Making fuzzy description logic more general

Petr Hájek*
Institute of Computer Science
Academy of Sciences of the Czech Republic
182 07 Prague, Czech Republic

Abstract

A version of fuzzy description logic based on the basic (continuous t-norm based) fuzzy predicate logic BL is presented. Problems of satisfiability, validity and subsumption of concepts are discussed and reduced to problems of fuzzy *propositional* logic known to be decidable for any continuous t-norm. For Łukasiewicz t-norm some stronger results are obtained.

1 Introduction

Description logic has became extensively studied in last years; the handbook [2] is an up-to-date reference. (See also [1].) It is known to be a fragment of classical predicate calculus and has several variants; we restrict ourselves to that named ALC. Its language consists of unary predicates $A_1, \ldots$ (atomic concepts), binary predicates $R_1, \ldots$ (roles), variables $x_1, \ldots$ and constants $a_1, \ldots$. Concepts are built from atomic concepts using connectives $\wedge, \vee, \neg$ and quantification constructs denoted $\forall R.C$, $\exists R.C$. Think of instances of atomic concepts as of formulas $A_i(t)$ (t being a fixed constant or variable); this extends to (instances of) concepts defined by connectives. Furthermore,

$$(\forall R.C)(t) \text{ is to be read as } (\forall y)(R(t,y) \rightarrow C(y))$$

$$(\exists R.C)(t) \text{ as } (\exists y)(R(t,y) \wedge C(y)).$$

(For each construct always a new variable y is used.) Axioms may have the form $C(a_i)$ (C a concept), $R(a_i, a_j)$ (R a role) and $(\forall x)(C(x) \rightarrow D(x))$ (subsumption of concepts). We simplify a little and will disregard (apriori) axioms. Typical *problems* are:

satisfiability: does $C(a)$ have a model?
subsumption: is $(\forall x)(C(x) \rightarrow D(x))$ true in all models?

*This paper is a part of the Project number 1ET100300419 of the Program of the Information Society of the Thematic Program II of the National Research Program of the Czech Republic.

Both problems are *decidable* (and are PSPACE-complete).

It is very natural, due to the intended applications, to make this system (and related systems) fuzzy.[1] The handbook [2] refers to papers [13, 11, 12]. These are rather non-trivial papers but they restrict themselves to a rather poor apparatus of fuzzy logic: just min and max as $\wedge$ and $\vee$, $1-x$ (Łukasiewicz negation) and $\max(1-x, y)$ (Kleene-Dienes implication)[2]. The use of "minimalistic" fuzzy apparatus has some technical advantages but makes the system(s) unnecessarily weak.

Technically most developed are *t*-norm based fuzzy logics; the corresponding propositional and predicate calculi are elaborated in the monograph [4] (basic fuzzy logic BL and several stronger logics, notably Łukasiewicz, Gödel and product logics).

The reader is assumed to be familiar with these logics, their standard and general semantics. We just recall the definitions of Łukasiewicz, Gödel and product *t*-norm and its residuum for $x \in [0, 1]$,

$$x *_{Ł} y = \max(x + y - 1, 0), x *_G y = \min(x, y), x *_\Pi y = x \cdot y$$

$$x \Rightarrow_{Ł} y = x \Rightarrow_G y = x \Rightarrow_\Pi y = 1 \text{ for } x \leq y,$$

$$x \Rightarrow_{Ł} y = 1 - x + y, x \Rightarrow_G y = y, x \Rightarrow_\Pi y = y/x \text{ for } x > y.$$

Note that in this paper we deal only with the *standard* semantics, given always by a fixed continuous t-norm; our set of truth values is just the unit real interval $[0, 1]$. Our aim is to reformulate the description logic (ALC-like) using this apparatus, formulate some typical problems and prove some preliminary results; we also correct an (unessential) error in [13]. Our main technical contribution is the systematic reduction of problems of satisfiability and validity of concepts to problems of satisfiability and validity of some theories in *propositional* logic, problems that are known to be decidable. The paper should serve as a programme of further deeper research, using more deeply results from the existing literature on description logic in combination with results of mathematical fuzzy logic.

