathfrak{f}}(A)} \mathsf{R}(D,Q)$, then the same holds of $\mathsf{R}(D,A)$.

If both D and A are infinite, then the canonical map π is not an embedding. The tools used to prove Theorems 3 and 4 allow us to identify a complete lattice R_ω—the limit $\lim_{Q\in\mathrm{Part}_{\mathfrak{f}}(A)} \mathsf{R}(D,Q)$—which is a unique generator for the variety generated by the finite $\mathsf{R}(E,B)$. The quest for a characterization of the equational theory of relational lattices might involve recognizing R_ω as a sublattice of $\mathsf{R}(D,A)$ and how equational properties extend from the smaller lattice to its envelope.

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