sedlar@cs.cas.cz
http://www.cs.cas.cz/sedlar/

Substructural propositional dynamic logics

Igor Sedlár[0000−0002−1942−7982]

The Czech Academy of Sciences, Institute of Computer Science
Pod Vodárenskou věží 271/2, 182 07 Prague 8, Czech Republic

Abstract. We prove completeness and decidability of a version of Propositional Dynamic Logic where the underlying non-modal propositional logic is a substructural logic in the vicinity of the Full Distributive Non-associative Lambek Calculus. Extensions of the result to stronger substructural logics are briefly discussed.

Keywords: Lambek calculus· Modal logic · Propositional Dynamic Logic· Relevant logic · Substructural logic.

1 Introduction

Propositional Dynamic Logic, introduced in [8] following the ideas of [24], is a multi-modal logic for reasoning about structured actions with applications in formal verification of programs [11], automated planning [26,33], dynamic epistemic logic [1] and deontic logic [19], for example.

In its standard formulation, PDL is a normal modal logic extending classical logic. Nevertheless, many non-classical versions of PDL—non-classical PDLs—have been explored as well, ranging from intuitionistic versions [6,21,36], to many-valued [5,4,12,15,16,34] and paraconsistent ones [30,29]. In [32], the landscape is extended with a study of propositional dynamic logic based on weak substructural logics in the vicinity of the Non-associative Lambek calculus. In that paper a formula-formula sequent system is used on the proof-theoretic side to complement a simple relational semantics extending frames for the Lambek calculus [7,20]. This approach, however, is not typical in all areas of substructural logic; especially in relevant logic a Hilbert-style proof theory combined with models based on partially ordered sets is preferred [28,25]. One naturally wonders if PDLs can be easily formulated in this setting as well.

In this paper we explore completeness of Hilbert-style formulations of substructural PDLs with respect to partially ordered models. We employ the technique of [31] (itself based on [3]), where a fragment of the present setting was studied, in combination with Nishimura's approach to intuitionistic PDL [21]. We show that the approach works for PDLs based on some weak substructural logics but it fails for some stronger logics (for instance, some undecidable relevant logics); these ramifications are similar to those pointed out in [32]. In addition, extensions of Hilbert-style PDLs with primitive existential modalities ("diamond" versions of the action-indexed modalities) are shown to be problematic. These observations suggest that the study of substructural propositional dynamic logic abounds with interesting challenges and, most probably, requires the development of novel techniques.

The paper is structured as follows. In Section 2 we give the necessary background on PDL and on substructural logics. Section 3 discusses the motivation for studying substructural PDLs (in addition to technical curiosity). Completeness and decidability of PDL based on a weak substructural logic close to the Non-associative Lambek calculus is established in Section 4. The ramifications of the technique used to obtain the result, along with a number of related open problems, are discussed in Section 5.

2 Preliminaries

In this section we give an outline of propositional dynamic logic based on classical logic (Sect. 2.1, where we build on [11]) and of substructural logics (Sect. 2.2, where we build mainly on [25]).

2.1 Classical PDL

Fix countable sets At of atomic formulas and Ac of atomic action expressions. *Formulas* and *action expressions* are defined by mutual induction as follows: 1. Each $p \in At$ is a formula, the truth constant $\bar{1}$ and the falsity constant $\bar{0}$ are formulas and each combination of formulas using Boolean connectives $\wedge, \vee, \rightarrow$ is a formula; moreover, if α is an action expression, then $[\alpha]\varphi$ is a formula. 2. Each $a \in Ac$ is an action expression; if α and β are action expressions, then so are $\alpha;\beta$ (expressing composition of actions, "doing α and then β"), $\alpha \cup \beta$ (expressing non-deterministic choice, "doing α or β"), α^* (representing iteration, "doing α some finite number of times"). Moreover, if φ is a formula, then $\varphi?$ is an action expression (expressing test, "testing whether φ holds"). This language will be called the *dynamic language*. We define $\top$ as $\bar{0} \rightarrow \bar{0}$, $\neg\varphi$ and $\varphi \leftrightarrow \psi$ are defined as usual. Conjunctions and disjunctions of finite sets of formulas are defined usual, with $\bigwedge \emptyset := \top$ and $\bigvee \emptyset := \bar{0}$.[1]

Formulas and action expressions are referred to jointly as *expressions*. The notions of subformula and action-subexpression are defined as expected. The relation of *subexpression* is defined as the least relation satisfying the following: 1. Each subformula of φ is a subexpression of φ; if φ is of the form $[\alpha]\psi$, then each subexpression of α is a subexpression of φ. 2. Each action-subexpression of α is a subexpression of α; if α is of the form $\varphi?$, then each subexpression of φ is a subexpression of α. Proofs by induction on subexpressions are often used in propositional dynamic logic.

Fix an axiomatization CPC of the classical propositional calculus in the language $\{\wedge, \vee, \rightarrow, \bar{1}, \bar{0}\}$ using axiom schemata and Modus Ponens ans the only

[1] We distinguish between $\bar{1}$ and $\top$ for the sake of presentation; these will not be equivalent in substructural logics. See Sect. 2.2. We need $\bar{1}$ in our language for a technical reason, see the proof of Lemma 6.

rule of inference. The axiom system PDL is obtained by adding to CPC the axiom schemata

$$\begin{array}{l} ([\alpha]\varphi \land [\alpha]\psi) \to [\alpha](\varphi \land \psi), \\ [\alpha \cup \beta]\varphi \leftrightarrow ([\alpha]\varphi \land [\beta]\varphi), \\ [\alpha;\beta]\varphi \leftrightarrow [\alpha][\beta]\varphi, \\ [\alpha^*]\varphi \to (\varphi \land [\alpha][\alpha^*]\varphi), \\ [\psi?]\varphi \land \psi \to \varphi; \end{array}$$

and the inference rules

$$\frac{\varphi \to \psi}{[\alpha]\varphi \to [\alpha]\psi}, \ \frac{\varphi \to [\alpha]\varphi}{\varphi \to [\alpha^*]\varphi} \text{ and } \frac{(\varphi \land \psi) \to \chi}{\varphi \to [\psi?]\chi}.$$

Let us refer to these additional modal axioms and rules as MAX. Theorems and derivability in PDL are defined in the usual way. (Hence, derivability is finitary: φ is derivable from Γ iff it is derivable from a finite subset of Γ.)

