Arithmetic interpretation of the monadic fragment of intuitionistic predicate logic and Casari's formula

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By a celebrated theorem of Solovay, the Gödel-Löb logic GL is the modal logic of the provability predicate of Peano Arithmetic PA. This entails arithmetic interpretation of the intuitionistic propositional calculus IPC as was shown by Goldblatt, by Boolos, and by Kuznetsov and Muravitsky in the late 1970's and early 1980's. To see this, first use the Gödel translation t to embed IPC into the modal logic Grz, and then use the splitting translation sp—that maps $\Box\varphi$ to $\varphi \wedge \Box\varphi$—to embed Grz into GL as in the following diagram.

$$\mathsf{IPC} \xrightarrow{\quad \mathsf{t} \quad} \mathsf{Grz} \xrightarrow{\quad \mathsf{sp} \quad} \mathsf{GL}$$

$$\mathsf{IPC} \vdash \varphi \qquad \text{iff} \qquad \mathsf{Grz} \vdash \mathsf{t}(\varphi) \qquad \text{iff} \quad \mathsf{GL} \vdash \mathsf{sp}(\mathsf{t}(\varphi))$$

Finally, use Solovay's theorem to interpret GL into PA. The aim of this talk is to lift the above correspondences to monadic extensions of the logics in question completing the work of Esakia [2, 3]. To motivate the exact statement, we recollect some obstacles one encounters when trying to extend the above correspondences to the predicate setting. Let QIPC, QGrz, and QGL be the full predicate extensions of IPC, Grz, and GL, respectively. As was shown by Montagna [9], the analogue of Solovay's theorem is no longer true for QGL. Regarding the remaining correspondences, the situation seems at least severely more complicated than in the propositional case. While it is a well-known result of Kripke [8] that QIPC is complete with respect to Kripke frames, neither QGL nor QGrz is complete with respect to Kripke frames (see [9] and [5]). So the standard proofs for the propositional case do not extend to the predicate setting since they make use of Kripke semantics for IPC, Grz, and GL, respectively.

Unlike the full predicate logics, their one-variable fragments often behave much nicer. We will refer to them as monadic fragments. Bull [1] showed that the intuitionistic bi-modal logic MIPC axiomatizes the monadic fragment of QIPC (by interpreting $\Box$ and $\Diamond$ as the universal and existential quantifiers, respectively). Esakia [2] introduced the monadic fragments MGL and MGrz of QGL and QGrz, respectively, and conjectured that—in contrast to the full predicate case—Solovay's theorem extends to MGL. This conjecture was verified by Japaridze [6, 7].

The (extended) Gödel translation embeds MIPC into MGrz. However, the (extended) splitting translation fails to embed MGrz into MGL. To remedy this, Esakia adopted Casari's formula Cas—a modified version of the rule of universal quantification—to the monadic setting.

$$(\mathsf{Cas}) \qquad \forall x[(p(x) \to \forall x p(x)) \to \forall x p(x)] \to \forall x p(x)$$

Let MCas be the monadic version of Casari's formula and let

$$\mathsf{M}^+\mathsf{IPC} = \mathsf{MIPC} + \mathsf{MCas} \quad \text{and} \quad \mathsf{M}^+\mathsf{Grz} = \mathsf{MGrz} + \mathsf{t}(\mathsf{MCas}).$$

Esakia anticipated that the desired correspondence can be lifted to $\mathsf{M}^+\mathsf{IPC}$, $\mathsf{M}^+\mathsf{Grz}$ and, MGL (note that $\mathsf{MGL} \vdash \mathsf{sp}(\mathsf{t}(\mathsf{MCas}))$, so $\mathsf{M}^+\mathsf{GL} = \mathsf{MGL}$):

$$\begin{array}{ccccc} \mathsf{M}^+\mathsf{IPC} & \xrightarrow{\quad \mathsf{t} \quad} & \mathsf{M}^+\mathsf{Grz} & \xrightarrow{\quad \mathsf{sp} \quad} & \mathsf{MGL} \\ \mathsf{M}^+\mathsf{IPC} \vdash \varphi & \text{iff} & \mathsf{M}^+\mathsf{Grz} \vdash \mathsf{t}(\varphi) & \text{iff} & \mathsf{MGL} \vdash \mathsf{sp}(\mathsf{t}(\varphi)) \end{array}$$

