

Topological Semantics for Modal Logic

Part I: Topological Completeness and Finite Model Property for S4

Igor Sedlár

Institute of Computer Science, CAS

Topology on a set X : A collection $\mathcal{T} \subseteq \mathcal{P}(X)$ such that

$$\mathbf{T(i)} \quad X \in \mathcal{T};$$

$$\mathbf{T(ii)} \quad U, V \in \mathcal{T} \implies U \cap V \in \mathcal{T};$$

$$\mathbf{T(iii)} \quad \{U_i\}_{i \in I} \subseteq \mathcal{T} \implies \left(\bigcup_{i \in I} U_i\right) \in \mathcal{T};^1$$

Topological space: $\langle X, \mathcal{T} \rangle$ where $\mathcal{T}$ is a topology on X . Elements of $\mathcal{T}$ are called ‘open sets’ of the topology. If $X \setminus A \in \mathcal{T}$, then A is a ‘closed set’.

Examples:

(a) Co-finite topology on X :

$$\mathcal{T} = \{U \subseteq X \mid U = \emptyset \text{ or } X \setminus U \text{ is finite}\}.$$

(b) Full up-set topology on a pre-ordered set $\langle X, \leq \rangle$:

$$\mathcal{T}_{\leq} = \{U \subseteq X \mid (\forall x, y \in X: x \in U \ \& \ x \leq y \implies y \in U)\}.$$

¹Hence, $\emptyset \in \mathcal{T}$.

Basis: A collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is a basis for a topology $\mathcal{T}$ on X iff

$$\mathcal{T} = \left\{ \bigcup \mathcal{A} \mid \mathcal{A} \subseteq \mathcal{B} \right\}$$

Basis Lemma 1

Let $\mathcal{B} \subseteq \mathcal{P}(X)$ and let $\mathcal{T}$ be a topology on X . The following are equivalent:

- (a) $\mathcal{B}$ is a basis for $\mathcal{T}$.
- (b) $\mathcal{B} \subseteq \mathcal{T}$ and for every $U \in \mathcal{T}$ there is $\mathcal{B}_U \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{B}_U$.
- (c) $\mathcal{B} \subseteq \mathcal{T}$ and for every $U \in \mathcal{T}$ and $x \in U$, there is $B_{x,U} \in \mathcal{B}$ such that $x \in B_{x,U} \subseteq U$.

Proof. (a $\implies$ b $\implies$ c) is clear. $\mathcal{B} \subseteq \mathcal{T}$ since obviously $\{B\} \subseteq \mathcal{B}$ for all $B \in \mathcal{B}$.

(c $\implies$ a) To prove that $\mathcal{T} \subseteq \left\{ \bigcup \mathcal{A} \mid \mathcal{A} \subseteq \mathcal{B} \right\}$, fix $U \in \mathcal{T}$. Clearly $U = \bigcup_{x \in U} B_{x,U}$. To prove the converse inclusion, take an arbitrary $\mathcal{A} \subseteq \mathcal{B}$. Since, $\mathcal{B} \subseteq \mathcal{T}$, we obtain $\bigcup \mathcal{A} \in \mathcal{T}$. □

Basis Lemma 2

A collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is a basis for *some* topology on X iff

B(i) For all $x \in X$ there is $B \in \mathcal{B}$ such that $x \in B$;

B(ii) For all $x \in X$, if $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then there is $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

Proof. ($\implies$) If $\mathcal{B}$ is a basis for $\mathcal{T}$, then B(i) holds by (c) of the Basis Lemma 1 and the fact that $X \in \mathcal{T}$. B(ii) follows from BL1(c) and the fact that $B_1 \cap B_2$ is open in $\mathcal{T}$.

($\impliedby$) Let $\mathcal{T} := \{U \subseteq X \mid \exists \{B_i\}_{i \in I} \subseteq \mathcal{B} : U = \bigcup_{i \in I} B_i\}$. This is a topology on X :

T(i) holds since $X = \bigcup \mathcal{B}$ by B(i). T(ii) holds since $(\bigcup_{i \in I} B_i) \cap (\bigcup_{j \in J} B_j) = \bigcup_{i \in I, j \in J} (B_i \cap B_j) = \bigcup \{C \in \mathcal{B} \mid C \subseteq B_i \cap B_j \text{ for some } i \in I, j \in J\}$ by B(ii). T(iii) holds since

$\bigcup_{j \in J} \left(\bigcup_{i \in I_j} B_i \right) = \bigcup_{k \in \bigcup_{j \in J} I_j} B_k$. Finally, $\mathcal{B}$ is a basis for $\mathcal{T}$ by definition. $\square$

Fact: If $\mathcal{B}$ satisfies B(i-ii), then it is a basis for a *unique* topology, called *the topology generated by $\mathcal{B}$* . However, one topology can have different bases.