A *standard frame* is a couple $F = \langle W, R\rangle$, where W is a non-empty set ("worlds" or "states") and R is a function from Ac to binary relations on W. The "accessibility" relation $R(a)$ represents actions of type a—$R(a)(x, y)$ can be read as "state y is accessible from x by performing an action of type a". A *standard model* is a triple $M = \langle W, R, V\rangle$, where V is a function from At to subsets of W. We also say that $\langle W, R, V\rangle$ is a model *based on* the frame $\langle W, R\rangle$.

For each M, we define the *evaluation function* $[\![\ ]\!]_M$ that assigns subsets of W to formulas and binary relations on W to action expressions in the following way (again, the definition is by mutual induction):

- $[\![p]\!]_M = V(p)$, $[\![\bar{1}]\!]_M = W$ and $[\![\bar{0}]\!]_M = \emptyset$; the usual set-theoretic clauses are used for Boolean combinations of formulas. Moreover, $[\![[\alpha]\varphi]\!]_M$ is the set of x such that, for all y, if $x[\![\alpha]\!]_M y$, then $y \in [\![\varphi]\!]_M$.
- $[\![a]\!]_M = R(a)$, $[\![\alpha \cup \beta]\!]_M$ is the union of $[\![\alpha]\!]_M$ and $[\![\beta]\!]_M$, $[\![\alpha;\beta]\!]_M$ is the composition of $[\![\alpha]\!]_M$ and $[\![\beta]\!]_M$, $[\![\alpha^*]\!]_M$ is the reflexive transitive closure of $[\![\alpha]\!]_M$, and $[\![\varphi?]\!]_M$ is the identity relation on $[\![\varphi]\!]_M$.

Infix notation $x[\![\alpha]\!]_M y$ is used for the fact that $\langle x, y\rangle$ is in the relation $[\![\alpha]\!]_M$. The subscript is often omitted.

Formula φ is *valid* iff $[\![\varphi]\!]_M$ is the set of worlds in M, for all models M. More generally, φ follows from a set of assumptions Γ iff $\left(\bigcap_{\psi\in\Gamma}[\![\psi]\!]_M\right) \subseteq [\![\varphi]\!]_M$, for all M.

Theorem 1. *φ is a theorem of PDL iff it is valid. The set of theorems of PDL is decidable.*

Decidability of the set of valid formulas was shown in [8] using a finite model constructuion. Completeness of a system equivalent to PDL without test and with a "converse" modality was shown in [13,23], using a finite model construction similar to the one used in [8]; it is noted in the papers that the proof strategy is compatible with adding test and removing converse. For the full proof of this fact consult [11].

A noteworthy feature of propositional dynamic logic is that it is not compact. To see this, note that $[a^*]p$ follows from $\{p\} \cup \{[a^n]p \mid n \in \omega\}$, where a^1 is a and a^{n+1} is $a; a^n$. However, $[a^*]p$ does not follow from any finite subset of that set of assumptions. Hence, $[a^*]p$ is not derivable from that set of assumptions in PDL. As a result, one cannot hope for a *strong* completeness theorem for PDL. In [14] an *infinitary* proof system PDL_ω is shown to be strongly complete with respect to the standard semantics.

2.2 Some substructural logics

For the sake of simplicity, we diverge somewhat from the usual presentation (e.g. [10,25]) and we discuss substructural logics in the language of CPC, that is, in $\{\wedge, \vee, \rightarrow, \bar{1}, \bar{0}\}$.[2] Substructural logics in this language can be seen as logics where $\rightarrow$ lacks some properties that implication has in classical logic. Some such properties are given in Figure 1.

B	$(\varphi \rightarrow \psi) \rightarrow ((\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi))$	Associativity
C	$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$	Commutativity
CI	$\varphi \rightarrow ((\varphi \rightarrow \psi) \rightarrow \psi)$	Weak commutativity
W	$(\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$	Contraction
WI	$(\varphi \wedge (\varphi \rightarrow \psi)) \rightarrow \psi$	Weak contraction
K	$\varphi \rightarrow (\psi \rightarrow \varphi)$	Weakening

Fig. 1. "Structural schemata" that fail in some substructural logics.

The reasons to avoid the respective properties of implication are related to various possible informal readings of $\rightarrow$. The Weakening axiom is usually avoided based on the assumption that ψ has to be *relevant* to φ in order for $\psi \rightarrow \varphi$ to be true. Note that the Weakening axiom entails that $\psi \rightarrow \varphi$ is derivable form the mere assumption that φ is the case, without assuming anything about ψ at all. These considerations led to the study of *relevant* logics; the main examples of such logics—for example the logic R—include all the other schemata. The contraction axiom is usually omitted when implication $\varphi \rightarrow \psi$ is read in terms of *resource use*, for instance as "by using a resource of type φ, outcome of type ψ may be produced". It is clear that some outputs require *several* pieces of resource of some type to be used. These considerations are central to *linear logic*, for instance. In addition, contraction is also avoided in some *fuzzy logics* (logics of graded truth). Note that Contraction is also not plausible when formulas are seen as expressing types of linguistic items (expressions) and $\varphi \rightarrow \psi$ represents the type of expression that, when concatenated with expression of type φ, results in an expression of type ψ, such as in the various versions of the *Lambek calculus*. This interpretation is also inconsistent with Commutativity—the order of expression concatenation usually matters. Finally, Associativity is

[2] This means that we do not include the fusion $\circ$ and the dual implication $\leftarrow$.

omitted in some versions of the Lambek calculus (not dealing with strings, but with some more general class of linguistic items). See the introductory chapter of [22] for more details on these motivations, for example. We note, in addition, that the Explosion principle, $(\varphi \wedge \neg\varphi) \to \psi$, follows from Weak contraction and $\bar{0} \to \psi$. *Paraconsistent* logics avoid the Explosion principle since it trivializes inconsistent sets of assumptions.

