Modelling sources of inconsistent information in paraconsistent modal logic

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Abstract

Epistemic logics based on normal modal logic are notoriously bad at handling inconsistent and yet non-trivial information. This fact motivates epistemic logics based on paraconsistent logic, examples of which can be traced back at least to the 1980s. These logics handle inconsistent and non-trivial information, but they usually do not articulate sources of the inconsistency. Yet, making the origin of an inconsistency present in a body of information explicit is important to assess the body—can we trace the mutually conflicting pieces of information to sources of information relevant to the body or is the inconsistency a result of an error unrelated to any outside sources? Is the inconsistency derived from various equally trustworthy sources or from a single source that is inconsistent itself? In this article we show that a paraconsistent modal logic, namely, the logic BK introduced by Odintsov and Wansing, is a first step toward a formalism capable of making these distinctions explicit. We interpret the accessibility relation between states in a model as a source relation—states accessible from a given state are seen as sources of potential justification of the information contained in the original state. This interpretation also motivates the study of a number of extensions of BK. We focus here on extensions of BK able to articulate the relation of compatibility between bodies of information and extensions working with labels explicitly differentiating between bodies of information. In the case of compatibility-based extensions a more detailed technical study including a completeness proof is provided; technical features of the simpler case of label-based extensions, on the other hand, are discussed without going into details.

1 Introduction

It is well known that epistemic logics[1] based on normal modal logic [12, 11] treat all *inconsistent* information as being equivalent. This is hard-wired into how information is treated within normal epistemic logic. Bodies of information, as represented by normal epistemic logics, are closed under all classically valid inference rules including *Ex falso quodlibet*

$$\frac{\varphi \quad {\sim}\varphi}{\psi}$$

This means that if a body of information contains (supports) a contradictory pair $\varphi, {\sim}\varphi$, then it contains (supports) every ψ. Consequently, all inconsistent bodies of information contain (support) *every* piece of information whatsoever and so they cannot be distinguished from each other. A related semantic point is that, on the *information-as-range* methodology adopted within normal epistemic logic [24], a body of information is represented by the set of possible worlds compatible with the body; however, the set of possible worlds compatible with each and every inconsistent body of information is the same, namely, the empty set. In many contexts, however, it is crucial to distinguish between inconsistent bodies of information and model logically the fact that different conclusions may be drawn from different inconsistent assumptions.

Paraconsistent logics invalidate *Ex falso* and so they allow us to differentiate between inconsistent bodies of information by drawing different conclusions from them. These logics are therefore a suitable basis for epistemic logic dealing with distinct inconsistent and yet non-trivial bodies of information.

Levesque [14] develops an epistemic logic based on the four-valued paraconsistent logic FDE of Belnap and Dunn [4, 3, 8, 9]. The idea is to replace possible worlds by more general "states" that allow formulas to be both true and false, or neither true nor false, while keeping most of the other aspects of epistemic logic intact. Similarly to the information-as-range approach, a body of information is represented by the set of states supporting all the information in the body. In a paraconsistent setting, however, the set of states may vary depending on the specific contradiction at hand.

Levesque's approach allows to distinguish between inconsistent bodies of information, but it fails to articulate *sources* of the inconsistency. Yet, making the origin of an inconsistency present in a body of information explicit is important to assess the body—can we trace the mutually conflicting pieces of information to sources of information relevant to the body or is the inconsistency a result of an error unrelated to any outside sources? Is the inconsistency derived from various equally trustworthy sources or from a single source that is inconsistent itself?

Example 1.1. In a conversation with a friend, Alice expresses contradictory beliefs about Carl, a prominent politician: she claims that Carl wants to grant asylum to a significant number of refugees but later in the discussion she mentions that Carl is xenophobic and does not want to allow any foreigners to enter

[1] To avoid complications unnecessary with respect to our motivation, we do not distinguish between "epistemic" and "doxastic" in this article and we use "epistemic" in a sense covering all relevant information-relative adjectives. This approach is taken in some literature on epistemic logic; see [15, xi], for example.

the country. Assume that Alice's belief about Carl's xenophobia is based on the opinions of her trusted friend Cathy and the report about refugees is taken from an article in her favorite newspaper. Alice does not realize that these two are in mutual contradiction.

In an unrelated situation, Bob, being questioned by the police, claims that he was not in town at the time of the crime, but admits to being home by himself later during the interrogation. Assume that neither side of the inconsistency in Bob's testimony can be substantiated in any reliable way; Bob cannot produce any evidence supporting either side of his inconsistent statement.

It seems natural to say that Bob's testimony is somewhat worse off than Alice's beliefs about Carl. Imagine further that Alice's friend Cathy is in fact the author of the article about refugees and that Alice knows this. This is again relevant to our assessment of the situation—Alice's source itself is inconsistent.

Making justice to the distinctions pointed out in our example requires us to make explicit the fact that, while assessing a body of information, one may *turn to other bodies of information*, understood as *sources* for the original body—sometimes an inconsistency is a result of following these sources too closely without reflection; sometimes no such excuse is available. This requires to make explicit use of *relations between bodies of information*, something not commonly present in epistemic logic, normal or paraconsistent.