2 Main notions; an algorithm

Definition 1 (1) Given atomic concepts $A_1, \ldots, A_n$ and roles $R_1, \ldots, R_m$, *concepts* are defined as follows: atomic concepts are concepts; $\overline{0}$ (falsum) is a concept; if C, D are concepts then so are $C\&D$, $C \to D$; if C is a concept and R a role then $\forall R.C$ and $\exists R.C$ are concepts.

[1] Trivial example: the concept of a person, whose all children are tall; and of a person having at least one tall child.

[2] [2] refers also to an older paper [14] dealing with different logical systems; [8] considers use of various conjunctions and implications.

(2) For each variable or constant t, the *instance* $C(t)$ of a concept is defined as follows: $A(t)$ is the atomic formula in which A is understood as unary predicate; $\overline{0}(t)$ is $\overline{0}$; $(C \to D)(t)$ is $C(t) \to D(t)$, similarly $(C\&D)(t)$. Furthermore, $(\forall R.C)(t)$ is $(\forall y)(R(t,y) \to C(y))$ and $(\exists R.C)(t)$ is $(\exists y)(R(r,y)\&C(y))$ where y is a variable not occurring in $C(t)$.

(3) *Nesting of quantifiers* in C (or $C(t)$) is defined inductively: $nest(C) = 0$ for atomic C, $nest(C\&D) = \max(nest(C), nest(D))$, $nest(\forall R.C) = nest(C)+1$, similarly for $\to, \exists$.

(4) *Generalized atoms* are instances of quantified concepts, i.e. $(\forall R.C)(t)$, $(\exists R.C)(t)$. Evidently, each instance $C(t)$ of any concept is a propositional combination of some atoms and generalized atoms; the latter will be called *generalized atoms of* $C(t)$.

Remark 1 Recall definable connectives $\neg, \wedge, \vee$; thus if C, D are concepts then also $\neg C$, $C \wedge D$, $C \vee D$ are concepts.

Example 1 $\forall R_1.\exists R_2.C \,\&\, \forall R_2.D$ is a concept; its instance with a constant a is $(\forall R_1.\exists R_2.C)(a)\&(\forall R_2.D)(a)$, i.e.

$$[(\forall y)(R_1(a,y) \to (\exists z)(R_2(y,z)\&C(z))]\&(\forall y)(R_2(a,y) \to D(y)].$$

Generalized atoms of our instance of this concept are $(\forall R_1.\exists R_2.C)(a)$ and $(\forall R_2.D)(a)$.

Definition 2 Assign to each generalized atom $G(a)$ (G atomic concept or a quantified concept, a a constant) a propositional variable $p_{G,a}$. Extend this to all instances $C(a)$ of concepts by defining

$prop(\overline{0}(a)) = \overline{0}$,

$prop(((C\&D)(a)) = prop(C(a))\&prop(D(a))$ and similarly for $(C \to D)(a)$.

If T is a set of formulas (instances of concepts) let $prop(T)$ be the set of all $prop(\alpha)$, $\alpha \in T$.

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Now we describe an algorithm (in the style of cited papers on description logic) assigning to each (closed) instance $C_0(a_0)$ of a concept C_0 with a constant a_0 a finite "witnessing" theory $T(C_0(a_0))$ important in what follows. The construction takes steps $0, \ldots, n$ where n is the degree of nesting of quantifiers in C_0. Each step will *process* some generalized atoms, *produce* some new constants and new axioms and *transfer* some instances of concepts for processing in the next step. The instances produced in step i will have degree of nesting $n - i$; after step n is completed the algorithm will stop. The theory T in question consists of all axioms produced in all steps. In details:

Definition 3 Given $C_0(a_0)$, step 0 just transfers it to further processing in step 1; the constant a_0 has level 0. For $i > 0$ step i processes generalized atoms of formulas transferred from step $i - 1$; they have the form $(QR.C)(b)$ where Q is

$\forall$ or $\exists$, R is a role, C a concept with nesting degree $\leq n-i$ and b is a constant of level i. For each generalized atom α in question do the following:

If α is $(\forall R.C)(b)$ then produce a new constant d_α and the axiom

$$(\forall R.C)(b) \equiv (R(b, d_\alpha) \to C(d_\alpha)).$$

If α is $(\exists R.C)(B)$ then produce d_α and axiom

$$(\exists R.C)(b) \equiv (R(b, d_\alpha) \& C(d_\alpha)).$$

In both cases call the generated axiom the *witnessing axiom* for α and d_α a constant *belonging to* R, b.