Let us turn to axiomatic presentations of substructural logics. In what follows, a *logic* will be any set of formulas in the language $\{\wedge, \vee, \to, \bar{1}, \bar{0}\}$ containing all the formulas of the form

$$\begin{array}{l}\bar{1},\ \varphi \to \varphi,\ (\varphi \wedge \psi) \to \varphi,\ (\varphi \wedge \psi) \to \psi,\ \varphi \to (\varphi \vee \psi),\ \psi \to (\varphi \vee \psi),\\ \varphi \wedge (\varphi \vee \chi) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \chi)),\ \bar{0} \to \varphi,\ \varphi \to (\bar{0} \to \bar{0}),\\ ((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi)),\\ ((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi);\end{array}$$

and closed under

$$\frac{\varphi}{\bar{1} \to \varphi},\ \frac{\varphi \to \psi \quad \varphi}{\psi},\ \frac{\varphi \quad \psi}{\varphi \wedge \psi},\quad \frac{\varphi \to \psi \quad \chi \to \theta}{(\psi \to \chi) \to (\varphi \to \theta)}.$$

We sometimes write $\vdash_\Lambda \varphi$ instead of $\varphi \in \Lambda$.

Let Λ_0 be the smallest logic; it can be seen as a Hilbert-style axiomatization of a fragment of the Distributive Full Non-associative Lambek calculus extended with the falsity constant $\bar{0}$ (or, as we may also say in the terminology of [10], the $\{\wedge, \vee, \to, \bar{1}, \bar{0}\}$-fragment of the "zero-bounded" DFNL).

A *Routley–Meyer frame* is a structure $\mathcal{F} = \langle S, \leq, L, T\rangle$ where $\langle S, \leq\rangle$ is a partially ordered set, L is an upwards closed subset of $\langle S, \leq\rangle$ and T is a ternary relation on S such that

$$Txyz, x' \leq x, y' \leq y, z \leq z' \implies Tx'y'z' \tag{1}$$

$$x \leq y \iff (\exists z)(z \in L \ \&\ Tzxy) \tag{2}$$

A *Routley–Meyer model* based on $\mathcal{F}$ is $\mathcal{M} = \langle \mathcal{F}, V\rangle$, where V is a function from At to upwards closed subsets of the frame $\mathcal{F}$. For each $\mathcal{M}$, we define the *evaluation function* $[\![]\!]_\mathcal{M}$ that assigns subsets of the frame on which $\mathcal{M}$ is based to formulas (states that "satisfy" the formulas) in the following way: $[\![p]\!]_\mathcal{M} = V(p)$, $[\![\bar{0}]\!]_\mathcal{M} = \emptyset$ and

$$[\![\bar{1}]\!]_\mathcal{M} = L; \tag{3}$$

the usual set-theoretic clauses are used for $\wedge, \vee$ and

$$[\![\varphi \to \psi]\!]_\mathcal{M} = \{x \mid (\forall yz)((Txyz \ \&\ y \in [\![\varphi]\!]_\mathcal{M}) \implies z \in [\![\psi]\!]_\mathcal{M})\} \tag{4}$$

Note that $[\![\top]\!]_\mathcal{M} = S$ and so $[\![\top]\!]_\mathcal{M} \neq [\![\bar{1}]\!]_\mathcal{M}$ if $L \neq S$.

The following well-known facts outline the reasons why Routley–Meyer frames contain L and $\leq$ and why (1–2) are assumed.

Lemma 1. *For all $\mathcal{M}$ and φ, $[\![\varphi]\!]_\mathcal{M}$ is an upwards closed set.*

Proof. Use (1) for $\varphi = \psi \to \chi$; other cases are trivial.

A formula φ is *valid in* $\mathcal{M}$ iff $L \subseteq [\![\varphi]\!]_{\mathcal{M}}$.

Lemma 2. *$\varphi \to \psi$ is valid in $\mathcal{M}$ iff $[\![\varphi]\!]_{\mathcal{M}} \subseteq [\![\psi]\!]_{\mathcal{M}}$.*

Proof. "If": Take $x \in L$ and assume that $Txyz$ and $y \in [\![\varphi]\!]_{\mathcal{M}}$. By (2), $y \leq z$ and by Lemma 1 $z \in [\![\varphi]\!]_{\mathcal{M}}$. Hence $z \in [\![\psi]\!]_{\mathcal{M}}$ by the assumption.

"Only if": Take $x \in [\![\varphi]\!]_{\mathcal{M}}$. By (2), we have $Tyxx$ for some $y \in L$ and so $x \in [\![\psi]\!]_{\mathcal{M}}$ by the assumption.

We note that formulas in Λ_0 are typically *not* satisfied in *all* states in a model. For instance, $p \to p$ may fail in x if there are y, z such that $Rxyz$ and $y \not\leq z$. However, thanks to Lemma 2, $p \to p$ is clearly satisfied in all $x \in L$.

Let Λ be a logic. A set of formulas Γ is a *non-trivial prime Λ-theory* iff 1. Γ is non-empty, 2. $\varphi \to \psi \in \Lambda$ and $\varphi \in \Gamma$ implies $\psi \in \Gamma$, 3. $\varphi, \psi \in \Gamma$ only if $\varphi \land \psi \in \Gamma$, 4. $\varphi \lor \psi \in \Gamma$ only if $\varphi \in \Gamma$ or $\psi \in \Gamma$.