This article shows that the "four-valued" modal logic BK introduced by Odintsov and Wansing [19] (and studied subsequently in [16, 17, 20, 18], for example) is a suitable starting point. We may see states in a four-valued Kripke model as potentially inconsistent (and incomplete) bodies of information and the accessibility relation as articulating a "source" relation between these bodies of information. On this reading, Rxy means that y is a source for the information contained in x. We can also say that sources for x serve as *potential justifications* for the information contained in x.[2] For instance, take a news article, x, and $R(x)$ consisting of all the materials, interviews etc. from which the information conveyed by the article derives. It is clear that the relation of the article to the sources may be strong, but also very loose. Some articles build on their sources very closely, some add information that cannot be traced to any sources whatsoever. As another example, take x consisting of a student's beliefs about the implications of the Special Theory of Relativity and assume that the student is required to justify these beliefs, for instance during an exam. In typical cases, the student can provide a number of sources (references, calculations etc.) and it is the job of the examiner to asses the relation of these sources to the student's beliefs. Or consider an empirical scientist writing an article, x, based on her experiments and experimental and theoretical work done by other scientists, $R(x)$. Bílková et al. [5] adopt a similar reading of the accessibility relation in models for substructural modal logics.

As shown in this article, the source-interpretation of the semantics for BK gives us the necessary tools for articulating the distinctions pointed out in Example 1.1 and it also motivates interesting extensions of BK. We focus here on extensions of BK able to articulate the relation of compatibility between bodies of information and extensions working with labels explicitly differentiating

[2] Justification logics [2] make the relation between information and justification explicit, but the setting is one where justifications are represented syntactically. In our framework, bodies of information and sources are *semantic* objects of the same kind.

between bodies of information.

The article is structured as follows. Section 2 provides the basic information about BK, the source interpretation of modal accessibility and shows how the framework articulates different ways inconsistencies may be related to sources. In Section 3 we add to models for BK a compatibility relation and show that specific formulas of our language define interesting classes of models in the standard modal correspondence sense. In Section 4 we introduce a new negation connective based on the compatibility relation (in the style of [10, 21]) and prove a completeness result for the extension of BK with the negation. Interesting technical and philosophical features of the logic are pointed out; for example, the logic is not closed under uniform substitution and there is a certain tension between the properties of the compatibility-based negation on one hand and a natural requirement concerning the compatibility relation on the other. In Section 5 we return to BK and we show how relative reliability of sources can be expressed using special formulas called labels.

2 The Belnapian modal logic BK

In this section we outline the paraconsistent modal logic BK (Section 2.1) and we show how it articulates the distinctions analogous to those pointed out in Example 1.1 (Section 2.2).

2.1 BK

This subsection outlines the technical features of BK, introduced by Odintsov and Wansing [19] and later studied, for example, in [16, 17, 20, 18]. We build mainly on the presentation in [19]. The logic BK is an FDE-based modal logic; it extends the Belnap–Dunn logic [4, 3, 8, 9] with "weak implication" $\to$ in the sense of [1], the falsum constant $\bot$ and normal modal operators $\Box, \Diamond$. BK can be seen as a conservative extension of the smallest normal modal logic K with a "strong" negation $\sim$ allowing truth-value gaps and gluts.

Formally, the language $\mathcal{L}$ (over a countable set of atomic formulas AT) contains a nullary connective $\bot$, unary connectives $\sim, \Box, \Diamond$ and binary connectives $\wedge, \vee, \to$. A second negation $\neg\varphi$ is defined as $\varphi \to \bot$, $\top$ is defined as $\neg\bot$ and $\varphi \leftrightarrow \psi$ is defined as $(\varphi \to \psi) \wedge (\psi \to \varphi)$. "Strong implication" $\varphi \Rightarrow \psi$ is defined as $(\varphi \to \psi) \wedge ({\sim}\psi \to {\sim}\varphi)$ and "strong equivalence" $\varphi \Leftrightarrow \psi$ is defined as $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$. We use "$\mathcal{L}$" also to denote the set of formulas of the language.

A basic $\mathcal{L}$-frame ("frame" for short) is a standard Kripke frame $F = \langle S, R \rangle$, that is, a non-empty set (of "states") S with a binary ("accessibility") relation R on S. A basic $\mathcal{L}$-model M ("model" for short) based on F adds to F two functions V^+ and V^- from AT to subsets of S ("the positive and the negative valuation"). [3]

For every M, we define two relations $\models^+_M, \models^-_M \subseteq \mathscr{P}(S \times \mathcal{L})$ as follows (we usually omit the subscript "M"):

1. $x \models^+ p$ iff $x \in V^+(p)$;
   $x \models^- p$ iff $x \in V^-(p)$

[3] Alternatively, equivalent semantics for BK can be formulated using four-valued valuations, i.e. functions from $AT \times S$ to $\{True, False, Neither, Both\}$ or to subsets of $\{T, F\}$—cf. [19].

2. $x \models^+ \bot$ for no x;
   $x \models^- \bot$ for all x

3. $x \models^+ {\sim}\varphi$ iff $x \models^- \varphi$;
   $x \models^- {\sim}\varphi$ iff $x \models^+ \varphi$

4. $x \models^+ \varphi \wedge \psi$ iff $x \models^+ \varphi$ and $x \models^+ \psi$;
   $x \models^- \varphi \wedge \psi$ iff $x \models^- \varphi$ or $x \models^- \psi$

5. $x \models^+ \varphi \vee \psi$ iff $x \models^+ \varphi$ or $x \models^+ \psi$;
   $x \models^- \varphi \vee \psi$ iff $x \models^- \varphi$ and $x \models^- \psi$

6. $x \models^+ \varphi \to \psi$ iff $x \not\models^+ \varphi$ or $x \models^+ \psi$;
   $x \models^- \varphi \to \psi$ iff $x \models^+ \varphi$ and $x \models^- \psi$