Examples:

(a) Let $\langle X, \leq \rangle$ be a pre-ordered set. Let $\uparrow x = \{y \in X \mid x \leq y\}$. The collection $\mathcal{B} = \{\uparrow x \mid x \in X\}$ is a basis for the full up-set topology on X .

(b) Let $0 < \delta \in \mathbb{R}$. A δ -ball around $(x_1, x_2) \in \mathbb{R}^2$ is

$$B_\delta(x_1, x_2) = \left\{ (y_1, y_2) \in \mathbb{R}^2 \mid \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} < \delta \right\}.$$

$\{B_\delta(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^2, \delta > 0\}$ is a basis for the *usual topology* on $\mathbb{R}^2$. Generalises to $\mathbb{R}^n$.

(c) Let A be a finite alphabet and A^∞ the set of all finite and countably infinite strings over A . Let $w \sqsubseteq u$ iff w is a prefix of u . Let $\uparrow w := \{u \in A^\infty \mid w \sqsubseteq u\}$. Then $\mathcal{B} = \{\uparrow w \mid w \text{ finite}\}$ is a basis for the *Scott topology* on A^∞ .

(c) is not a special case of (a). If $u \in A^\infty$ is infinite, then $\{u\}$ is open in (a), but not in (c). Open sets in (c) need to be ‘verifiable’ by a finite prefix.

(d) Let $\mathbf{A}$ be a Boolean algebra and let $Uf(\mathbf{A})$ be the collection of ultrafilters on $\mathbf{A}$. Let $\hat{a} := \{F \in Uf(\mathbf{A}) \mid a \in F\}$. Then $\mathcal{B} = \{\hat{a} \mid a \in \mathbf{A}\}$ is a basis for a topology on $Uf(\mathbf{A})$, known as the *Stone topology*.

Interior of a set $P \subseteq X$, given a topology $\mathcal{T}$ on X :

$$\text{Int}(P) = \bigcup \{U \in \mathcal{T} \mid U \subseteq P\}.$$

(Largest open set contained in P .)

Interior Lemma

Int has the following properties in any topological space:

- (a) $\text{Int}(X) = X$;
- (b) $\text{Int}(P \cap Q) = \text{Int}(P) \cap \text{Int}(Q)$;
- (c) $\text{Int}(P) \subseteq P$;
- (d) $\text{Int}(P) \subseteq \text{Int}(\text{Int}(P))$.

Proof. (a) and (d) follow from the fact that $\text{Int}(U) = U$ for all open U . (c) is obvious. (b, $\subseteq$) follows from the fact that Int is monotonic. (b, $\supseteq$) follows from T(ii). □

Modal language: Based on Pr , a countable set of prop. variables. Boolean connectives $\neg, \wedge$ (the rest defined as usual); modal operator $\Box$ (the dual $\Diamond$ defined as usual, $\Diamond\phi := \neg\Box\neg\phi$). Set of formulas Fm .

The modal logic S4 is the smallest set of formulas in Fm that

- S4(i)** contains all classical tautologies and is closed under Modus ponens;
- S4(ii)** is closed under liftings $\hat{\sigma}: Fm \rightarrow Fm$ of all substitutions $\sigma: Pr \rightarrow Fm$;
- S4(iii)** is a normal modal logic; that is, contains all instances of $\Box(\phi \wedge \psi) \leftrightarrow (\Box\phi \wedge \Box\psi)$, is closed under Necessitation (if $\phi \in S4$, then $\Box\phi \in S4$) and is closed under Replacement of Equivalents (if $\phi \leftrightarrow \psi \in S4$, then $\Box\phi \leftrightarrow \Box\psi \in S4$);
- S4(iv)** contains all instances of $\Box\phi \rightarrow \phi$ (the T-axiom) and $\Box\phi \rightarrow \Box\Box\phi$ (the 4-axiom).

