Generalized bunched implication algebras

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1 Introduction

A *residuated lattice* is an algebra $(A, \wedge, \vee, \cdot, \backslash/, 1)$, where $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid and $x \cdot y \leq z$ iff $x \leq y/z$ iff $y \leq x\backslash z$, for all $x, y, z \in A$. If $\cdot$ is equal to $\wedge$, then $\mathbf{A}$ is called a Brouwerian algebra (these are the bottom-free subreducts of Heyting algebras) and in this case we write $x \to y$ for $x\backslash y$; it also follows that $y/x = x\backslash y$ so we suppress this operation. A *generalized bunch implication algebra*, or *GBI-algebra*, is an algebra $\mathbf{A} = (A \wedge, \vee, \cdot, \backslash, /, 1, \to, \top)$, where $(A, \wedge, \vee, \cdot, \backslash/, 1)$ is a residuated lattice and $(A, \wedge, \vee, \to, \top)$ is a Brouwerian algebra.

Commutative and bounded GBI-algebras are known as BI-algebras and they form algebraic semantics for bunched implication logic. The later is of interest in computer science and it is used in proving correctness of concurrent programs.

2 Decidability and FMP

We present a Gentzen calculus for GBI, which enjoys cut elimination; the proof proceeds by considering distributive residuated frames, two-sorted structures that form relational semantics for GBI-algebras. This allows to prove cut elimination for any extension of GBI with equations over the signature $\{\vee, \wedge, \cdot, 1\}$. In particular we recover the known cut elimination for the system of bunched implication (BI) logic as a special case.

We further prove decidability of GBI. The decidability of the $\to$-free fragment can be shown by defining an appropriate complexity measure on sequents. We demonstrate that this complexity measure fails to be decreasing for the $\to$ rules of the calculus and also discuss the difficulties in finding any complexity measure that is decreasing. Nevertheless, we prove the decidability by defining a binary graph on the sequent tree of each sequent and showing that certain aspects of these graphs are reduced as we trace a proof upwards. This can be combined with the fact that we can restrict our attention to special types of sequents in a proof (3-reduced) to put a bound on the overall search space, thus yielding decidability.

We further prove that from the termination of the proof search we not only obtain decidability but also the finite model property. We do that by creating a distributive residuated frame whose dual GBI-algebra is finite.

3 Congruences

Congruences in residuated lattices are determined by certain subsets (in a way similar to the fact that congruences in groups are determined by normal subgroups). Given $a, x \in A$ we define $\rho'_a x = ax/a$ and $\lambda'_a(x) = a\backslash xa$ (which are akin to conjugates in group theory). A subset is called *normal* if it is closed under ρ'_a and λ'_a for all $a \in A$. A *(RL)-deductive filter* of a residuated lattice $\mathbf{A}$ is defined to be a normal upward closed subset of A that is closed under

multiplication and meet and contains the element 1. It is known that if θ is a congruence on $\mathbf{A}$ then $\uparrow[1]_\theta$, the upset of the equivalence class of 1, is a deductive filter. Conversely, if F is a deductive filter of a residuated lattice $\mathbf{A}$, then the relation θ_F is a congruence on $\mathbf{A}$, where $a\ \theta_F\ b$ iff $a\backslash b \wedge b\backslash a \in F$.

Note that if A is a Brouwerian or a Heyting algebra, then deductive filters are usual lattice filters.

We prove that the GBI-deductive filters are exactly the RL-deductive filters that are further closed under $r_{a,b}(x) = (a \to b)/(xa \to b)$ and $s_{a,b}(x) = (a \to bx)/(a \to b)$, for all a, b.

Alternatively, congruences are characterized by their equivalence classes of $\top$. These are usual lattice filters that are closed under the $t_{a,b}(x) = a/b \to (a \wedge x)/b$, $t'_{a,b}(x) = b/a \to b/(a \wedge x)$, $u_{a,b}(x) = a/(b \wedge x) \to a/b$, $u'_{a,b}(x) = (b \wedge x)\backslash a \to b\backslash a$, $v_{a,b}(x) = ab \to (a \wedge x)b$ and $v'_{a,b}(x) = ab \to a(b \wedge x)$ for all a, b.

4 Examples

A *weak conucleus* on a residuated lattice $\mathbf{A}$ is an interior operator σ on $\mathbf{A}$ such that $\sigma(x)\sigma(y) \leq \sigma(xy)$, for all $x, y \in \mathbf{A}$. Then $\sigma[\mathbf{A}] = (\sigma[A], \wedge_\sigma, \vee, \cdot, \backslash_\sigma, /_\sigma)$ is a residuated lattice-ordered semigroup, where $x \bullet_\sigma y = \sigma(x \bullet y)$, where $\bullet \in \{\wedge, \backslash, /\}$; we are interested in the cases where this algebra also has an identity element e and hence $(\sigma[A], e)$ is a residuated lattice. A *topological weak conucleus* on a GBI-algebra $\mathbf{A}$ is a conucleus on both the residuated lattice and the Brouwerian algebra reducts of $\mathbf{A}$.

Given a residuated lattice $\mathbf{A}$ and a positive idempotent element p we define the map σ_p by $\sigma_p(p) = p\backslash x/p$. Then σ_p is a topological weak conucleus (which we call the *double division conucleus by* p), and p is the identity element $\sigma_p(\mathbf{A})$; we denote the resulting residuated lattice $(\sigma_p(\mathbf{A}), p)$ by $p\backslash\mathbf{A}/p$. If $\mathbf{A}$ is involutive then so is $p\backslash\mathbf{A}/p$ and the latter is a subalgebra of $\mathbf{A}$ with respect to the operations $\wedge, \vee, \cdot, +, \sim, -$. Recall that an *involutive* residuated lattice is an expansion of a residuated lattice with an extra constant 0 such that $\sim(-x) = x = -(\sim x)$, where $\sim x = x\backslash 0$ and $-x = 0/x$; we also define $x + y = \sim(-y \cdot -x)$.

Given a poset $\mathbf{P} = (P, \leq)$, we define the set $Wk(\mathbf{P})$ of all binary relations R on P such that $a \leq b\ R\ c \leq d$ implies $a\ R\ d$, for all $a, b, c, d \in P$; these are called $\leq$-*weakening relations*. In other words $Wk(\mathbf{P}) = \mathcal{O}(\mathbf{P} \times \mathbf{P}^\partial)$, where $\mathcal{O}$ denotes the downset operator, and it supports a structure of a GBI-algebra, under union and intersection, and composition of relations.

We note that we also have that $Wk(\mathbf{P}) \cong Res(\mathcal{O}(\mathbf{P}))$, where for a complete join semilattice $\mathbf{L}$, $Res(\mathbf{L})$ denotes the residuated lattice of all residuated maps on $\mathbf{L}$; recall that a map on f on a poset $\mathbf{P}$ is called *residuated* if there exists a map f^* on P such that $f(x) \leq y$ iff $x \leq f^*(y)$, for all $x, y \in P$.

Given a poset $\mathbf{P} = (P, \leq)$, we set $\mathbf{A} = Rel(P)$, to be the involutive GBI algebra of all binary relations on the set P. Note that $p = \leq$ is a positive idempotent element of $\mathbf{A}$. It is easy to see that $p\backslash\mathbf{A}/p$ is exactly $Wk(\mathbf{P})$. Since $\mathbf{A}$ is an involutive GBI-algebra, so is $Wk(\mathbf{P})$.