7. $x \models^+ \Box\varphi$ iff for all y, if Rxy, then $y \models^+ \varphi$
   $x \models^- \Box\varphi$ iff there is y such that Rxy and $y \models^- \varphi$

8. $x \models^+ \Diamond\varphi$ iff there is y such that Rxy and $y \models^+ \varphi$
   $x \models^- \Diamond\varphi$ iff for all y, if Rxy, then $y \models^- \varphi$

The positive and negative extension of a formula φ is defined as $[\![\varphi]\!]^+_M = \{x \mid x \models^+_M \varphi\}$ and $[\![\varphi]\!]^-_M = \{x \mid x \models^-_M \varphi\}$, respectively. Formula φ is valid in a model M (M-valid) iff $[\![\varphi]\!]^+_M = S_M$; φ is valid in a frame F iff it is M-valid for all M based on F. A set of formulas Γ entails φ in M ($\Gamma \models_M \varphi$) iff $\bigcap_{\psi\in\Gamma}[\![\psi]\!]^+_M \subseteq [\![\varphi]\!]^+_M$; Γ entails φ in frame F ($\Gamma \models_F \varphi$) iff $\Gamma \models_M \varphi$ for all M based on F.

The relation $\models^+$ is seen as a *verification* relation, i.e. $x \models^+ \varphi$ means that state x verifies φ. We shall sometimes write $x \models \varphi$ instead of $x \models^+ \varphi$. Similarly, $\models^-$ is seen as the *falsification* relation, thus $x \models^- \varphi$ means that x falsifies φ. The crucial feature of the semantics is that a formula can be *both* verified and falsified in a state (the semantics allows truth value gluts) and, similarly, a formula can be *neither* verified nor falsified in a state (the semantics allows truth value gaps). A quick look at the verification/falsification conditions specified above reveals that this is, in fact, the only difference between the present semantics and the Kripke semantics for normal modal logics; the clauses for the connectives in our language mimic the Kripke-style clauses (the negation of a formula is verified iff the negated formula is falsified and it is falsified iff the negated formula is verified, a conjunction is verified iff both conjuncts are verified and it is falsified iff at least one conjunct is falsified etc.).

The logic BK is the least set of formulas containing the axioms of classical propositional logic in the language $\{\wedge, \vee, \to, \bot\}$, the strong negation axioms

$$
\begin{array}{rll}
{\sim}{\sim}p \leftrightarrow p & {\sim}(p \vee q) \leftrightarrow ({\sim}p \wedge {\sim}q) & {\sim}(p \to q) \leftrightarrow (p \wedge {\sim}q) \\
{\sim}\bot & {\sim}(p \wedge q) \leftrightarrow ({\sim}p \vee {\sim}q) &
\end{array}
$$

the K axioms

$$
\Box(p \to q) \to (\Box p \to \Box q) \qquad \Box\top
$$

and the modal interaction axioms

$$\neg\Box p \leftrightarrow \Diamond\neg p \qquad \Box p \Leftrightarrow {\sim}\Diamond{\sim}p$$
$$\neg\Diamond p \leftrightarrow \Box\neg p \qquad \Diamond p \Leftrightarrow {\sim}\Box{\sim}p,$$

closed under uniform substitution, modus ponens and the monotonicity rules

$$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi} \qquad \frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}$$

Formula φ is *derivable* in BK from the set of assumptions Γ, in symbols $\Gamma \vdash_{\mathsf{BK}} \varphi$, if φ can be obtained from Γ and BK by means of modus ponens.

Theorem 2.1 (Odintsov and Wansing [19])**.** *$\Gamma \vdash_{\mathsf{BK}} \varphi$ iff $\Gamma \models_F \varphi$ for each frame F.*

It is clear that *Ex falso quodlibet* fails in BK, i.e. $p, {\sim}p \nvdash_{\mathsf{BK}} q$. Another remarkable feature of BK is that it is not closed under the Replacement rule

$$\frac{\varphi \leftrightarrow \psi}{\chi(p/\varphi) \leftrightarrow \chi(p/\psi)}$$

For example, ${\sim}(p \to q) \leftrightarrow (p \land {\sim}q)$ is in BK, but ${\sim}{\sim}(p \to q) \leftrightarrow {\sim}(p \land {\sim}q)$ is not.[4] Nevertheless, BK is closed under the Positive replacement rule

$$\frac{\varphi \leftrightarrow \psi}{\chi(p/\varphi) \leftrightarrow \chi(p/\psi)} \quad \chi \text{ is } {\sim}\text{-free}$$

and the Weak replacement rule

$$\frac{\varphi \Leftrightarrow \psi}{\chi(p/\varphi) \leftrightarrow \chi(p/\psi)}$$

It is easy to check that $x \models \neg\varphi$ iff $x \not\models \varphi$ and that $\varphi \models_M \psi$ iff $\models_M \varphi \to \psi$. The first fact can be seen as stating that $\neg\varphi$ is the Boolean negation of φ but we need to be careful here. For example, if p is both verified and falsified, then $\neg\neg p$ is verified but not falsified and if p is neither verified nor falsified, then $\neg\neg p$ is only falsified. Hence, the "value" of $\neg\neg p$ is not necessarily identical to the "value" of p.

2.2 Sources and contradictions

We now show that the source-related interpretation of the semantics for BK can be used to articulate the different ways an inconsistency present in a body of information may be related to the sources for the body of information.

