The periodic sequence property

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In 1984, Wim Ruitenburg [19] published a surprising result[1] about the intuitionistic propositional calculus (IPC). It does not seem well-known: one of the few researchers making extensive use of it was the late Sergey Mardaev [9–13]; apart from it, some recent references quoting Ruitenburg's paper include Ghilardi et al. [6] or Humberstone's monograph [7]. Moreover, most of these references use it in the context of definability (eliminability) of fixpoints, where it is just one of possible lines of attack (the other being via uniform interpolation [16]; see [6] for a discussion). The property established by Ruitenburg deserves more attention though: to begin with, it turns out to be a natural generalization of *local finiteness*.

Consider a propositional formula A. Fix a propositional variable p, which can be thought of as representing the context hole or the argument of A taken as a polynomial (other propositional variables being additional constants). Given any other formula B, write $A(B)$ for the result of substituting B for p. Also, write $A \equiv_L B$ for $\vdash_L A \leftrightarrow B$. Now define the obvious iterated substitution operation $A^0(p) := p, A^{n+1}(p) := A(A^n(p))$. Such a sequence turns almost immediately into a cycle modulo $\equiv_{\mathsf{CPC}}$:

Lemma 1 ([19], Lemma 1.1)**.** *For any A,* $\quad A(p) \equiv_{\mathsf{CPC}} A^3(p)$.

The above observation can be reformulated as asserting that CPC has *uniformly globally periodic sequences* (ugps). A logic L has this property if there exist b, c s.t. for any formula A, $A^b(p) \equiv_L A^{b+c}(p)$. However, ugps has still a rather strong logical form: two existential quantifiers preceding an universal one. Hence one can consider changing the order of quantifiers to weaken the property:

(eventually) periodic sequences:

	globally	locally
uniformly	$\exists b. \quad \exists c. \quad \forall A. \quad A^b(p) \equiv_L A^{b+c}(p)$	$\exists c. \quad \forall A. \quad \exists b. \quad A^b(p) \equiv_L A^{b+c}(p)$
parametrically	$\exists b. \quad \forall A. \quad \exists c. \quad A^b(p) \equiv_L A^{b+c}(p)$	$\forall A. \quad \exists b. \quad \exists c. \quad A^b(p) \equiv_L A^{b+c}(p)$

So, do standard non-classical propositional calculi, IPC in particular, have at least *plps* (*parametrically locally periodic sequences*)?[2] To begin with, we have an obvious observation:

Lemma 2. *Any locally finite logic has plps.*

It is, however, well-known that IPC is not locally finite: even in one propositional variable, there are infinitely many nonequivalent formulas. And one can show that (uniformly or parametrically) globally periodic sequences would be too much to expect, at least when formulas are allowed to contain other variables than p itself [19, §2]. But we do have

Theorem 3 ([19], Theorem 1.9)**.** IPC *has the ulps property: for any A, there exists b s.t. $A^b(p) \equiv_{\mathsf{IPC}} A^{b+2}(p)$. Moreover, b is linear in the size of A.*

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[1] I would like to thank Albert Visser for attracting my attention to this work and for his comments on this abstract.

[2] Ruitenburg himself was using the term *finite order*.

In fact, Ruitenburg's theorem is effective: the proof provides an algorithm to compute b in question.[3] Moreover, as the periodic sequence property (in all its incarnations) transfers from sublogics to extensions in the same signature (just like local finiteness and unlike uniform interpolation), we also get that all superintuitionistic logics (*si-logics*) have ulps. This shows that unlike local finiteness, ulps does *not* guarantee the fmp, or even Kripke completeness.

As it turns out, however, finding other natural examples of logic enjoying plps without local finiteness is a very challenging task. First let us consider intuitionistic or classical normal modal logics (with $\Box$ only), with superscript $\cdot^{\mathsf{cl}}$ denoting the CPC propositional base:

Theorem 4. *A normal extension of* $\mathsf{K4}^{\mathsf{cl}}$ *has plps iff it is locally finite.*