After this is done for all α in question (in the present step) consider each α once more and do the following:

If α is $(\forall R.C)(b)$ and d_β is any constant belonging to R, b and different from d_α, produce the axiom

$$(\forall R.C)(b) \to (R(b, d_\beta) \to C(d_\beta)).$$

Similarly for α being $(\exists R.C)(b)$, produce

$$(R(b, d_\beta) \& C(d_\beta)) \to (\exists R.C)(b).$$

All produced axioms and constants are of level i. Formulas $C(d_\alpha), C(d_\beta)$ from the axioms produced at this level and having positive nesting of quantifiers are transferred for processing in the next level (if any). Observe that finitely many constants, axioms and formulas for next processing have been generated and that all the generated formulas for next processing have nesting of quantifiers $\leq n-i$.

Clearly, the algorithm stops after the n steps are performed. As stated above, T is the set of all axioms produced. We shall investigate properties of T and of its propositional counterpart $prop(T)$.

From now on, fix a continuous t-norm $*$ (and its residuum $\Rightarrow$); this gives semantics of your fuzzy propositional and predicate calculus. In particular, each evaluation e of propositional atoms by truth values from the real interval $[0, 1]$ defines uniquely the truth value $e_*(\varphi)$ of each propositional formula built from these atoms.

For each evaluation e of atoms of formulas in $prop(T)$ define an interpretation $\mathbf{M}_e$ of the predicate language as follows: M_e is the set of all constants occurring in formulas in T. For each atomic concept A let $r_A(d) = e(prop(A(d)))$ if $prop(A(d))$ occurs in $prop(T)$; otherwise $r_A(d) = 0$. Similarly, $r_R(c, d) = e(prop(R(c, d))$ if $prop(R(c, d))$ occurs in $prop(T)$, otherwise $= 0$.

Lemma 1 Let $*$ be a continuous t-norm. Assume that e is a $*$-model of $prop(T)$. Then for each formula $\alpha = (\forall R.C)(c)$ with $prop(\alpha)$ occurring in $prop(T)$,

$$e(prop(\alpha)) = \|\alpha\|^*_{\mathbf{M}_e} = \|R(c, d_\alpha) \to C(d_\alpha)\|^*_{\mathbf{M}_e}$$

and for $\alpha = (\exists R.C)(c)$ with $prop(\alpha)$ occurring in $prop(T)$,

$$e(prop(\alpha)) = \|\alpha\|^*_{\mathbf{M}_e} = \|R(c, d_\alpha) \& C(d_\alpha)\|_{\mathbf{M}_e}$$

Consequently, $\mathbf{M}_e$ is a finite $*$-model of T (in the sense of predicate logic) and $e_*(prop(C_0(a_0)) = \|C_0(a_0)\|^*_{\mathbf{M}_e}$.

Proof: The proof is by induction on the degree of nesting of α. Assume α is $(\forall R.C)(b)$; the proof for $\exists R.C)(b)$ is dual. First let nesting degree of α be 1, i.e. C does not contain any quantifiers. Then

$$\|(\forall R.C)(b)\|^*_{\mathbf{M}_e} = \|(\forall y)(R(b, y) \to C(y)\|^*_{\mathbf{M}_e} = \\ = \inf_{d \in M_e} (\|R(b, d)\| \to C(d)\|^*_{\mathbf{M}_e}).$$

Observe that $prop(R(b, d))$ occurs in $prop(T)$ if *and only if* d belongs to R, b in the sense of our algorithm; then $\|R(b, d)\|^*_{\mathbf{M}_e} = e(prop(R(b, d))$ and $\|C(d)\|^*_{\mathbf{M}_e} = e(prop(C(d))$ by the definition of $\mathbf{M}_e$ since $prop(R(b, d))$ is a propositional variable and $prop(C(d))$ is a propositional combination of propositional variables (due to the assumption on nesting.)