Take a frame $F = \langle S, R\rangle$. Recall that, on our interpretation outlined in the Introduction, S is a set of bodies of information (or as we shall equivalently say, "information states") and R is the source relation (Rxy means that y is a source for x). The relations $\models^+$ and $\models^-$ are seen as relations of support and rejection, respectively, between information states (represented by states of a model M) and pieces of information (represented by formulas of $\mathcal{L}$).

Our main motivation is to express the relation between information supported by a state x and the information supported by the sources for x, that is,

[4] If it were, then we could show that $(p \to q) \leftrightarrow ({\sim}p \lor q)$ is in BK using De Morgan laws and the Double negation law.

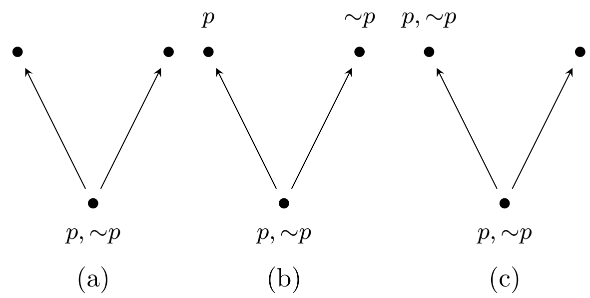

Figure 1: Three kinds of contradictory information.

states in $R(x)$. The goal is to differentiate between situations where a contradiction in x cannot be traced to sources for x, situations where a contradiction in x derives from two mutually inconsistent sources, and situations where the inconsistency derives from a single inconsistent source for x (recall Example 1.1).

On our interpretation of the semantics, $x \models^+ \varphi \wedge \Diamond\varphi$ iff φ is supported by the body of information x and also by some source for x. This is all we need to differentiate between the three situations mentioned above. Figure 1 depicts these three situations. Let x be the bottom state in all three models. In the model (a), x supports contradictory information about p, but the contradiction is not related to the sources for x in any way as no source for x supports either p or $\sim p$. In the model (b), the contradictory information supported by x is supported also by the sources for x: one of them gives positive information about p and the other one gives negative information. In the model (c), the contradiction in x is supported by a single contradictory source for x. To simplify notation, we define

$$\Diamonddot\varphi := \varphi \wedge \Diamond\varphi$$

($\Diamonddot\varphi$ may be read as "φ is supported (by the present state) and justified by a source (of the present state)" or "φ is supported and justifiable", for example.) Formally,

$$\begin{array}{ll} \text{(a)} & x \models \neg(\Diamonddot p \wedge \Diamonddot{\sim} p) \wedge \neg\Diamonddot(p \wedge \sim p) \\ \text{(b)} & x \models (\Diamonddot p \wedge \Diamonddot{\sim} p) \wedge \neg\Diamonddot(p \wedge \sim p) \\ \text{(c)} & x \models \Diamonddot(p \wedge \sim p) \end{array}$$

We can see $\Diamonddot$ as a kind of epistemic operator; it is thus natural to look at familiar properties of epistemic operators in normal epistemic logic and to compare them to the properties of $\Diamonddot$ in our setting. This is the content of the following proposition.

Proposition 2.2. *The following are not valid in* BK*, resp. do not preserve validity in* BK*:*

(a-) $\varphi / \Diamonddot\varphi$ *($\Diamonddot$-necessitation)*

(b-) $(\Diamonddot\varphi \wedge \Diamonddot\psi) \to \Diamonddot(\varphi \wedge \psi)$ *($\Diamonddot$-regularity)*

(c-) $(\Diamonddot\varphi \wedge \Diamonddot{\sim}\varphi) \to \Diamonddot\psi$ *($\Diamonddot$-explosion 1)*

(d-) $\Diamonddot(\varphi \wedge {\sim}\varphi) \to \Diamonddot\psi$ *($\Diamonddot$-explosion 2)*

(e-) $\Diamonddot(\varphi \vee \psi) \to (\Diamonddot\varphi \vee \Diamonddot\psi)$ *($\Diamonddot$-primeness)*

(f-) $\Diamonddot\varphi \to \Diamonddot\Diamonddot\varphi$ *(positive $\Diamonddot$-introspection)*

(g-) ${\sim}\Diamonddot\varphi \to \Diamonddot{\sim}\Diamonddot\varphi$ *(negative $\Diamonddot$-introspection)*

The following are valid in BK*:*

(a+) $\Diamonddot(\varphi \wedge \psi) \to (\Diamonddot\varphi \wedge \Diamonddot\psi)$ *($\Diamonddot$-monotonicity)*

(b+) $\Diamonddot\varphi \to \varphi$ *($\Diamonddot$-factivity)*

Proof. (a-) Take a model M where $V^+(p) = S$ and $R(x) = \emptyset$ for some x. Then $x \not\models \Diamond p$. (b-) Let $S = \{x, y, z\}$, $V^+(p) = \{x, y\}$, $V^+(q) = \{x, z\}$ and $R = \{\langle x, y\rangle, \langle x, z\rangle\}$. Then $x \models \Diamond p \wedge \Diamond q$, but $x \not\models \Diamond(p \wedge q)$. (c-) Let $S = \{x, y, z\}$, $V^+(p) = \{x, y\}$, $V^-(p) = \{x, z\}$, $V^+(q) = \emptyset$ and $R = \{\langle x, y\rangle, \langle x, z\rangle\}$. Then $x \models \Diamonddot p \wedge \Diamonddot{\sim}p$, but $x \not\models \Diamonddot q$. (d-) Similar to (c-); let $V^+(p) = V^-(p)$. (e-) Let $S = \{x, y\}$, $R = \{\langle x, y\rangle\}$, $V^+(p) = \{x\}$ and $V^+(q) = \{y\}$. Then $x \models (p \vee q) \wedge \Diamond(p \vee q)$, but $x \not\models \Diamonddot p \vee \Diamonddot q$. The rest follows from the $\models^+$-condition for $\vee$. (f-) and (g-) are easy, it is sufficient to note that they boil down to