Topological model. $\mathbf{M} = \langle X, \mathcal{T}, \mathcal{V} \rangle$ where $\langle X, \mathcal{T} \rangle$ is a topological space and $\mathcal{V}: Pr \rightarrow \mathcal{P}(X)$ (valuation).

Satisfaction and truth-sets. Given $\mathbf{M} = \langle X, \mathcal{T}, \mathcal{V} \rangle$:

- $\mathbf{M}, x \models p$ iff $x \in \mathcal{V}(p)$;
- $\mathbf{M}, x \models \neg\phi$ iff $\mathbf{M}, x \not\models \phi$;
- $\mathbf{M}, x \models \phi \wedge \psi$ iff $\mathbf{M}, x \models \phi$ and $\mathbf{M}, x \models \psi$;
- $\mathbf{M}, x \models \Box\phi$ iff $\exists U \in \mathcal{T}: U \subseteq \|\phi\|_{\mathbf{M}} \text{ \& } x \in U$;

where $\|\phi\|_{\mathbf{M}} = \{y \in X \mid \mathbf{M}, y \models \phi\}$. Hence,

$$\|\Box\phi\|_{\mathbf{M}} = \text{Int}(\|\phi\|_{\mathbf{M}}).$$

(Consequently, $\|\Diamond\phi\|_{\mathbf{M}} = \text{Cl}(\|\phi\|_{\mathbf{M}}) := \bigcap \{C \subseteq X \mid \|\phi\|_{\mathbf{M}} \subseteq C \text{ is closed}\}.$)

Validity. $\mathbf{M} \models \phi$ iff $\|\phi\|_{\mathbf{M}} = X$. $\langle X, \mathcal{T} \rangle \models \phi$ iff $\langle X, \mathcal{T}, \mathcal{V} \rangle \models \phi$ for all $\mathcal{V}: Pr \rightarrow \mathcal{P}(X)$.

Theorem 1 (Soundness of S4)

If $\phi \in \text{S4}$, then $\mathbf{M} \models \phi$ for all topological models $\mathbf{M}$.

Proof. We need to show that $\{\phi \mid \mathbf{M} \models \phi \text{ for all } \mathbf{M}\}$ satisfies S4(i–iv):

S4(i–ii) are standard.

S4(iii) boils down to $\text{Int}(P \cap Q) = \text{Int}(P) \cap \text{Int}(Q)$ and $\text{Int}(X) = X$.

S4(iv) follows from $\text{Int}(P) \subseteq P$ and $\text{Int}(P) = \text{Int}(\text{Int}(P))$. □

S4-theory (consistent & maximally so): $\Gamma \subseteq Fm$ closed under Modus ponens such that $S4 \subseteq \Gamma$. Consistent: If $\phi \in \Gamma$, then $\neg\phi \notin \Gamma$. Maximally consistent: Γ consistent and if $\Gamma \subseteq \Gamma'$ consistent, then $\Gamma = \Gamma'$.

Lindenbaum Lemma (for S4)

If Γ is an S4-theory such that $\phi \notin \Gamma$, then there is a maximally consistent S4-theory Γ' such that $\Gamma \subseteq \Gamma'$ and $\neg\phi \in \Gamma'$.

Canonical S4-model: $M = \langle X, \mathcal{T}, \mathcal{V} \rangle$ where

- X is the set of all maximally consistent S4-theories, where
 $|\phi| := \{\Gamma \in X \mid \phi \in \Gamma\}$;
- $\mathcal{T}$ is the topology on X generated by the basis $\{|\Box\phi|\}_{\phi \in Fm}$;²
- $\mathcal{V}(p) = |p|$.

²This is a basis for a topology on X since $|\Box T| = X$ by $\Box T \in S4$ for any tautology T , and $|\Box(\phi \wedge \psi)| = |\Box\phi| \cap |\Box\psi|$ by $\Box(\phi \wedge \psi) \leftrightarrow (\Box\phi \wedge \Box\psi) \in S4$.

Truth Lemma

In the canonical S4-model $\mathbf{M}$, we have $|\phi| = \|\phi\|_{\mathbf{M}}$.

Proof. Structural induction on ϕ . Base case by definition and induction step cases for $\neg$, $\wedge$ by basic properties of maximally consistent sets. Step for $\phi = \Box\psi$:

$(|\Box\psi| \subseteq \|\Box\psi\|)$ $|\Box\psi|$ is a basic open set and $|\Box\psi| \subseteq |\psi|$ by the T-axiom.