In particular, for the constant d_α we get

$$e(prop(\forall R.C)(b))) = \|R(b, d_\alpha) \to C(d_\alpha)\|^*_{\mathbf{M}_e},$$

for any other constant d belonging to R, b we get

$$e(prop(\forall R.C)(b))) \leq \|R(b, d) \to C(d)\|^*_{\mathbf{M}_e}$$

(since e gives value 1 to all axioms concerning R, b) and if d does not belong to R, b then $e(prop(R(b, d))) = 0$, $\|R(b, d) \to C(d)\|^*_{\mathbf{M}_e} = 1$) and we get again

$$e(prop(\forall R.C)(b))) \leq \|R(b, d) \to C(d)\|^*_{\mathbf{M}_e}.$$

Thus $\inf_{d \in \mathbf{M}_e} \|R(b, d) \to C(d)\|^*_{\mathbf{M}_e} = e(prop((\forall R.C)(b)))$ and hence

$$e(prop(\forall R.C)(b)) = \|(\forall R.C)(b)\|^*_{\mathbf{M}_e}.$$

Similarly for nesting $(i + 1)$, nesting i having been verified; now $C(d)$ is a propositional combination of atoms and generalized atoms with nesting $\leq i$ and for all such formulas the e-value equals to the $*$-value in $\mathbf{M}_e$. □

Observe that the $\mathbf{M}_e$-value of $(\forall R.C)(c)$ is $*$-*witnessed* in the following sense: for any $\mathbf{M}$, a closed formula $(\forall y)\varphi(y)$ is *-witnessed if $\|(\forall y)\varphi(y)\|^*_{\mathbf{M}}$ (which is the infimum of values of M-instances of $\varphi(y)$) is in fact equal to $\|\varphi(u)\|^*_{\mathbf{M}}$ for some $u \in M$, thus the infimum is in fact the *minimum.* Similarly for witnessed $\|(\exists y)\varphi(y)\|^*_{\mathbf{M}}$ (*maximum*). More generally, $(\forall y)\varphi(y, x_1, \ldots, x_n)$ is *-witnessed in $\mathbf{M}$ if for any choice $v_1, \ldots, v_n \in M$ of values of $x_1, \ldots, x_n$, the truth value $\|(\forall y)\varphi(y, v_1, \ldots, v_n)\|^*_{\mathbf{M}}$ equals $\|\varphi(u, v_1, \ldots, v_n)\|^*_{\mathbf{M}}$ for some $u \in M$. Similarly for $(\exists y)\varphi(y, x_1, \ldots, x_n)$ and maximum. $\mathbf{M}$ is **-witnessed* if all quantified formulas are *-witnessed in $\mathbf{M}$.

Lemma 2 Conversely, let $\mathbf{M}$ be a witnessed interpretation; expand the language by constants from T and expand $\mathbf{M}$ by interpreting the added constants as respective witnesses. For atoms α of formulas in $prop(T)$ define $e_{\mathbf{M}}(\alpha) = \|\alpha\|^*_{\mathbf{M}}$. Then $e_{\mathbf{M}}$ is a $*$-model of $prop(T)$ and hence $e_{\mathbf{M}}(C_0(a_0)) = \|C_0(a_0)\|^*_{\mathbf{M}}$.

Proof: The constant a_0 is assumed to be interpreted in $\mathbf{M}$; in step $i+1$ assume the constants of level i have been interpreted in $\mathbf{M}$. For each generalized atom $\alpha = (\forall R.C)(c)$ processed in step i choose an element $u \in M$ witnessing that atom and interpret d_α by u (calling the expansion of $\mathbf{M}$ by these constants again $\mathbf{M}$). This means that $\|(\forall R.C)(c) \equiv (R(c, d_\alpha) \to C(d_\alpha))\|_{\mathbf{M}} = 1$ and analogously for $\alpha = (\exists R.C)(c)$. Also the remaining axioms produced in this step are obviously true in $\mathbf{M}$. Thus setting, for each atom and generalized atom α, $e(prop(\alpha)) = \|\alpha\|^*_M$, we get $e_*(prop(\varphi)) = \|\varphi\|^*_{\mathbf{M}}$ for each propositional combination of atoms and generalized atoms occurring in T. In particular, the (expanded) model $\mathbf{M}$ is a model of T and e is a model of $prop(T)$. □

Remark 2 Let $\mathbf{M}$ be the (expanded) model as above and let $\mathbf{M'}$ be its restriction to the (interpretations of) all constants of all levels. Then evidently $\|C(d)\|^*_{\mathbf{M}} = \|C(d)\|^*_{\mathbf{M'}}$ for all formulas $C(d)$ processed in the construction of T (proof by induction on the complexity of C; in particular, $\|C_0(a_0)\|_{\mathbf{M}} = \|C_0(a_0)\|_{\mathbf{M'}}$ and obviously, $\mathbf{M'}$ is finite.