Lemma 3 (Pair Extension). *Let Λ be any logic extending Λ_0. Assume that there is no conjunction γ of elements of Γ and a disjunction δ of elements of Δ such that $\gamma \to \delta \in \Lambda$. Then there is a non-trivial prime Λ-theory Σ extending Γ and disjoint from Δ.*

Proof. See [25, 92–94].

A formula is *valid in* $\mathcal{F}$ iff it is valid in all models based on $\mathcal{F}$.

Theorem 2. *$\varphi \in \Lambda_0$ iff φ is valid in all Routley–Meyer frames.*

Proof. Canonical model construction, the argument uses the Pair Extension Lemma; see [25] for details.

We note that a simpler semantics (without L and $\leq$) is sufficient for some formula-formula sequent presentations of some substructural logics, i.e. where a logic is defined as a set of ordered pairs of formulas, not as a set of formulas.

3 Motivation

The previous section suggests an obvious way to produce proof systems for substructural propositional dynamic logics—take a substructural logic and add MAX. Semantics for these proof systems do not seem hard to come by as well. Following the lead of the literature on modal relevant logics [9,17,18,27], the idea is to add to Routley–Meyer frames a function R from atomic action expressions Ac to binary relations on S satisfying a tonicity condition in the style of (1) and then define the evaluation function on complex action expressions in the style of classical PDL. It is to be expected that if Λ is sound and complete with respect to a class of Routley–Meyer frames, then PDL_{Λ}, an extension of Λ with MAX, is sound and complete with respect to suitable "modal extensions" of frames in the class. We will show in Section 4 that this is indeed the case for PDL_{Λ_0} and we will point out some problems that pop up when stronger logics are considered in Section 5.

But first, we need to address another question, namely, *why* is it interesting to consider such substructural PDLs. We will not go into a detailed discussion of this important question here. We just point out some relations of the present question to the original motivations for omitting some of the structural schemata of Fig. 1.

One of the crucial properties of actions expressed in the language of PDL are *partial correctness assertions* of the type

$$\varphi \to [\alpha]\psi,$$

read "if φ is the case, then each (terminating) execution of action α leads to a state where ψ holds"; see [11]. One may insist that such assertions express meaningful properties of actions only if φ is *relevant*, in some sense close to the motivations of relevant logic, to $[\alpha]\psi$ (or to ψ).[3] This motivates the study of PDL without the Weakening axiom. We note that most non-classical PDLs studied in the literature so far (intuitionistic and fuzzy PDLs) assume Weakening.

In general, omissions of the structural schemata from PDL can be motivated by the goal of formulating logics of structured actions that modify the types of objects related to non-modal substructural logics without the respective structural schemata. For instance, assume that we want to study a logic for reasoning about structured actions modifying linguistic items (expressions) of some kind. It is reasonable to take a PDL based on some version of the Lambek calculus. Similarly, reasoning about actions in a setting where graded truth values are admitted (e.g. situations where graded predicates play an important role), requires a fuzzy version of PDL without Contraction.

4 The basic substructural PDL

In this section we prove completeness and decidability of the basic substructural propositional dynamic logic PDL_{Λ_0}, which we denote simply as PDL_0. To be more precise, PDL_0 is the least set of formulas of the dynamic language 1. containing all the formulas of the forms used in the definition of Λ_0 and closed under all the Λ_0-inference rules; and 2. containing (or closed under) all elements of MAX.

A *dynamic Routley–Meyer frame* is a structure $\mathfrak{F} = \langle S, \leq, L, T, R\rangle$ where $\langle S, \leq, L, T\rangle$ is a Routley–Meyer frame and R is a function from Ac to binary relations on S such that

$$R(a)xy, x' \leq x, y \leq y' \implies R(a)x'y' \tag{5}$$

A *dynamic Routley–Meyer model based on* $\mathfrak{F}$ is $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ where V is as in Routley–Meyer models. The evaluation function $[\![\ ]\!]_{\mathfrak{M}}$ assigning subsets of

[3] For instance, on may wonder if $p \to [\alpha]\top$ express a meaningful specification of α.

S to formulas and binary relations on S to action expressions is defined as in dynamic and Routley–Meyer models, respectively (the clause for $\to$ is the one used in Routley–Meyer models), with one exception:

$$[\![\varphi?]\!]_{\mathfrak{M}} = \{\langle x, y\rangle \mid x \leq y \ \ \& \ \ y \in [\![\varphi]\!]_{\mathfrak{M}}\} \tag{6}$$

Lemma 4. *1. Each $[\![\varphi]\!]_{\mathfrak{M}}$ is upwards closed.*
2. For all α, $x[\![\alpha]\!]_{\mathfrak{M}}y$, $x' \leq x$ and $y \leq y'$ imply $x'[\![\alpha]\!]_{\mathfrak{M}}y'$.

Proof. The claims are established simultaneously by induction on subexpressions (in each case, the induction hypothesis is that *both* claims hold for all proper subexpressions of the expression at hand). 1. The only new claim is the one concerning formulas of the form $[\alpha]\psi$—and that claim is easily seen to follow from 2. for α (a subexpression of $[\alpha]\psi$). 2. We give details of the case $\psi?$ as it hinges on the non-standard evaluation condition (6). If $x \leq y$ and $y \in [\![\psi]\!]_{\mathfrak{M}}$, then $x' \leq x$ and $y \leq y'$ imply $x' \leq y'$ and $y' \in [\![\psi]\!]_{\mathfrak{M}}$ by transitivity of $\leq$ and claim 1. for ψ (a subexpression of $\psi?$).