$$(\varphi \wedge \Diamond\varphi) \to (\varphi \wedge \Diamond\varphi \wedge \Diamond(\varphi \wedge \Diamond\varphi))$$

and

$${\sim}(\varphi \wedge \Diamond\varphi) \to ({\sim}(\varphi \wedge \Diamond\varphi) \wedge \Diamond{\sim}(\varphi \wedge \Diamond\varphi))$$

(a+) follows from the $\models^+$-condition for conjunction, (b+) is trivial. □

Proposition 2.2 shows that $\Diamonddot$ is quite weak. Let us discuss the points of the proposition in turn: (a-) a valid formula may not be justifiable since we may have a body of information without any sources; (b-) justifiable support may not be closed under conjunctions as there may be distinct sources justifying the conjuncts, but no single one justifying the conjunction; (a+) the converse, however, holds as support is closed under conjunction elimination; (c-) two distinct sources may provide conflicting information about a formula φ without any of them providing positive information about some completely unrelated formula ψ; and similarly for one inconsistent source (d-).

We note that an axiom analogous to (e-) was a source of some concern about the framework of [5]. In our setting, there is even a model M such that R_M is a function and (e-) is not valid in M. The converse of (e-) is valid in every M.

It is not hard to observe, however, that (a-) preserves validity in every model with a serial R (i.e., $\forall x \exists y(Rxy)$); (b-) is valid in every model where R is a function (i.e., $\forall xyz(Rxy \wedge Rxz \to y = z)$; and (c-) is valid in every model where every state is consistent (i.e., $V^+(p) \cap V^-(p) = \emptyset$ for all p) (though trivially so, as $[\![\varphi \wedge {\sim}\varphi]\!]^+ = \emptyset$ in such models for all φ). The formulas usually characterizing positive and negative introspection have a slightly different meaning in our source based approach—they are related to "second-order confirmation". For example, (f-) says that if φ is supported by x and justified by a source y, then there is a source z supporting the information that φ is supported and justified.

We note that similar facts would hold if we defined $\Diamonddot$ within the normal modal logic K and not within BK. Notable exceptions, of course, are the explosion axioms (d) and (e) of Proposition 2.2.

3 Compatibility

Admittedly, our requirements concerning sources were quite weak so far. One way to put more flesh on these bones is to relate sources to the notions of compatibility. One may require, for example, that sources be self-compatible; that sources for a state be compatible with that state; or that sources for a given state be mutually compatible.

But, one may ask, what does it take for two information states to be mutually compatible? A necessary condition for two states x and y being compatible, we submit, is

(NC) There is no φ such that x supports φ and y supports $\sim\varphi$.

Our semantics can be made to comply with (NC) as follows. For non-modal φ, (NC) is enforced by requiring that, for all $p \in AT$, if x supports (rejects) p, then y does not reject (support) p, and vice versa. For modal formulas, however, an additional condition is needed. The easiest way to enforce the condition is to add a binary *compatibility* relation on states of the frame. In what follows, (when M is clear form the context) let $V^{\pm}(x) = \{p \mid x \in V^{\pm}(p)\}$ and $[\![x]\!]^{\pm} = \{\varphi \mid x \in [\![\varphi]\!]^{\pm}\}$, for $\pm \in \{+, -\}$.

Definition 3.1. A *compatibility frame* is a tuple $F = \langle S, R, C\rangle$ where $S \neq \emptyset$ and R, C are binary relations on S such that

$$Cxy \implies Cyx \tag{1}$$
$$Cxy \text{ and } Rxz \implies (\exists w)(Ryw \;\&\; Czw) \tag{2}$$

A *compatibility $\mathcal{L}$-model M* based on a compatibility frame F is a tuple $\langle F, V^+, V^-\rangle$, where V^+, V^- are functions from AT to $\mathscr{P}(S)$ such that

$$Cxy \implies V^+(x) \cap V^-(y) = \emptyset \tag{3}$$

Validity of formulas in compatibility models is defined as usual; φ is valid in a compatibility frame iff it is valid in each compatibility model based on the frame. Entailment in a model and a class of frames is defined as usual.

It is clear that since Cxy implies Cyx, Cxy implies $V^-(x) \cap V^+(y) = \emptyset$.

Proposition 3.2. *If Cxy, then $[\![x]\!]^+ \cap [\![y]\!]^- = \emptyset$ and $[\![x]\!]^- \cap [\![y]\!]^+ = \emptyset$.*

Proof. The first claim is established by induction on the complexity of formulas; the second claim follows from the first one and (1).

The base case follows immediately from Definition 3.1 and the clauses for non-modal connectives are established by easy induction.