Corollary 1 If there is a $*$-witnessed $*$-model $\mathbf{M}$ with $\|C_0(a_0)\|^*_{\mathbf{M}} = v \in [0,1]$ then there is a finite $*$-model $\mathbf{M'}$ with $\|C_0(a_0)\|^*_{\mathbf{M'}} = v$. Consequently, if $(\forall x)C_0(x)$ is true in all finite models then it is true in all witnessed models (in particular, for Łukasiewicz logic $(\forall x)C_0(x)$ is then true in *all* models, see Sect. 3 for details).

Example 2 We show that in Gödel fuzzy logic we can have a concept satisfiable in degree 1 in an infinite model but having value 0 in each finite model (and in each witnessed model). The example in $(\neg\forall R.A)\,\&\,(\neg\exists R.\neg A)$. Let N be the set of natural numbers, let $a = 0, \|R(m,n)\| = 1$ iff $m = 0$ and $n > 0$, for $n > 0$ let $\|A(n)\| = \frac{1}{n}$. Then

$$\|(\forall x)(R(a,x) \to A(x)\| = \inf_n \|R(a,n) \to A(n)\| = 0, \text{ thus } \|\neg(\forall R.A)(a)\| = 1,$$

$$\|(\exists x)(R(a,x) \& \neg A(x)\| = \sup_n \|R(a,n) \& \neg A(n)\| = 0, \text{ thus } \|\neg(\exists R.\neg A)(a)\| = 1,$$

Now assume a model $\mathbf{M}$ in which an element a satisfies our concept in degree 1 and the formula $(\forall R.A)(a)$ is witnessed, i.e. for some $d \in M$, $\|R(a,d) \to A(d)\| = 0$. It follows $\|R(a,d)\| > 0$ and $\|A(d)\| = 0$ (Gödel implication!). Then $\|R(a,d) \wedge \neg A(d)\| = \|R(a,d) \wedge 1\| = \|R(a,d)\| > 0$, thus $\|(R(a,d) \wedge \neg A(d)\| = 0$ and $\|(\neg\exists R.\neg C)(a)\| = 0$, thus our concept has value 0, a contradiction.

Remark 3 Let T be a theory over $G\forall$ and φ a (closed) formula. Let $\|\varphi\|^G_T$ be $\inf\{\|\varphi\|^G_{\mathbf{M}} \,|\, \mathbf{M}$ a $[0,1]_G$-model of $T\}$. Observe that $\|\varphi\|^G_T$ is 0 or 1. Indeed, if there is an $\mathbf{M}$ such that $\|\varphi\|^G_M$ is an element of the open interval $(0,1)$ then applying an increasing $1-1$ mapping of $[0,1]$ onto itself you can produce an isomorphic model $\mathbf{M}'$ of T in which $\|\varphi|^G_{\mathbf{M}'}$, is as small (positive) as you like; thus $\inf\{\|\varphi\|^G_{\mathbf{M}}|\mathbf{M}$ an $[0,1]^*_G$-model of $T\} = 0$. This applies to the infimum of degrees of satisfiability of a concept: it is 1 or 0.

3 Satisfiability, validity, subsumption of concepts

Definition 4 A concept C is $*$-*satisfiable* if there is an interpretation $\mathbf{M}$ such that $\|C(a)\|^*_{\mathbf{M}} = 1$ (a being a constant); C is $*$-*valid* if for all $\mathbf{M}$, $\|(\forall x)C(x)\|^*_{\mathbf{M}} = 1$; in particular, C is $*$-*subsumed* by D if $C \to D$ is $*$-valid.

Note that C is $*$-valid iff for all $\mathbf{M}$, $\|C(a)\|^*_{\mathbf{M}} = 1$ where a is a constant (since $C(x)$ contains no constants). Here we allow no apriori axioms; if you would have some, a would have to be a new constant not occuring in your axioms. Our aim is to discuss decidability of the three notions just introduced. How can we use the algorithm from the preceding section? What is the impact of *witnessing* quantified formulas?