The proof of Lemma 4 provides the justification for our choice of the non-standard evaluation condition (6)—note that the second claim of the lemma would fail if $[\![\varphi?]\!]_{\mathfrak{M}}$ was defined, classically, as the identity relation on $[\![\varphi]\!]_{\mathfrak{M}}$. Nevertheless, this definition seems to bring about a significant shift in "meaning" of the test action when compared to the classical case. Instead of "Test whether φ is satisfied; do not change state", we now have "Move to an arbitrary bigger state that supports φ", i.e. something along the lines of "Assume, *ceteris paribus*, that φ is satisfied". Should we even call this action *test*?

We note only that the classical definition of test is a special case, obtained under particular assumptions concerning the notion of a state, of the new definition. Note that if only the maximal elements in the partial ordering are considered (assume, for the sake of discussion, that we have a model where such maximal elements exist), then the newly defined $[\![\varphi?]\!]_{\mathfrak{M}}$ is in fact the identity relation on $[\![\varphi]\!]_{\mathfrak{M}}$—moving to an arbitrary bigger state supporting φ amounts to staying in the present state if φ is satisfied there and "aborting" otherwise, just as in the classical case. Hence, the shift here is not in the meaning of test, but in the kind of state allowed.[4]

Validity in dynamic Routley–Meyer models and frames, respectively, is defined in the same way as in Routley–Meyer models.

Lemma 5. *$\varphi \to \psi$ is valid in $\mathfrak{M}$ iff $[\![\varphi]\!]_{\mathfrak{M}} \subseteq [\![\psi]\!]_{\mathfrak{M}}$.*

Proof. Similar to the proof of Lemma 2, using Lemma 4.

Theorem 3. *Each element of PDL_0 is valid in all dynamic Routley–Meyer frames.*

[4] Another interesting observation is that, on the present evaluation condition, $[\varphi?]\psi$ is equivalent to intuitionistic implication $\varphi \to_{IL} \psi$. Hence, in a sense, our substructural PDLs contain intuitionistic PDL.

Proof. Induction on the length of the proof; Lemma 5 provides a useful shortcut.

Completeness is established by a finitary method related to the standard proofs for PDL. Out of convenience we chose a combination of the method outlined in [31] with Nishimura's approach given in [21]. (It is also possible to obtain the result by combining the technique of [31] with the standard approach of [11,23] that uses non-standard models, but we have opted for a more direct approach that we deem more elegant.)

Definition 1 (Closure). *Let Σ be a set of formulas of the dynamic language. The* closure *of Σ is the least set $\Sigma^c \supseteq \Sigma$ closed under subformulas such that:*

- $\bar{0} \to \bar{0} \in \Sigma^c$ *and* $\bar{1} \in \Sigma^c$
- $[\alpha \cup \beta]\varphi \in \Sigma^c$ *implies* $[\alpha]\varphi \in \Sigma^c$ *and* $[\beta]\varphi \in \Sigma^c$
- $[\alpha ; \beta]\varphi \in \Sigma^c$ *implies* $[\alpha][\beta]\varphi \in \Sigma^c$
- $[\alpha^*]\varphi \in \Sigma^c$ *implies* $[\alpha][\alpha^*]\varphi \in \Sigma^c$
- $[\psi?]\varphi \in \Sigma^c$ *implies* $\psi \in \Sigma^c$

Σ *is* closed *iff* $\Sigma = \Sigma^c$.

We say that a pair of sets of formulas $\underline{\Gamma} = \langle \underline{\Gamma}^+, \underline{\Gamma}^- \rangle$ is an *independent Λ-pair* (member of IP_Λ) iff there is no conjunction γ^+ of elements of $\underline{\Gamma}^+$ and disjunction γ^- of element of $\underline{\Gamma}^-$ such that $\gamma^+ \to \gamma^-$ is in Λ. (We note that both $\underline{\Gamma}^+$ and $\underline{\Gamma}^-$ may be empty or infinite.) Recall the Pair Extension Lemma 3 saying that for each $\underline{\Gamma} \in IP_\Lambda$ there is a non-trivial prime Λ-theory $\Delta \supseteq \underline{\Gamma}^+$ disjoint from $\underline{\Gamma}^-$.

Definition 2 (Canonical model). *Let Φ be a finite closed set. The* Λ-canonical model for Φ *is defined as consisting of the following elements:*

- S^Φ *is the set of all* $\underline{\Gamma} \in IP_\Lambda$ *such that* $\underline{\Gamma}^+ \cup \underline{\Gamma}^- = \Phi$
- $\underline{\Gamma} \leq^\Phi \underline{\Delta}$ *iff* $\underline{\Gamma}^+ \subseteq \underline{\Delta}^+$
- L^Φ *is the set of* $\underline{\Gamma}$ *such that* $\bar{1} \in \underline{\Gamma}^+$
- $T^\Phi \underline{\Gamma \Delta \Sigma}$ *iff there are non-trivial prime Λ-theories Γ', Δ' and Σ' such that* $\underline{\Gamma}^+ \subseteq \Gamma'$, $\underline{\Delta}^+ \subseteq \Delta'$, $(\Sigma' \cap \Phi) \subseteq \underline{\Sigma}^+$ *such that*

$$(\forall \varphi, \psi)(\varphi \to \psi \in \Gamma' \;\;\&\;\; \varphi \in \Delta' \implies \psi \in \Sigma') \tag{7}$$

- $R^\Phi(a)\underline{\Gamma \Delta}$ *iff, for all* $[a]\varphi \in \Phi$, *if* $[a]\varphi \in \underline{\Gamma}^+$, *then* $\varphi \in \underline{\Delta}^+$
- *If* $p \in \Phi$, *then* $V^\Phi(p) = \{\underline{\Gamma} \mid p \in \underline{\Gamma}^+\}$; $V^\Phi(p) = \emptyset$ *otherwise.*

The canonical evaluation function $[\![\ ]\!]^\Phi$ *is defined as in dynamic Routley–Meyer models.*