$(\|\Box\psi\| \subseteq |\Box\psi|)$ If $\Gamma \in U$ and $U \subseteq |\psi|$ ($= \|\psi\|$ by the induction hypothesis) for an open U , then there is χ such that $\Gamma \in |\Box\chi|$ and $|\Box\chi| \subseteq U$. Thus, $\Box\chi \rightarrow \psi \in \text{S4}$ by Lindenbaum, and so $\Box\chi \rightarrow \Box\psi \in \text{S4}$ by $\Box$ -monotonicity and the 4-axiom. Since $\Box\chi \in \Gamma$, we obtain $\Box\psi \in \Gamma$. $\square$

Theorem 2 (Completeness of S4)

If $\mathbf{M} \models \phi$ in all topological models $\mathbf{M}$, then $\phi \in \text{S4}$.

Proof. If $\phi \notin \text{S4}$, then $\neg\phi \in \Gamma$ for some maximally consistent S4-theory by Lindenbaum, and so $\mathbf{M} \not\models \phi$ in the canonical model $\mathbf{M}$. $\square$

Quotient topology. Let $\langle X, \mathcal{T} \rangle$ be a topological space and $f: X \rightarrow Y$ surjective. The quotient topology on Y induced by f is the collection $\mathcal{T}_f \subseteq \mathcal{P}(Y)$ such that

$$U \in \mathcal{T}_f \iff f^{-1}(U) \in \mathcal{T}.$$

This is indeed a topology on Y :

T(i) holds: $f^{-1}(Y) = \{x \in X \mid f(x) \in Y\} = X$;

T(ii) holds: $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$;

T(iii) holds: $f^{-1}(\bigcup_{i \in I} U_i) = \bigcup_{i \in I} f^{-1}(U_i)$.

Example. If $\sim$ is equivalence relation on X and $Y = X/\sim$ is the set of $\sim$ -equivalence classes $[x]_\sim$ of all $x \in X$, then $q: x \mapsto [x]_\sim$ is a surjective map $X \rightarrow Y$, inducing a quotient topology $\mathcal{T}_\sim$ on Y . Note that for any $U \subseteq Y$, its pre-image $q^{-1}U$ is $\sim$ -saturated (if $x \in q^{-1}U$ and $x \sim y$, then $y \in q^{-1}U$). Hence, $U \in \mathcal{T}_\sim$ iff its saturated pre-image is in $\mathcal{T}$.

Saturated sets: Take $\langle X, \mathcal{T} \rangle$ and surjective $f : X \rightarrow Y$. Then $P \subseteq X$ is f -saturated iff, for all $x, y \in X$,

$$x \in P \text{ \& } f(x) = f(y) \implies y \in P.$$

Lemma

Take $\langle X, \mathcal{T} \rangle$ and $\langle Y, \mathcal{T}_f \rangle$, induced by some surjective $f : X \rightarrow Y$. Then

$$\mathcal{T}_f = \{f(U) \mid U \in \mathcal{T} \text{ \& } U \text{ is } f\text{-saturated}\}.$$

Proof. ($\subseteq$) If $V \in \mathcal{T}_f$, then $f^{-1}(V) \in \mathcal{T}$. Note that $f^{-1}(V)$ is f -saturated.

($\supseteq$) If U is f -saturated, then $f^{-1}(f(U)) = U$. Hence, if $U \in \mathcal{T}$ is f -saturated, then $f(U) \in \mathcal{T}_f$. $\square$

Quotient by Σ . Let $\Sigma \subseteq Fm$. Given a model $\mathbf{M} = \langle X, \mathcal{T}, \mathcal{V} \rangle$, define

$$x \sim_{\Sigma}^{\mathbf{M}} y \iff (\forall \phi \in \Sigma: x \in \|\phi\|_{\mathbf{M}} \iff y \in \|\phi\|_{\mathbf{M}})$$

Let $\mathbf{M}^{\Sigma} = \langle X^{\Sigma}, \mathcal{T}^{\Sigma}, \mathcal{V}^{\Sigma} \rangle$:

- X^{Σ} is the set of equivalence classes $[x] = \{y \mid x \sim_{\Sigma}^{\mathbf{M}} y\}$ for all $x \in X$;
- $\mathcal{T}^{\Sigma}$ is the quotient topology on X^{Σ} induced by $[\cdot]: x \mapsto [x]$;
- $\mathcal{V}^{\Sigma}(p) = \begin{cases} \{[x] \mid x \in \mathcal{V}(p)\} & p \in \Sigma^3 \\ \emptyset & p \notin \Sigma \end{cases}$

Note that if Σ is finite, then X^{Σ} is finite (size at most $2^{\#(\Sigma)}$).