The goal of this talk is to verify this. Our main technical contribution consists in proving the finite model property (fmp) for the logics $\mathsf{M}^+\mathsf{IPC}$ and $\mathsf{M}^+\mathsf{Grz}$. We prove this by carefully modifying the selective filtration method for MIPC as presented in [4, Section 10.3].

Theorem 1. *The logics* $\mathsf{M}^+\mathsf{IPC}$ *and* $\mathsf{M}^+\mathsf{Grz}$ *have the fmp.*

Using that finite $\mathsf{M}^+\mathsf{IPC}$-frames coincide with finite $\mathsf{M}^+\mathsf{Grz}$-frames, we can now show:

Corollary 2. *The Gödel translation embeds* $\mathsf{M}^+\mathsf{IPC}$ *into* $\mathsf{Q}^+\mathsf{Grz}$*, and the splitting translation embeds* $\mathsf{M}^+\mathsf{Grz}$ *into* MGL*.*

Using Theorem 1, we can also draw the connection to the full predicate case. Let

$$\mathsf{Q}^+\mathsf{IPC} = \mathsf{QIPC} + \mathsf{Cas} \quad \text{and} \quad \mathsf{Q}^+\mathsf{Grz} = \mathsf{QGrz} + \mathsf{t}(\mathsf{Cas}).$$

Using a semantic criterion from [10], we derive:

Corollary 3. $\mathsf{M}^+\mathsf{IPC}$ *is the monadic fragment of* $\mathsf{Q}^+\mathsf{IPC}$ *and* $\mathsf{M}^+\mathsf{Grz}$ *is the monadic fragment of* $\mathsf{Q}^+\mathsf{Grz}$*.*

Recall that by Japaridze's results, MGL is arithmetically complete. We therefore obtain arithmetic interpretation of the one-variable fragment of $\mathsf{Q}^+\mathsf{IPC}$.

References

[1] R. A. Bull. MIPC as the formalisation of an intuitionist concept of modality. *J. Symbolic Logic*, 31:609–616, 1966.

[2] L. L. Esakia. Provability logic with quantifier modalities. In *Intensional logics and the logical structure of theories (Russian) (Telavi, 1985)*, pages 4–9. "Metsniereba", Tbilisi, 1988.

[3] L. L. Esakia. Quantification in intuitionistic logic with provability smack. *Bull. Sect. Logic*, 27:26–28, 1998.

[4] D. M. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-dimensional modal logics: theory and applications*, volume 148 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 2003.

[5] S. Ghilardi. Incompleteness results in Kripke semantics. *J. Symbolic Logic*, 56(2):517–538, 1991.

[6] G. K. Japaridze. Arithmetic completeness of provability logic with quantifier modalities. *Soobshch. Akad. Nauk Gruzin. SSR*, 132(2):265–268, 1988.

[7] G. K. Japaridze. Decidable and enumerable predicate logics of provability. *Studia Logica*, 49(1):7–21, 1990.

[8] S. A. Kripke. Semantical analysis of intuitionistic logic. I. In *Formal Systems and Recursive Functions (Proc. Eighth Logic Colloq., Oxford, 1963)*, pages 92–130. North-Holland, Amsterdam, 1965.

[9] F. Montagna. The predicate modal logic of provability. *Notre Dame J. Formal Logic*, 25(2):179–189, 1984.

[10] H. Ono and N.-Y. Suzuki. Relations between intuitionistic modal logics and intermediate predicate logics. *Reports on Mathematical Logic*, 22:65–87, 1988.