Lemma 6. *For each Λ and finite Φ, $\mathfrak{M}^\Phi_\Lambda$ is a dynamic Routley–Meyer model.*

Proof. We show that the "if" implication of (2) holds. Let $\varphi \in \underline{\Gamma}^+$ (recall that $\varphi \in \Phi$ as a result) and $T^\Phi \underline{\Sigma \Gamma \Delta}$ for some $\underline{\Sigma} \in L^\Phi$. The latter means that $\bar{1} \in \underline{\Sigma}^+ \subseteq \Sigma'$ for some non-trivial prime theory Σ' such that there are non-trivial prime theories Γ' and Δ', where $\underline{\Gamma}^+ \subseteq \Gamma'$ and $(\Delta' \cap \Phi) \subseteq \underline{\Delta}^+$, for which it holds that if $\varphi \to \psi \in \Sigma'$, then $\psi \in \Delta'$. But $\vdash_\Lambda \bar{1} \to (\varphi \to \varphi)$, so $\varphi \to \varphi \in \Sigma'$, so $\varphi \in \Delta'$, so $\varphi \in \underline{\Delta}^+$.

It is noteworthy that this argument could not have been simulated without $\bar{1}$ in the language. Then the only plausible definition of $\underline{\Sigma} \in L^\Phi$ is that $\underline{\Sigma}^+ \subseteq \Sigma'$ for some $\Sigma' \supseteq \Lambda$. However, the fact that $T^\Phi \underline{\Sigma}\underline{\Gamma}\underline{\Delta}$ allows us to infer that there is *some* $\Sigma'' \supseteq \underline{\Sigma}^+$, possibly different from $\Sigma' \supseteq \Lambda$, such that a version of (7) holds for some $\Gamma' \supseteq \underline{\Gamma}^+$ and $(\Delta' \cap \Phi) \subseteq \underline{\Delta}^+$. Hence, we cannot infer that $\varphi \to \varphi \in \Sigma''$.

Clearly if $\varphi \notin \Lambda$, then $\bar{1} \to \varphi \notin \Lambda$ and so $\langle \{\bar{1}\}, \{\varphi\} \rangle \in IP_\Lambda$. We want to show now that there is a state in $\mathfrak{M}_\Lambda^{\{\bar{1} \to \varphi\}^c}$ that satisfies $\bar{1}$ (i.e. it is a logical state), but not φ. Then we will have shown that φ is not valid in all dynamic Routley–Meyer frames. This yields a completeness result for PDL_0 right away as, in this case, it is not necessary to show that the frame underlying $\mathfrak{M}_{PDL_0}^{\{\bar{1} \to \varphi\}^c}$ satisfies any additional frame conditions.

Lemma 7. *Let $\underline{\Gamma} \in IP_\Lambda$ such that $\underline{\Gamma}^+ \cup \underline{\Gamma}^- \subseteq \Phi$. Then there is $\underline{\Delta} \in IP_\Lambda$ such that $\underline{\Gamma}^+ \subseteq \underline{\Delta}^+$, $\underline{\Gamma}^- \subseteq \underline{\Delta}^-$ and $\underline{\Delta}^+ \cup \underline{\Delta}^- = \Phi$.*

Proof. Similar to the proof of the Pair Extension Lemma, see [25, 92–94].

Let X, Y be subsets of S^Φ. We define αX as the set of such $\underline{\Gamma}$ where $\underline{\Gamma}[\![\alpha]\!]^\Phi \underline{\Delta}$ implies $\underline{\Delta} \in X$. Let us also define $f(\underline{\Gamma}) = \bigwedge \underline{\Gamma}^+$ and $f(\{\underline{\Gamma_1}, \dots, \underline{\Gamma_n}\}) = \bigvee_{i \leq n} f(\Gamma_i)$.

Lemma 8. *1. For all $\varphi \in \Phi$, $\varphi \in [\![\Sigma]\!]^\Phi$ iff $\varphi \in \underline{\Sigma}^+$.*
2. If $[\alpha]\varphi \in \Phi$ for some φ, then $X \subseteq \alpha Y$ implies that $\vdash_\Lambda f(X) \to [\alpha] f(Y)$.

Proof. See the Technical appendix.

Theorem 4. *Each formula valid in all dynamic Routley–Meyer frames belongs to PDL_0.*

Proof. If $\nvdash_{PDL_0} \varphi$, then $\nvdash_{PDL_0} \bar{1} \to \varphi$, so $\langle \{\bar{1}\}, \{\varphi\} \rangle$ in IP_{PDL_0}. By Lemma 6, $\mathfrak{M}_{PDL_0}^{\{\bar{1} \to \varphi\}^c}$ is a dynamic Routley–Meyer model. By Lemma 8, φ is not valid in the model. Hence, φ is not valid in all dynamic Routley–Meyer frames.

Theorem 5. *PDL_0 is a decidable set.*

Proof. Note that if Γ is finite, then so is Γ^c. The number of models in $\mathfrak{M}_{PDL_0}^{\Gamma^c}$ is at most $2^{|\Gamma^c|}$.

5 Beyond the minimal substrucutral PDL

In this section we discuss the applicability of our technique to some extensions of PDL_0.

5.1 Axiomatic extensions

Let us refer to the schemata shown in Figure 1 as "structural schemata". Similarly to modal logic, structural schemata *define* various properties of Routley–Meyer frames in the sense of correspondence theory—the defining schema holds in a frame iff the frame has the defined property. Figure 2 shows the frame properties defined by the structural schemata, see [25, ch. 11] for proofs and details ($T(xy)zw$ means $(\exists u)(Txyu \ \& \ Tuzw)$, $Tx(yz)w$ means $(\exists u)(Tyzu \ \& \ Txuw)$.)