First, clearly each *finite* model is $*$-witnessed, whatever your t-norm $*$ is (finite suprema are maxima, analogously infima). Also clearly, there are infinite models with some formulas not witnessed: if $\mathbf{M}$ interprets a unary predicate A, for all elements $u \in M$ the value $\|A(u)\|_{\mathbf{M}}$ is positive but the infimum over all u is 0 then in $\mathbf{M}$ the formula $(\forall x)A(x)$ is not witnessed. Similarly for $(\exists x)A(x)$, if $\sup \|A(u)\|_{\mathbf{M}}$ is bigger than each $\|A(u)\|_{\mathbf{M}}$.

Here we mention an *error* in [13]: In the proof of Lemma 13 point (1) part 4 (page 10) the authors assume that in a model $(\mathfrak{A}, \tau)$ the formula $(\exists R.C)(x)$ has a value t and conclude that there is an element ξ in the model such that t is the minimum (conjunction) of the values of $R(x,\xi)$ and (ξ), i.e. that the formula $(\exists R.C)(x)$, thus $(\exists y)(R(x,y) \wedge C(y))$ has a witness. This may not be the case as we have seen. Below we shall see how to overcome this defect.

Lemma 3 For Łukasiewicz t-norm $*$ and each $*$-model $\mathbf{M}$ (of a given predicate language) there is an $*$-witnessed model $\mathbf{M}'$ such that $\mathbf{M}$ is a submodel of $\mathbf{M}'$ and $\mathbf{M}$ is elementarily equivalent to $\mathbf{M}'$ (in the sense that for any closed formula α, $\|\alpha\|^*_{\mathbf{M}} = 1$ iff $\|\alpha\|^*_{\mathbf{M}'} = 1$).

Proof: Clearly it is enough to prove it for one fixed formula $(\exists x)\varphi(x)$ (since in Łukasiewicz logic, $(\forall x)\varphi(x)$ is equivalent to $\neg(\exists x)\neg\varphi(x)$) and then to iterate the construction countably many times. We use the extension of Ł$\forall$ by rational truth constants $\bar{r}$ for any rational $r \in [0,1]$ called RPL (rational Pavelka logic) in [4]. Assume that the language of $\mathbf{M}$ contains names of all elements of M and the crisply interpreted equality predicate ($r_=(a,b) = 1$ iff $a = b$). The *rational theory* $RTh(\mathbf{M})$ of $\mathbf{M}$ is the union over all closed formulas α of the sets

$$L_\alpha = \{\bar{r} \to \alpha \,|\, r \leq \|\alpha\|^{\text{Ł}}_{\mathbf{M}}, r \text{ rational}\}$$

$$U_\alpha = \{\alpha \to \bar{r}, | \, r \leq \|\alpha\|^{\text{Ł}}_{\mathbf{M}}, r \text{ rational}\}.$$

In particular, $RTh(\mathbf{M})$ contains these sets for each atomic closed formula $P(m_1, \ldots m_n)$, P being a predicate and m_i being (names of) particular elements of M. $RTh(\mathbf{M})$ is a consistent theory over RPL$\forall$ and adding a new constant c and the witnessing axiom $(\exists x)\varphi(x) \equiv \varphi(c)$ we get a conservative extension $\hat{T}$ of $RTh(\mathbf{M})$ (see Appendix). By [4] 5.4.17 $\hat{T}$ has a standard model $\mathbf{M}'$. We may identify each element $m \in M$ with the corresponding element of $\mathbf{M}'$ (named with the same constant); this is a one-one relation due to the presence of crisp equality. And due to the presence of the axioms from L_α, U_α we see that $\|\alpha\|^{\text{Ł}}_{\mathbf{M}} = \|\alpha\|^{\text{Ł}}_{\mathbf{M}'}$ for each α. □

But e.g. for Gödel t-norm this fails: think of a model $\mathbf{M}$ in which $\|(\forall x)P(x)\|^G_{\mathbf{M}} = 0$ but $\|P(u)\|^G_{\mathbf{M}} > 0$ for all $u \in M$. Then $\|(\exists x)\neg P(x)\|^G_{\mathbf{M}} = 0$ (Gödel negation), hence no $\mathbf{M}' \supseteq \mathbf{M}$ elementarily equivalent to $\mathbf{M}$ can contain a witness for $(\forall x)P(x)$. The same for product logic and any logic having Gödel negation.