³This is well-defined: if $y \in [x]$, then $x \in \mathcal{V}(p)$ iff $y \in \mathcal{V}(p)$. That is, $\mathcal{V}(p)$ is a $[\cdot]$ -saturated set.

Quotient Lemma

Let $\Sigma \subseteq Fm$ be closed under subformulas (if $\phi \in \Sigma$ and $\psi \in Sf(\phi)$, then $\psi \in \Sigma$).
Then, for all $\mathbf{M}, x$ and all $\phi \in \Sigma$:

$$\mathbf{M}, x \models \phi \iff \mathbf{M}^\Sigma, [x] \models \phi.$$

Proof. Structural induction on ϕ . Base case by definition and induction step for $\neg, \wedge$ straightforward from IH (closure under subf. needed). Step $\phi = \Box\psi$:

($\implies$) If $x \in Int(\|\psi\|_{\mathbf{M}})$, then $[x] \in [Int(\|\psi\|_{\mathbf{M}})] \subseteq [\|\psi\|_{\mathbf{M}}]$ (since $A \subseteq B$ implies $[A] \subseteq [B]$)
 $= \|\psi\|_{\mathbf{M}^\Sigma}$ (by IH). Moreover, $[Int(\|\psi\|_{\mathbf{M}})]$ is open in $\mathcal{T}_{[\cdot]}$ since $Int(\|\psi\|_{\mathbf{M}})$ is $[\cdot]$ -saturated:
 $\mathbf{M}, x \models \Box\psi$ and $[x] = [y]$ implies $\mathbf{M}, y \models \Box\psi$ since $\Box\psi \in \Sigma$.

($\impliedby$) If $\mathbf{M}^\Sigma, [x] \models \Box\psi$, then $[x] \in V$ open in $\mathcal{T}_{[\cdot]}$ s.t. $V \subseteq \|\psi\|_{\mathbf{M}^\Sigma}$. Then $x \in U = [\cdot]^{-1}V$
open in $\mathcal{T}$. Moreover, $U \subseteq \|\psi\|_{\mathbf{M}}$ (If $y \in U$, then $[y] \in V \subseteq \|\psi\|_{\mathbf{M}^\Sigma}$; so $\mathbf{M}, y \models \psi$ by IH.) □

Note: The lemma establishes that $\|\phi\|_{\mathbf{M}}$ is $[\cdot]$ -saturated for all $\phi \in \Sigma$.

Theorem (Strong Finite Topological Model Property for S4)

If $\phi \notin \text{S4}$, then $\mathbf{M} \not\models \phi$ for some finite topological model $\mathbf{M}$ whose size bound is computable from the size of ϕ .

Proof. If $\phi \notin \text{S4}$, then $\mathbf{M} \not\models \phi$ by the Completeness Theorem. Take $\Sigma = \text{Sf}(\phi)$ and form $\mathbf{M}^\Sigma$. The number of states of $\mathbf{M}^\Sigma$ is at most $2^{\#(\phi)}$ which allows us to compute the upper bound on the size of the model. By the Quotient Lemma, $\mathbf{M}^\Sigma \not\models \phi$. Hence, S4 is decidable. $\square$

Thus, we obtain completeness with respect to a special kind of spaces:

Alexandroff spaces: $\langle X, \mathcal{T} \rangle$ where $\mathcal{T}$ is closed under arbitrary intersections.

Example. If $\langle X, R \rangle$ is a Kripke frame for S4 (a pre-ordered set), then $\langle X, \mathcal{T}_R \rangle$ (featuring the full up-set topology) is an Alexandroff space.