B	$T(xy)zw \to Tx(yz)w$
C	$T(yx)zw \to Tx(yz)w$
CI	$Txyz \to Tyxz$
W	$Txyz \to T(xy)yz$
WI	$Txxx$
K	$Txyz \to x \leq z$

Fig. 2. Frame properties defined by the structural schemata shown in Fig. 1.

Let us denote as $\Lambda_{\mathsf{S}_1 \dots \mathsf{S}_n}$ the extension of Λ_0 with $\mathsf{S}_1 \dots \mathsf{S}_n$ as extra axiom schemata (in the obvious sense); $PDL_{\mathsf{S}_1 \dots \mathsf{S}_n}$ is the extension of $\Lambda_{\mathsf{S}_1 \dots \mathsf{S}_n}$ with MAX. A plausible conjecture is the following:

Conjecture 1. $PDL_{\mathsf{S}_1 \dots \mathsf{S}_n}$ is sound and complete with respect to the class of dynamic Routley–Meyer models with the properties defined by $\mathsf{S}_1 \dots \mathsf{S}_n$.

As it happens, the present technique can be used to establish only some special cases of Conjecture 1.

Theorem 6. *Let* $\mathsf{S}_1 \dots \mathsf{S}_n$ *be any combination of* CI, WI *and* K. *Then* $PDL_{\mathsf{S}_1 \dots \mathsf{S}_n}$ *is sound and complete with respect to the corresponding class of dynamic Routley–Meyer models.*

Proof. It is sufficient to show that the frame underlying $\mathfrak{M}^{\Phi}_{PDL_{\mathsf{S}_1 \dots \mathsf{S}_n}}$ has the corresponding frame properties. Let $\mathsf{S}_1 \dots \mathsf{S}_n = \mathsf{CI}$. Assume that $T^{\Phi}\underline{\Gamma}\underline{\Delta}\underline{\Sigma}$; hence, (7) holds for some appropriate Γ', Δ' and Σ'. Now assume that $\varphi \to \psi \in \Delta' \supseteq \underline{\Delta}^+$ and $\varphi \in \Gamma' \supseteq \underline{\Gamma}^+$. Using the axiom schema CI and the fact that Γ' is a prime theory, we have $(\varphi \to \psi) \to \psi \in \Gamma'$. Hence, $\psi \in \Sigma'$ by (7). The argument is similar in the remaining cases.

It is easy to see that our "finitary" technique *cannot* be used for some combinations of structural schemata. For instance, Λ_{BC} and Λ_{BCW} are fragments of the *undecidable* relevant logics $R{-}W$ and R [35], [25, ch. 15], so we cannot hope for a finite model property for these logics. However, our proof always produces a *finite* countermodel for a unprovable formula.

It is not straightforward to imagine a modification of our technique that would yield a proof of Conjecture 1 in these undecidable cases. We leave this as a curious open problem.

A surrogate strategy that might look promising at first is to work at least with an *infinitary* proof system $PDL^{\omega}_{\mathsf{S}_1 \dots \mathsf{S}_n}$ in the problematic cases (infinitary in the sense of containing an inference rule for α^* with a countable set of assumptions). It is shown in [14] that, in the case based on classical logic, using an infinitary proof system allows to construct a well-behaved *infinite* canonical model. This sounds promising for logics without the finite model property. However, as shown in [2], the Pair Extension Lemma does not hold for infinitary logics. We do not see how our proof could be rephrased without using the Pair Extension Lemma.

5.2 Adding diamonds

Another way to extend PDL_0 is to add to the language *primitive* existential modalities $\langle \alpha \rangle$ with evaluation defined as follows:

– $[\![\langle \alpha \rangle \varphi]\!]_{\mathfrak{M}} = \{x \mid (\exists y)(x [\![\alpha]\!]_{\mathfrak{M}} y \ \ \& \ \ y \in [\![\varphi]\!]_{\mathfrak{M}})\}$[5]

Without going into details we note two problems that arise from such an addition; both are related to the canonical model construction. Firstly, the presence of primitive diamonds requires to modify the definition of $R^{\Phi}(a)$ by adding the requirement that $R^{\Phi}(a)\underline{\Gamma}\underline{\Delta}$ only if, for all $\langle a \rangle \varphi \in \Phi$, if $\varphi \in \underline{\Delta}^+$, then $\langle a \rangle \varphi \in \underline{\Gamma}^+$. This modification makes it problematic to prove a version of Lemma 8; we have failed to provide a proof without the extra assumption that $R(a)$ is a serial relation.

Secondly, neither (5) nor any other condition presently assumed entail that $[\![\langle \alpha \rangle \varphi]\!]_{\mathfrak{M}}$ be an upset. Some additional frame condition is required, for example:

$$R(a)xy \ \ \& \ \ x \le x' \implies (\exists y')(y \le y' \ \ \& \ \ R(a)x'y') \tag{8}$$

or the stronger

$$R(a)xy \ \ \& \ \ x \le x' \implies R(a)x'y \tag{9}$$

On the assumption of either one of these conditions, however, the proof of Lemma 6 seems to fail.

A Technical appendix

Lemma 8.

1. For all $\varphi \in \Phi$, $\varphi \in [\![\Sigma]\!]^{\Phi}$ iff $\varphi \in \underline{\Sigma}^+$.
2. If $[\alpha]\varphi \in \Phi$ for some φ, then $X \subseteq \alpha Y$ implies that $\vdash_{\Lambda} f(X) \to [\alpha] f(Y)$.

Proof. Induction on subexpressions. 1. holds for $p \in At$ by definition and the inductive steps for the rest of the Boolean connectives are easy. The case for $[\alpha]\varphi$ is more complicated. Note that we have to prove that $[\alpha]\varphi \in \underline{\Gamma}^+$ iff $\underline{\Gamma} \in \alpha[\![\varphi]\!]^{\Phi}$. The "if" part is established using claim 2. for α (a subexpression of

[5] Note that $\langle \alpha \rangle \varphi$ can be defined as $\neg[\alpha]\neg\varphi$ in the present setting, but the defined modality does not yield the same truth condition—recall that $\neg\psi$ is defined as $\psi \to \bar{0}$.