Corollary 2 (1) Over Łukasiewicz logic, a concept C is satisfiable iff it is satisfiable by a finite model iff the associated finite set $prop(C(a)) \cup prop(T(C(a)))$ is propositionally satisfiable.

(2) Over Łukasiewicz logic, a concept C is valid iff it is valid in all finite models iff the propositional theory $prop(T(C(a))$ entails $prop(C(a))$, i.e. each evaluation e of propositional variables which is an Ł-model of $prop(T(C(a)))$ gives $prop(C(a))$ the Ł-value 1.

Corollary 3 Over Łukasiewicz logic, the (standard) satisfiability and (standard) validity of a concept are decidable problems. (See the Appendix for details.)

Remark 4 (1) Propositional satisfiability of a formula (or a finite set of formulas) is NP in its size; but the size of the theory $prop(T)$ is *not* polynomial in the size of $C(a)$). Similarly for entailment and co-NP. Recall that satisfiability of a concept over classical logic is PSPACE-complete; thus our problems are PSPACE-hard too.

(2) Note that the "minimalistic" logic of min, max, $1 - x$ is a sublogic of Łukasiewicz logic and the Kleene-Dienes implication is definable; thus our proof can be adapted to get proofs of (known) facts on the "minimalistic" fuzzy ALC.

(3) Clearly there are concepts satisfiable over Łukasiewicz but not our classical logic; the simplest example is the concept $A \equiv \neg A$ (i.e. $(A \to \neg A)\&(\neg A \to A)$).

4 On arbitrary continuous t-norms

In the preceding section we focused our attention to Łukasiewicz logic; here we formulate consequences of our observations for a logic given by an arbitrary fixed continuous t-norm $*$. Here the reader is assumed to know the (Mostert-Shields) representation of $*$ as an ordered sum of copies of Łukasiewicz, Gödel and product t-norms. The ordered set of copies may not have a first element; if it does have a first element and if it is Łukasiewicz we say that $*$ *begins* by Ł. Every $*$ not beginning by Ł has Gödel negation ($\neg 0 = 1, \neg x = 0$ for $x > 0$). The following is a very useful observation (see [6]):

Lemma 4 (1) If $*$ begins by Ł then, for any propositional formula φ, φ is $*$-satisfiable iff it is Ł-satisfiable.

(2) If $*$ does not begin by Ł then for each φ, φ is $*$-satisfiable if it is satisfiable in the classical Boolean logic.

Theorem 1 Let $*$ begin by Ł. Then the following are equivalent, for any concept C:

(1) C is satisfiable by a witnessed $*$-model,

(2) C is satisfiable by a finite $*$-model,

(3) $prop(C(a)) \cup prop(T(C(a)))$ is $*$-satisfiable

(4) $prop(C(a)) \cup prop(T(C(a)))$ is Ł-satisfiable

Theorem 2 If $*$ does not begin by Ł then the following are equivalent for any concept C:

(1)–(3) as in the preceding theorem

(4) $prop(C(a)) \cup prop(T(C(a)))$ is boolean satisfiable.

Theorem 3 For any continuous t-norm $*$ an any concept C, the following are equivalent:

(1) C is valid in all witnessed $*$-models

(2) C is valid in all finite $*$-models

(3) $prop(T(C(a)))$ $*$-entails $prop(C(a))$.

All those there theorems are immediate consequences of our lemmas.

Corollary 4 For an arbitrary continuous t-norm $*$, witnessed $*$-satisfiability and witnessed $*$-validity of a concept is decidable.

This is because propositional $*$-satisfiability and propositional $*$-entailment are decidable, see the appendix.

Remark. Properties of witnessed models are further studied in the paper [5] (in preparation).

Problems. (1) Analyze computational complexity results from existing papers on (fuzzy) description logic and their impact to our approach.

(2) In particular, analyze the problem of *degree of satisfaction* of a concept, i.e. $\inf_{\mathbf{M}}\{\|C(a)\|^{\text{Ł}}_{\mathbf{M}}\}$

(3) Some proofs of decidability of problems of description logic used the fact that they can be expressed in classical predicate logic having only two object variables, which is decidable. Is Łukasiewicz predicate logic (Gödel, product logic, BL) with only two object variables decidable? (Observe that Łukasiewicz with two variables does *not* have finite model property: consider the formula

$$(\exists x)(P(x) \equiv \neg P(x))\&(\forall x)(\exists y)(P(x) \equiv (P(y)\&P(y))).$$

5 Appendix

First we discuss computational complexity of propositional $*$-satisfiability and $*$-entailment. Then we prove a theorem on witnessing axioms used for a proof of Lemma 3 above.