Kripke completeness from Alexandroff completeness

Let $\langle X, \mathcal{T}, \mathcal{V} \rangle$ be an Alexandroff model. We define $R_{\mathcal{T}}$ as the *specialisation pre-order* on X :

$$xR_{\mathcal{T}}y \iff (\forall U \in \mathcal{T} : x \in U \implies y \in U).$$

Hence, $R_{\mathcal{T}}(x) = \bigcap \{U \in \mathcal{T} \mid x \in U\}$, which is open in $\mathcal{T}$. Hence:

$$R_{\mathcal{T}}(x) \subseteq P \iff (\exists U \in \mathcal{T} : x \in U \ \& \ U \subseteq P) \iff x \in \text{Int}_{\mathcal{T}}(P).$$

It follows that $\langle X, \mathcal{T}, \mathcal{V} \rangle \models \phi$ iff $\langle X, R_{\mathcal{T}}, \mathcal{V} \rangle \models \phi$, for all $\phi \in \text{Fm}$.

Alexandroff completeness from Kripke completeness

Let $\langle X, R, \mathcal{V} \rangle$ be a Kripke model and take the Alexandroff model $\langle X, \mathcal{T}_R, \mathcal{V} \rangle$. Note that

$$x \in \text{Int}(P) \iff (\exists y : x \in R(y) \ \& \ R(y) \subseteq P) \iff R(x) \subseteq P.$$

Hence, $\langle X, R, \mathcal{V} \rangle \models \phi$ iff $\langle X, \mathcal{T}_R, \mathcal{V} \rangle \models \phi$, for all $\phi \in \text{Fm}$.

Interior map: Function $f : X \rightarrow Y$ with $\mathcal{T}_X$ and $\mathcal{T}_Y$ fixed such that

- $\forall V \in \mathcal{T}_Y : f^{-1}(V) \in \mathcal{T}_X$ (f is **continuous**)
- $\forall U \in \mathcal{T}_X : f(U) \in \mathcal{T}_Y$ (f is **open**).

Note that $f^{-1}(\text{Int}_Y P) = \text{Int}_X(f^{-1}P)$:

$$\begin{aligned}x \in \text{Int}_X(f^{-1}P) &\iff \exists U \in \mathcal{T}_X : x \in U \text{ \& } U \subseteq f^{-1}P \\&\iff \exists U \in \mathcal{T}_X : x \in U \text{ \& } f(U) \subseteq P \\&\overset{*}{\iff} \exists V \in \mathcal{T}_Y : f(x) \in V \text{ \& } V \subseteq P \\&\iff f(x) \in \text{Int}_Y P \\&\iff x \in f^{-1}(\text{Int}_Y P)\end{aligned}$$

$\overset{*}{\implies}$ holds since f is open and $\overset{*}{\impliedby}$ holds since it is continuous (plus $V \subseteq P \implies f^{-1}V \subseteq f^{-1}P$).

Interior Map Lemma 1

Let $\mathbf{M}_X = \langle X, \mathcal{T}_X, \mathcal{V}_X \rangle$ and $\mathbf{M}_Y = \langle Y, \mathcal{T}_Y, \mathcal{V}_Y \rangle$ be two topological models and $f: X \rightarrow Y$ an interior map such that $x \in \mathcal{V}_X(p) \iff f(x) \in \mathcal{V}_Y(p)$. Then, for all $\phi \in Fm$:

$$\mathbf{M}_X, x \models \phi \iff \mathbf{M}_Y, f(x) \models \phi.$$

Proof. Structural induction on ϕ . [...] Step for $\phi = \Box\psi$: $f(x) \in \text{Int}_Y(\|\psi\|_Y)$ iff $x \in \text{Int}_X(f^{-1}\|\psi\|_Y)$ iff $x \in \text{Int}_X(\|\phi\|_X)$ (by IH). □

Interior Map Lemma 2

Let $\mathbf{S}_X = \langle X, \mathcal{T}_X \rangle$ and $\mathbf{S}_Y = \langle Y, \mathcal{T}_Y \rangle$ be two topological spaces and $f: X \rightarrow Y$ a surjective interior map. Then, for all $\phi \in Fm$:

$$\mathbf{S}_X \models \phi \implies \mathbf{S}_Y \models \phi.$$

Proof. If $\langle \mathbf{S}_X, \mathcal{V} \rangle \not\models \phi$, then define $\mathcal{V}_X: p \mapsto f^{-1}\mathcal{V}(p)$ and use Lemma 1. □

Lemma (Morphisms equivalence)

(a) If $f : \langle X, \mathcal{T}_X \rangle \rightarrow \langle Y, \mathcal{T}_Y \rangle$ is an int. map, then it is a p-morphism $\langle X, R_{\mathcal{T}_X} \rangle \rightarrow \langle Y, R_{\mathcal{T}_Y} \rangle$. Recall that $f : X \rightarrow Y$ is a p-morphism $\langle X, R_X \rangle \rightarrow \langle Y, R_Y \rangle$ iff (i) $xR_X x' \implies f(x)R_Y f(x')$, and (ii) $f(x)R_Y y \implies (\exists x' \in X : xR_X x' \ \& \ y = f(x'))$.