$[\alpha]\varphi$) as follows. If $\underline{\Gamma} \in \alpha[\![\varphi]\!]^{\Phi}$, then $\vdash_{\Lambda} f(\underline{\Gamma}) \to [\alpha]f([\![\varphi]\!]^{\Phi})$ by 2. Note that $\vdash_{\Lambda} f([\![\varphi]\!]^{\Phi}) \to \varphi$ by the induction hypothesis ($\varphi \in \underline{\Delta}^{+}$ for all $\underline{\Delta} \in [\![\varphi]\!]^{\Phi}$) and so $\vdash_{\Lambda} [\alpha]f([\![\varphi]\!]^{\Phi}) \to [\alpha]\varphi$ using the fact that Λ contains MAX. Hence, $\vdash_{\Lambda} f(\underline{\Gamma}) \to [\alpha]\varphi$. Now $[\alpha]\varphi$ is assumed to be in Φ, so if it were the case that $[\alpha]\varphi \notin \underline{\Gamma}^{+}$, then $\underline{\Gamma} \notin IP_{\Lambda}$ contrary to our assumption. Hence, $[\alpha]\varphi \in \underline{\Gamma}^{+}$. The "only if" part is established by induction on the complexity of α using the MAX axioms for the action operators; we skip the details.

2. Assume that $X \subseteq aY$. Take an arbitrary $\underline{\Gamma} \in X$. Suppose, for the sake of contradiction, that

$$\nvdash_{\Lambda} f(\underline{\Gamma}) \to [a]f(Y).$$

Let $Z = \{\psi \mid [a]\psi \in \underline{\Gamma}^{+}\}$. It follows that

$$\nvdash_{\Lambda} \bigwedge Z \to f(Y).$$

Hence, by the Pair Extension Lemma, there is a non-trivial prime Λ-theory Δ such that $Z \subseteq \Delta$ and $f(Y) \notin \Delta$. Now consider $\underline{\Sigma} = \langle \Delta \cap \Phi, \bar{\Delta} \cap \Phi \rangle$ (where $\bar{\Delta}$ is the complement of Δ). Obviously $\underline{\Sigma} \in IP_{\Lambda}$ and $R^{\Phi}(a)\underline{\Gamma}\underline{\Sigma}$. Hence, by our assumption, $\underline{\Sigma} \in Y$, so

$$\vdash_{\Lambda} f(\underline{\Sigma}) \to f(Y),$$

but also

$$\vdash_{\Lambda} \bigwedge Z \to f(\underline{\Sigma})$$

(by the construction of $\underline{\Sigma}$), so

$$\vdash_{\Lambda} \bigwedge Z \to f(Y),$$

contrary to our assumption. Consequently, it has to be the case that $\vdash_{\Lambda} f(\underline{\Gamma}) \to [a]f(Y)$. The same argument can be repeated for all $\underline{\Gamma}_i \in X$. Hence, $\vdash_{\Lambda} f(X) \to [a]f(Y)$.

The inductive steps for concatenation and choice are easily established using the MAX axioms characterising these action operators. It is worthwhile to go through the cases for α^* and $\varphi?$. Assume first that $X \subseteq \alpha^* Y$. Hence

$$\vdash_{\Lambda} f(X) \to f(\alpha^* Y)$$

It is easily seen that $\alpha^* Y \subseteq \alpha(\alpha^* Y)$. Hence, using induction hypothesis for α,

$$\vdash_{\Lambda} f(\alpha^* Y) \to [\alpha]f(\alpha^* Y).$$

So, by the MAX rule characterizing α^*,

$$\vdash_{\Lambda} f(\alpha^* Y) \to [\alpha^*]f(\alpha^* Y).$$

Yet, we have

$$\vdash_{\Lambda} [\alpha^*]f(\alpha^* Y) \to [\alpha^*]f(Y)$$

(note that $\alpha^* Y \subseteq Y$ and use the monotonicity MAX rule). Therefore,

$$\vdash_{\Lambda} f(X) \to [\alpha^*]f(Y).$$

The case for $\varphi?$ is established as follows. Assume that $X \subseteq (\varphi?)Y$. Take an arbitrary $\underline{\Gamma} \in X$ and assume, for the sake of contradiction, that

$$\nvdash_\Lambda f(\underline{\Gamma}) \to [\varphi?]f(Y).$$

Hence, using the MAX rule for $\varphi?$,

$$\nvdash_\Lambda (f(\underline{\Gamma}) \wedge \varphi) \to f(Y).$$

Using the Pair Extension Lemma, there is a non-trivial prime theory Δ containing $f(\underline{\Gamma}) \wedge \varphi$ but not containing $f(Y)$. Now take $\underline{\Delta} = \langle \Delta \cap \Phi, \bar{\Delta} \cap \Phi \rangle$. It is clear that $\underline{\Gamma} \leq \Phi \underline{\Delta}$ and that $\underline{\Delta} \in [\![\varphi]\!]^\Phi$ (by induction hypothesis 1. applied to φ, the subexpression of $\varphi?$). By the definition of $[\![\varphi?]\!]^\Phi$, it follows that $\underline{\Gamma}[\![\varphi?]\!]^\Phi \underline{\Delta}$ and, hence, $\underline{\Delta} \in Y$. Consequently,

$$\vdash_\Lambda f(\underline{\Delta}) \to f(Y).$$

But this contradicts the observation that the prime theory Δ does not contain $f(Y)$. Hence, it must be the case that $\vdash_\Lambda (f(\underline{\Gamma}) \wedge \varphi) \to f(Y)$. Similar reasoning can be applied to each element of X, so

$$\vdash_\Lambda f(X) \to [\varphi?]f(Y).$$

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