The fact that the set of all $*$-satisfiable is NP-complete was first proved for Łukasiewicz $*$ by Mundici [10] (see also [4] for an alternative proof). Thus for each $*$, we get the same result by Lemma 4 above. The co-NP completeness of all $*$-tautologies was proved for Łukasiewicz in [10]; for Gödel and product in [4] and for arbitrary $*$ in [7] as well as in [6]. For $*$-entailment the result seems to be new (but easy from known facts).

Theorem 4 For each continuous t-norm $*$, the set of all pairs (φ, ψ) of (propositional) formulas such that φ $*$-entails ψ is co-NP-complete.

Proof: For Łukasiewicz inspect in [4] the proof of the fact that Ł-tautologies are co-NP: modify the algorithm assigning to each formula ψ a mixed integer problem such that solutions of the problem give evaluation e such that $e(\psi) < 1$, getting a very similar algorithm assigning to each pair φ, ψ a MIP-problem whose solutions give $e(\varphi) = 1$ and $e(\psi) < 1$. For Gödel use classical deduction theorem (valid for $G\forall$) reducing entailment of ψ from φ to tautologicity of $\varphi \to \psi$. For product inspect the proof in [4] analogously to Łukasiewicz above. Having all this inspect the proof of Haniková from [7, 6] and make corresponding modifications. □

$*$

In the rest the reader is assumed to know basic facts on theories over the fuzzy predicate logic BL$\forall$ and stronger logics. Recall that T' is a *conservative extension* of T if $T' \supseteq T$ and each formula φ in the language of T provable in T' is provable in T.

Lemma 5 Each theory T over Łukasiewicz logic Ł$\forall$ has a conservative extension T' containing witnessing axioms for all closed formulas beginning by a quantifier.

Proof: Clearly, it is enough to show that adding just one witnessing axiom is conservative (and then iterate); moreover, it suffices to discuss only $\exists$ (due to definability of $\forall$ from $\exists$ in Ł$\forall$). Thus let c be a new constant and assume

$$T, (\exists x)\varphi(x) \to \varphi(c) \vdash \alpha$$

where α does not contain c. Then for some n,

$$T \vdash [(\exists x)\varphi(x) \to \varphi(c)]^n \to \alpha$$

thus for a new variable y,

$$T \vdash (\forall y)([(\exists x)\varphi(x) \to \varphi(y)]^n \to \alpha)$$

(just replace c by y in the proof)

$T \vdash (\exists y)[(\exists x)\varphi(x) \to \varphi(y))^n] \to \alpha$; then by [4] 5.1.18 (10),

$T \vdash [(\exists y)((\exists x)\varphi(x) \to \varphi(y)]^n \to \alpha$ and, by [4], 5.4. 15 (i),

$T \vdash [(\exists x)\varphi(x) \to (\exists y)\varphi(y)]^n \to \alpha$ (here Ł$\forall$-provability is used!), and

$T \vdash \alpha$, since the formula $[\ldots]$ is evidently provable. □

Our Example 2 above shows that adding a witnessing axiom may be non-conservative over $G\forall$ (and over $\Pi\forall$-the same example). Thus an analogon of Lemma 3 does not hold for $G\forall, \Pi\forall$; one has to work with witnessed models. (Recall once more that each finite model is witnessed; also if you work with Gödel logic over a fixed finite subset of truth values ($v \subseteq [0,1], 0 \in v, 1 \in v$) all models (finite or infinite) are witnessed. Also note in passing that BL$\forall$ (and hence each stronger logic) has the following conservation property.

Lemma 6 Let T be a theory such that $T \vdash (\exists x)\varphi(x)$, let c be a new constant. Then $T \cup \{\varphi(c)\}$ is a conservative extension of T.

Conclusion

We hope to have shown two things: description logic can profit from advanced fuzzy logic getting richer expressive possibilities (still decidable) and fuzzy logic may profit from description logic by getting new problems and inspirations.

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