A duality for involutive bisemilattices

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It is a common trend in mathematics to study (natural) dualities for general algebraic structures and, in particular, for those arising from mathematical logic. The first step towards this direction traces back to the pioneering work by Stone for Boolean algebras [12]. Later on, Stone duality has been extended to the more general case of distributive lattices by Priestley [8], [9]. The two above mentioned are the prototypical examples of natural dualities and will be both recalled and constructively used in the present work.

A natural duality, in the sense of [2], is built using a schizophrenic object living in two different categories and has an intrinsic value: it is a way of describing the very same mathematical object from two different perspectives, the target category and its dual.

The starting point of our analysis is the duality established by Gierz and Romanowska [4] between distributive bisemilattices and compact totally disconnected partially ordered left normal bands with constants, which we refer to as GR spaces. Such duality is natural; however, its relevance mainly lies in the use of the technique of Płonka sums [6], [7], as an essential tool for proving the duality [11], [10].

Our aim is to provide a duality between the categories of involutive bisemilattices and certain topological spaces, here christened as GR spaces with involution. The former consists of a class of algebras introduced and extensively studied in [1] as algebraic semantics (although not equivalent[1]) for paraconsistent weak Kleene logic. Involutive bisemilattices are strictly connected to Boolean algebras as they are representable as Płonka sums of Boolean algebras.

The present work consists of two main results. On one hand, taking advantage of the Płonka sums representation in terms of Boolean algebras and Stone duality, we are able to describe the dual space of an involutive bisemilattice as a strongly inverse system of Stone spaces (the use of this terminology is borrowed from [5]). On the other hand, we generalize Gierz and Romanowska duality by considering GR spaces with involution as an additional operation: the duality cannot be constructed using the usual techniques for natural dualities. As a byproduct of our analysis we get a topological description of *strongly inverse systems* of Stone spaces.

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