Consider modal formulas. Assume Cxy. $x \models \Diamond\varphi$ iff there is $z \in R(x)$ such that $z \models \varphi$. By (2), there is $w \in R(y)$ such that Czw. By the induction hypothesis, $w \not\models \sim\varphi$, so $y \not\models \sim\Diamond\varphi$. The fact that $x \models \sim\Diamond\varphi$ implies $y \not\models \Diamond\varphi$ is established similarly, using (1) in addition to (2). The case for $\Box$ is analogous. $\square$

Condition (1) corresponds to the intuitive assumption that any reasonable compatibility relation should be symmetric. Conditions (2)–(3) establish (NC), as witnessed by Proposition 3.2.

Frame condition	*Frame class label*
(SC) $Rxy \implies Cxy$	**SC**
(SSC) $Rxy \implies Cyy$	**SSC**
(SMC) Rxy and $Rxz \implies Cyz$	**SMC**

Figure 2: Some intuitive frame classes.

As mentioned earlier, intuitive considerations may lead us to adopting additional frame conditions. Fig. 2 lists some of these conditions. The class **SC** (‘*s*ources are *c*ompatible’) requires every source for a state to be compatible with that state. As a result, the information supported by a state never contradicts the sources for the state. The class **SSC** (‘*s*ources are *s*elf-*c*ompatible’) requires every source to be self-compatible. As a result, no reliable source supports an explicit contradiction $\varphi \wedge {\sim}\varphi$. The class **SMC** (‘*s*ources are *m*utually *c*ompatible’) requires every pair of sources for a given state to be mutually compatible. As a result, contradictions are never implicitly supported. Note that **SMC** $\subset$ **SSC**.

We say that a formula *defines a class of frames* $\mathbf{F}$ in case $F \in \mathbf{F}$ iff the formula is valid in F.

Proposition 3.3.

(a) **SC** *is defined by (*SC*)* $\Diamond{\sim}p \to \neg p$

(b) **SSC** *is defined by (*SSC*)* $\neg\Diamond(p \wedge {\sim}p)$

(c) **SMC** *is defined by (*SMC*)* $\neg(\Diamond p \wedge \Diamond{\sim}p)$

Proof. It is easy to check that the formulas are valid in the respective frame classes. We give a detailed proof of the converse implication in case of (a) only, the other cases are established similarly. Assume that we have a frame F such that Rxy and not Cxy for some x, y. Define a model M on F as follows: $V^{-}(p) = \{y\}$ and $V^{+}(p) = \{x\}$ (this is possible since not Cxy). It is clear that $x \models^{+}_{M} \Diamond{\sim}p$ but also $x \models^{+}_{M} p$. $\square$

Proposition 3.3 has a number of interesting consequences. For instance, the following schemes are valid in **SC**:

$$(\Diamond\varphi \vee \Diamond{\sim}\varphi) \to \neg(\varphi \wedge {\sim}\varphi) \tag{4}$$

$$(\Diamond\varphi \wedge \Diamond{\sim}\varphi) \to \neg(\varphi \vee {\sim}\varphi) \tag{5}$$

As a consequence of (4), if a contradiction is supported by a state, then it cannot be supproted by its sources in any way; if a state supports $\varphi \wedge {\sim}\varphi$, then the conflicting conjuncts are not supported by sources.

Remark 3.4. States x in our model are seen as bodies of information and $R(x)$ as a set of sources that can be used to justify the information contained in x. Informally, we may also think about x as being the information believed by (available to, entertained by...) an agent. The question is what is the formal counterpart, on this interpretation, of the information state of the agent. Is it x, $\{x\} \cup R(x)$, $\{x\} \cup R(x) \cup R^2(x)$...? In other words, are sources (sources of sources, ...) a part of the information state? We do not decide the issue here;

we just note that the "distances" between a state supporting φ and a source justifying φ can be formalized using iterated $\Diamond$. Generally, $\varphi \wedge \Diamond^k \varphi$ says that φ is supported and justified by a source of ... a source (k times). Of course, our basic language cannot express the fact that there is a finite path from the present state to a source of ... a source of the present state. To this end, we would need to employ the reflexive transitive closure modality $\Diamond^*$ (the semantic conditions associated with this modality involve the reflexive transitive closure R^* of the accessibility relation R). Such a modality is studied in the context of paraconsistent Propositional Dynamic Logic in [22].

Let $\mathsf{BK}{+}\mathsf{SC}$ be the least set of formulas containing BK and the formula SC closed under uniform substitution, modus ponens and the monotonicity rules; $\mathsf{BK}{+}\mathsf{SSC}$ and $\mathsf{BK}{+}\mathsf{SMC}$ are defined similarly. It is relatively straightforward to show that $\Gamma \models_{\mathbf{SC}}$ iff φ is derivable from Γ and $\mathsf{BK}{+}\mathsf{SC}$ by means of modus ponens; and similarly for **SSC** and **SMC**. We omit the details.

4 Compatibility negation

We have seen in the previous section that the compatibility relation C is useful in articulating interesting relations between sources. However, the relation usually plays additional roles as well. In the relational semantics for substructural logics, C is used to interpret a negation connective [10, 21]. It is interesting to look at the interaction between such a connective and the Belnapian negation $\sim$.

Let $\mathcal{L}^\frown$ be $\mathcal{L}$ extended with a new unary connective '$\frown$', called the *compatibility negation*. Intuitively, $\frown\varphi$ says that φ is incompatible with the present information. In other words, no state compatible with the present state supports φ (every state that supports φ is incompatible with the present state). We will interpret the language in compatibility models. The new verification and falsification conditions are (the verification condition is standard in the literature on substructural logics [10, 21]):

- $x \models^+ \frown\varphi$ iff for all y, if Cxy, then $y \not\models^+ \varphi$
- $x \models^- \frown\varphi$ iff there is y such that Cxy and $x \models^+ \varphi$

Validity and entailment in compatibility models and classes of frames are defined as usual.