(b) If $f : \langle X, R_X \rangle \rightarrow \langle Y, R_Y \rangle$ is a p-morphism, then it is an int. map $\langle X, \mathcal{T}_{R_X} \rangle \rightarrow \langle Y, \mathcal{T}_{R_Y} \rangle$.

Proof. (a, i) Assume that $xR_{\mathcal{T}_X} y$ and $f(x) \in U \in \mathcal{T}_Y$. Now $f^{-1}U \in \mathcal{T}_X$ since f is continuous and $x \in f^{-1}U$ by the second assumption. By the first assumption, $f(y) \in U$.

(a, ii) Assume that $f(x)R_{\mathcal{T}_Y} y$. We must find an $x' \in X$ such that $xR_{\mathcal{T}_X} x'$ and $f(x') = y$.

Because X is an Alexandroff space, x has a minimal open neighborhood

$U_x = \{z \in X \mid xR_{\mathcal{T}_X} z\}$. (Officially, $U_x = \bigcap \{U \in \mathcal{T} \mid x \in U\}$.) Since f is an open map, $f(U_x) \in \mathcal{T}_Y$. Obviously, $x \in U_x$, so $f(x) \in f(U_x)$. Since $f(U_x)$ is an open set containing $f(x)$, we obtain $y \in f(U_x)$ by the assumption. It follows that there is some $x' \in U_x$ such that $f(x') = y$. Since $x' \in U_x$, we have $xR_{\mathcal{T}_X} x'$.

Proof (cont'd). Assume that $f : \langle X, R_X \rangle \rightarrow \langle Y, R_Y \rangle$ is a p-morphism. We need to show that it is continuous and open with respect to $\mathcal{T}_{R_X}$ and $\mathcal{T}_{R_Y}$.

(cont.) Let $U \in \mathcal{T}_{R_Y}$, i.e. U is an upper set in $\langle Y, R_Y \rangle$. We have to show that $f^{-1}U \in \mathcal{T}_{R_X}$, i.e. that it is an upper set in $\langle X, R_X \rangle$. Hence, assume that $x \in f^{-1}U$ and xR_Xx' for some $x' \in X$. It follows that $f(x) \in U$ and, since f is a p-morphism, $f(x)R_Yf(x')$. Since U is an upper set, $x' \in f^{-1}U$.

(open) Let $U \in \mathcal{T}_{R_X}$, i.e. U is an upper set in $\langle X, R_X \rangle$. We have to show that $f(U) \in \mathcal{T}_{R_Y}$, i.e. that it is an upper set in $\langle Y, R_Y \rangle$. Hence, assume that $y \in f(U)$ and yR_Yy' . It follows that $y = f(x)$ for some $x \in U$. Since f is a p-morphism, $f(x)R_Yy'$ implies that there is $x' \in X$ such that xR_Xx' and $y' = f(x')$. Since $x \in U$ and U is an upper set, we obtain $x' \in U$. This means that $y' \in f(U)$. □

Lemma (Objects equivalence)

- (a) Let $\langle X, \mathcal{T} \rangle$ be an Alexandroff space. Then $\mathcal{T} = \mathcal{T}_{R_{\mathcal{T}}}$.
- (b) Let $\langle X, R \rangle$ be a pre-ordered set. Then $R = R_{\mathcal{T}_R}$.

Proof. (a, $\subseteq$) Let $U \in \mathcal{T}$. Assume $x \in U$ and $xR_{\mathcal{T}}y$. By definition of $R_{\mathcal{T}}$, $y \in U$. Thus, U is an $R_{\mathcal{T}}$ -upper set, so $U \in \mathcal{T}_{R_{\mathcal{T}}}$.

