A proper Multi-type display calculus for Semi De Morgan Logic

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Semi De Morgan algebras form a variety of normal distributive lattice expansions [7] introduced by H.P. Sankappanavar [16] as a common abstraction of De Morgan algebras and distributive pseudocomplemented lattices. A fully selfextensional logic SDM naturally arises from semi De Morgan algebras, which has been studied from a duality-theoretic perspective [13], from the perspective of canonical extensions [15], and from a proof-theoretic perspective [14]. Related to the proof theoretic perspective, the G3-style sequent calculus introduced in [14] is shown to be cut-free. However, the proof of cut elimination is quite involved, due to the fact that, along with the standard introduction rules for conjunction and disjunction, this calculus includes also introduction rules under the scope of structural connectives. These difficulties can be explained by the fact that the axiomatization of SDM is not analytic inductive in the sense of [10, Definition 55], due to the presence of the following axioms

$$(a \wedge b)'' = a'' \wedge b'' \qquad a' = a'''.$$

In order to address these difficulties, an analytic calculus for SDM is introduced in [9], which is sound, complete, conservative, and enjoys cut elimination and subformula property proved by means of a general Belnap-style method.

This calculus is a proper multi-type display calculus according to the definition of [12, Definition A.1]. The methodology of multi-type calculi has been introduced in [8, 3], motivated by proof-theoretic semantic considerations [5], and further developed in [6, 4, 1, 11].

Our main insights come from algebra. Specifically, we introduce an equivalent representation of semi De Morgan algebras as the following heterogeneous algebras (in the sense of [2]): structures $\mathbb{H} = (\mathbb{L}, \mathbb{D}, f, h)$ such that:
$\mathbb{L}$ is a bounded distributive lattice,
$\mathbb{D}$ is a De Morgan algebra,
$h : \mathbb{L} \to \mathbb{D}$ is a surjective lattice homomorphism,
$f : \mathbb{D} \to \mathbb{L}$ is a finitely meet-preserving order embedding which preserves the bottom element,
$h(f(\alpha)) = \alpha$ for every $\alpha \in \mathbb{D}$.
We show that any semi De Morgan algebra $\mathbb{A}$ gives rise to one such heterogeneous algebra $\mathbb{A}^+$, and conversely any heterogeneous algebra $\mathbb{H}$ as above gives rise to one semi De Morgan algebra $\mathbb{H}_+$, so that

$$\mathbb{A} \cong (\mathbb{A}^+)_+ \qquad \mathbb{H} \cong (\mathbb{H}_+)^+.$$

This equivalence motivates a reformulation of the logic SDM into the multi-type language canonically interpreted in the heterogeneous algebras defined above. In this reformulation, all the axioms are analytic inductive. This makes it possible to obtain a proper multi-type display calculus for SDM by suitably generalizing the method introduced in [10].

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*This research is supported by the NWO Vidi grant 016.138.314, the NWO Aspasia grant 015.008.054, and a Delft Technology Fellowship awarded to the second author in 2013.

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