An $\mathcal{L}^\frown$-substitution is any map $AT \to \mathcal{L}^\frown$. An $\mathcal{L}^\frown$-substitution instance of a formula φ is the result of replacing each occurrence of p in φ (for all $p \in AT$) by $\varsigma(p)$ (where ς is an $\mathcal{L}^\frown$-substitution).

The logic $\mathsf{BK}^\frown$ is the smallest set of $\mathcal{L}^\frown$-formulas containing

1. all $\mathcal{L}^\frown$-substitution instances of classical tautologies in the language $\{\wedge, \vee, \to, \bot\}$,
2. the strong negation axioms
$$\begin{array}{ccc} \sim\sim\varphi \leftrightarrow \varphi & \sim(\varphi \vee \psi) \leftrightarrow (\sim\varphi \wedge \sim\psi) & \sim(\varphi \to \psi) \leftrightarrow (\varphi \wedge \sim\psi) \\ \sim\bot & \sim(\varphi \wedge \psi) \leftrightarrow (\sim\varphi \vee \sim\psi) & \end{array}$$
3. the K axioms
$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \qquad \Box\top$$

4. the modal interaction axioms

$$\begin{array}{ll} \neg\Box\varphi \leftrightarrow \Diamond\neg\varphi & \Box\varphi \Leftrightarrow {\sim}\Diamond{\sim}\varphi \\ \neg\Diamond\varphi \leftrightarrow \Box\neg\varphi & \Diamond\varphi \Leftrightarrow {\sim}\Box{\sim}\varphi, \end{array}$$

5. the compatibility negation axioms

$$\begin{array}{rrl} \varphi \to \frown\frown\varphi & \chi \to \frown{\sim}\chi & \chi \text{ is } \frown\text{-free} \\ (\frown\varphi \wedge \frown\psi) \to \frown(\varphi \vee \psi) & \neg\frown\varphi \to {\sim}\frown\varphi & \\ \Diamond\frown\varphi \to \frown\Box\varphi & {\sim}\frown\varphi \to \neg\frown\varphi & \end{array}$$

and closed under modus ponens, the monotonicity rules

$$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi} \qquad \frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}$$

and the transposition rule

$$\frac{\varphi \to \psi}{\frown\psi \to \frown\varphi}$$

(where φ, ψ range over $\mathcal{L}^\frown$ formulas).

The derivability relation $\vdash_{\mathsf{BK}^\frown}$ is defined as expected: $\Gamma \vdash_{\mathsf{BK}^\frown} \varphi$ iff φ can be obtained from Γ and $\mathsf{BK}^\frown$ by means of modus ponens.

It is a matter of easy checking to establish the validity of each $\varphi \in \mathsf{BK}^\frown$ in all compatibility frames. An axiom schema that deserves special attention, however, is $\chi \to \frown{\sim}\chi$. Proposition 3.2 entails that each instance of the schema where χ is $\frown$-free is valid in each compatibility frame. The restriction is essential, as shown by the following example.

Example 4.1. Take a frame where $S = \{x, y, z\}$, $R = \emptyset$ and C is the symmetric closure of $\{\langle x, y\rangle, \langle y, z\rangle\}$. Let $V^+(p) = \{z\}$, $V^-(p) = \emptyset$ and $V^+(q) = V^-(q) = \emptyset$ for all other $q \in AT$. This is obviously a compatibility model. It is clear that $x \models^+ \frown p$ and $y \not\models^+ \frown p$. The latter (together with the falsification condition for $\frown$) entails that $y \models^- \frown p$ and so $y \models^+ {\sim}\frown p$. This means that $x \not\models^+ \frown{\sim}\frown p$. Hence, $\frown p \to \frown{\sim}\frown p$ is not valid in all compatibility frames.

Example 4.1 shows that the set of formulas valid in a compatibility frame is not in general closed under uniform $\mathcal{L}^\frown$-substitution. This observation motivates our definition of $\mathsf{BK}^\frown$ in terms of schemes

We will prove now that $\Gamma \vdash_{\mathsf{BK}^\frown} \varphi$ iff $\Gamma \models_F \varphi$ for each compatibility frame F. We use the techniques of [21] and [19].

A *prime* $\mathsf{BK}^\frown$*-theory* is a subset of $\mathcal{L}^\frown$ that contains $\mathsf{BK}^\frown$, does not contain $\bot$, is closed under $\vdash_{\mathsf{BK}^\frown}$ (i.e. Δ is a prime $\mathsf{BK}^\frown$-theory only if $\Gamma \subseteq \Delta$ and $\Gamma \vdash_{\mathsf{BK}^\frown} \varphi$ imply $\varphi \in \Delta$) and contains $\varphi \vee \psi$ only if it contains either φ or ψ.

Let $\Delta, \Sigma \subseteq \mathcal{L}^\frown$. We say that $\langle \Delta, \Sigma\rangle$ is a $\mathsf{BK}^\frown$*-pair* iff there are no finite $\Delta' \subseteq \Delta$ and $\Sigma' \subseteq \Sigma$ such that

$$\bigwedge \Delta' \to \bigvee \Sigma' \quad \in \quad \mathsf{BK}^\frown$$

We will use 'pair' instead of '$\mathsf{BK}^\frown$-pair' and 'theory' instead of '$\mathsf{BK}^\frown$-theory' in this section.