(a, $\supseteq$) Let $V \in \mathcal{T}_{R_{\mathcal{T}}}$. Since $\mathcal{T}$ is an Alexandroff topology, every $x \in X$ has a minimal open neighborhood $U_x \in \mathcal{T}$. By definition of $R_{\mathcal{T}}$, $U_x = \{y \in X \mid xR_{\mathcal{T}}y\}$. Because V is an $R_{\mathcal{T}}$ -upper set, $V = \bigcup_{x \in V} U_x$. It follows that $V \in \mathcal{T}$.

(b, $\subseteq$) Assume xRy . Let $U \in \mathcal{T}_R$ and $x \in U$. Since U is an upper set, we have $y \in U$. Since U was arbitrary, $xR_{\mathcal{T}_R}y$.

(b, $\supseteq$) Assume $xR_{\mathcal{T}_R}y$. The principal upper set $\uparrow x = \{z \in X \mid xRz\}$ is an R -upper set, meaning $\uparrow x \in \mathcal{T}_R$. Obviously, $x \in \uparrow x$. By definition of $R_{\mathcal{T}_R}$, $y \in \uparrow x$, which means that xRy . $\square$

The previous two lemmas show that Alexandroff spaces + interior maps are ‘the same thing’ as Kripke frames + p-morphisms. More precisely: They form isomorphic categories.

T_0 -spaces: $\langle X, \mathcal{T} \rangle$ where for all $x \neq y$ there is $U \in \mathcal{T}$ such that $x \in U$ and $y \notin U$.

Note that $\mathcal{T}_R$ is T_0 iff R is a partial order.

Theorem (T_0 -completeness of S4)

For all ϕ , if ϕ is valid in all T_0 -spaces, then it is valid in all Kripke frames for S4.

Proof. Let $\mathbf{F} = \langle X, R \rangle$ be a Kripke frame. Take $\langle \mathbf{F}^*, \mathcal{T}_{\sqsubseteq} \rangle$, where $\mathbf{F}^*$ is the set of all finite paths over $\langle X, R \rangle$, that is all $\langle x_1, \dots, x_n \rangle$ where $x_i R x_{i+1}$, and $\sqsubseteq$ is the prefix ordering on $\mathbf{F}^*$. Note that $\sqsubseteq$ is a partial order, so $\langle \mathbf{F}^*, \mathcal{T} \rangle$ is T_0 .

Define a map $f : \mathbf{F}^* \rightarrow X$ by $f(\langle x_1, \dots, x_n \rangle) = x_n$. It is surjective and interior (where X is endowed with $\mathcal{T}_R$):

- (cont.) Let V be an upper set in $\langle X, R \rangle$. Note that $f^{-1}(V) \in \mathcal{T}$ iff $f^{-1}(V)$ is an upper set under $\sqsubseteq$. Let $f(w) \in V$ and $w \sqsubseteq u$. Since R is transitive and V is an upper set, $f(u) \in V$.
- (open) Let U be an upper set in $\langle \mathbf{F}^*, \sqsubseteq \rangle$. Note that $f(U) \in \mathcal{T}_R$ iff $f(U)$ is an upper set in $\langle X, R \rangle$. Let $f(w) R y$ for $w = \langle x_1, \dots, x_n \rangle \in U$; we have to show that $y = f(v)$ for some $v \in U$. Take $v = \langle x_1, \dots, x_n, y \rangle$. This is a 'legal' path and, since U is an upper set, $v \in U$.

Hence, if $\mathbf{F} \not\models \phi$, then $\mathbf{F}^* \not\models \phi$. □

Notes

Good introductions to general topology include [Willard, 1970] and [Munkres, 2000]. Marius Furter's [2022] video lecture series is also a valuable resource.

McKinsey [1941] established completeness of S4 with respect to all finite topological spaces and all T_1 -spaces using a different method. Our proof of completeness of S4 with respect to all spaces follows the direct method of Aiello et al. [2003]. The latter paper is a good introduction to topological semantics for modal logics, along with [van Benthem and Bezhanishvili, 2007].

McKinsey and Tarski [1944] established that S4 is complete with respect to any dense-in-itself metrizable space (e.g. the real line $\mathbb{R}$), a result known as the *McKinsey-Tarski Theorem*.⁴

⁴They originally proved this assuming also that the space is separable. This assumption was later shown to be unnecessary by Rasiowa and Sikorski.

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