Corollary 5. *All extensions of* $\mathsf{K}^{\mathsf{int}}_{\Box}$ *contained in either* $\mathsf{S4Grz.3}^{\mathsf{cl}}$ (*including, for example,* K^{cl}, $\mathsf{K4}^{\mathsf{cl}}$, $\mathsf{S4}^{\mathsf{cl}}$, T^{cl}, $\mathsf{K4}^{\mathsf{int}}_{\Box}$, $\mathsf{T}^{\mathsf{int}}_{\Box}$, $\mathsf{S4}^{\mathsf{int}}_{\Box}$, $\mathsf{S4Grz.3}^{\mathsf{int}}_{\Box}$ *or* $\mathsf{S4Grz.3}^{\mathsf{int}}_{\Box}$) *or* $\mathsf{GL.3}^{\mathsf{cl}}$ (*including, for example,* $\mathsf{GL}^{\mathsf{cl}}$, $\mathsf{GL}^{\mathsf{int}}_{\Box}$ *or* $\mathsf{GL.3}^{\mathsf{int}}_{\Box}$) *fail to have locally periodic sequences.*[4]

Some intuitionistic modal logics of computational interest have "degenerate" classical counterparts and hence Corollary 5 cannot be used to disprove they have periodic sequences. This includes $\mathsf{S}^{\mathsf{int}}_{\Box} := \mathsf{K}^{\mathsf{int}}_{\Box} \oplus A \to \Box A$, i.e., the Curry-Howard logic of *applicative functors*, also known as *idioms* [14]. Its classical counterpart S^{cl} and all its two consistent proper extensions are finite logics enjoying ulps. In contrast, not only does $\mathsf{S}^{\mathsf{int}}_{\Box}$ have uncountably many propositional extensions, but the failure of plps remains a common phenomenon among them:

Theorem 6. *No sublogic of* $\mathsf{KM.3}^{\mathsf{int}}_{\Box}$, *also denoted as* $\mathsf{KM}_{\mathsf{lin}}$ [4] *has parametrically locally periodic sequences; this in particular applies to* $\mathsf{SL.3}^{\mathsf{int}}_{\Box} := \mathsf{S}^{\mathsf{int}}_{\Box} \oplus \mathsf{GL.3}^{\mathsf{int}}_{\Box}$, $\mathsf{SL}^{\mathsf{int}}_{\Box} := \mathsf{S}^{\mathsf{int}}_{\Box} \oplus \mathsf{GL}^{\mathsf{int}}_{\Box}$ *or* $\mathsf{S}^{\mathsf{int}}_{\Box}$.

To contrast this with Theorem 4, note that $\mathsf{KM.3}^{\mathsf{int}}_{\Box}$, the propositional fragment of the logic of the Mitchell-Bénabou logic of the *topos of trees* [2, 4, 8], is prefinite (*pretabular*). Turning to substructural logics:

Theorem 7. *The product logic* Π, *the infinite valued Łukasiewicz logic* Ł_∞ *or the logic of the* heap model *of* BBI (*boolean logic of bunched implications* [3, 15, 17, 18]) *fail to have plps. Consequently, the property fails in all their sublogics, including* $(\mathsf{In}-)\mathsf{FL}_{(\mathsf{ew})}$, *multiplicative-additive fragment of linear logic* MALL (*and its intuitionistic fragment* IMALL) *and fuzzy logics like* BL *or* MTL.[5]

Presently, I am running out of ideas how to obtain an example of a natural non-locally-finite logic with plps which is not a si-logic. Here are the remaining lines of attack I can think of:

Open Problem 1. *Do any extensions of the relevance logic* R *have periodic sequences without being locally finite? How about the propositional lax logic* $\mathsf{PLL}^{\mathsf{int}}_{\Box}$*?*

For the latter case, note that si-logics can be identified with extensions of $\mathsf{PPL}^{\mathsf{int}}_{\Box}$ satisfying $p \leftrightarrow \Box p$, so the question here is if Ruitenburg's result can be extended in a *nontrivial* way. And at any rate, we need an in-depth algebraic investigation why plps tends to collapse to local finiteness so often—and why varieties of Heyting algebras do not follow the trend.

[3] A formally verified proof in the Coq proof assistant allowing computation of b using either programming features of Coq itself or via extraction to other languages is available at `git://git8.cs.fau.de/ruitenburg1984`, with a web front end at `https://git8.cs.fau.de/redmine/projects/ruitenburg1984`.

[4] The reader is referred to the extensive literature [1, 8, 20–23] for basic information about intuitionistic modal logics, including axiomatizations of systems mentioned in this theorem.

[5] See Galatos et al. [5] for substructural systems mentioned in the statement of this theorem.

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