Lemma 4.2 (Pair Extension Theorem). *If $\langle \Delta, \Sigma\rangle$ is a pair, then there is a prime theory Γ such that $\Delta \subseteq \Gamma$ and $\Sigma \cap \Gamma = \emptyset$.*

Proof. (This is a standard proof of the result based on [21, 92–95].) Let $\langle \varphi_n \mid n \in \omega \rangle$ be an enumeration of $\mathcal{L}^\frown$. Define a sequence

$$\langle \langle \Delta_n, \Sigma_n \rangle \mid n \in \omega \text{ and } \Delta_n, \Sigma_n \subseteq \mathcal{L}^\frown \rangle$$

as follows: $\langle \Delta_0, \Sigma_0 \rangle = \langle \Delta, \Sigma \rangle$ and

$$\langle \Delta_{n+1}, \Sigma_{n+1} \rangle = \begin{cases} \langle \Delta_n \cup \{\varphi_n\}, \Sigma_n \rangle & \text{if } \langle \Delta_n \cup \{\varphi_n\}, \Sigma_n \rangle \text{ is a pair} \\ \langle \Delta_n, \Sigma_n \cup \{\varphi_n\} \rangle & \text{otherwise.} \end{cases}$$

Claim 4.3. If $\langle \Delta_n, \Sigma_n \rangle$ is a pair, then so is $\langle \Delta_n \cup \{\varphi_n\}, \Sigma_n \rangle$ or $\langle \Delta_n, \Sigma_n \cup \{\varphi_n\} \rangle$.
If both are not pairs, then there are conjunctions χ, χ' of some formulas in Δ_n and disjunctions ψ, ψ' of some formulas in Σ_n such that

$$\chi \wedge \varphi_n \to \psi \qquad \chi' \to \varphi_n \vee \psi'$$

are in $\mathsf{BK}^\frown$. But then $\chi \wedge \chi' \to \psi \vee \psi'$ is in $\mathsf{BK}^\frown$ (we may reason as in classical propositional logic); so $\langle \Delta_n, \Sigma_n \rangle$ is not a pair. But this is a contradiction with the assumption, so the claim is established and it follows that $\langle \Delta_n, \Sigma_n \rangle$ is a pair for each $n \in \omega$.

Claim 4.4. $\langle \bigcup_{n\in\omega} \Delta_n, \bigcup_{n\in\omega} \Sigma_n \rangle$ is a pair; $\bigcup_{n\in\omega} \Delta_n \cup \bigcup_{n\in\omega} \Sigma_n = \mathcal{L}^\frown$; and $\bigcup_{n\in\omega} \Delta_n$ is a prime theory.
These facts are straightforward. We show that $\bigcup_{n\in\omega} \Delta_n$ is a prime theory if the other facts hold. The set $\bigcup_{n\in\omega} \Delta_n$ does not contain $\bot$. Otherwise there would be some Δ_n containing $\bot$ and then $\langle \Delta_n, \Sigma_n \rangle$ would not be a pair, contradicting Claim 4.3. The set $\bigcup_{n\in\omega} \Delta_n$ contains $\mathsf{BK}^\frown$. If not, then there is $\chi \in \mathsf{BK}^\frown$ such that, for some n, $\langle \Delta_n \cup \{\chi\}, \Sigma_n \rangle$ is not a pair. This means that $\delta \wedge \chi \to \sigma$ is in $\mathsf{BK}^\frown$ (for a conjunction δ of a finite subset of Δ_n and a disjunction σ of a finite subset of Σ_n). But then $\delta \to \sigma$ is in $\mathsf{BK}^\frown$ (since, obviously, $\delta \to \chi \vee \sigma$ is in $\mathsf{BK}^\frown$) and so $\langle \Delta_n, \Sigma_n \rangle$ is not a pair, contradicting Claim 4.3. It can be shown in a similar fashion that $\bigcup_{n\in\omega} \Delta_n$ is closed under $\vdash_{\mathsf{BK}^\frown}$. Finally, if $\varphi, \varphi' \notin \bigcup_{n\in\omega} \Delta_n$, then $\varphi, \varphi' \in \bigcup_{n\in\omega} \Sigma_n$. Therefore $\bigcup_{n\in\omega} \Delta_n \nvdash \varphi \vee \varphi'$ and $\varphi \vee \varphi' \notin \bigcup_{n\in\omega} \Delta_n$.

Finally, note that $\bigcup_{n\in\omega} \Delta_n$ contains Δ and is disjoint from Σ. □

The *canonical* $\mathsf{BK}^\frown$*-frame* ('the canonical frame') is $F^\vdash = \langle S^\vdash, R^\vdash, C^\vdash \rangle$ where

- $S^\vdash$ is the set of all prime theories,
- $R^\vdash = \{\langle \Gamma, \Delta \rangle \mid \{\varphi \mid \Box\varphi \in \Gamma\} \subseteq \Delta\}$ and
- $C^\vdash = \{\langle \Gamma, \Delta \rangle \mid \{\varphi \mid \frown\varphi \in \Gamma\} \cap \Delta = \emptyset\}$.

Proposition 4.5. *The canonical frame is a compatibility frame.*

Proof. Firstly, we have to show that $C^\vdash$ is symmetric. Assume that $C^\vdash \Gamma \Delta$ and $\frown\varphi \in \Delta$. Then $\frown\frown\varphi \notin \Gamma$ and so $\varphi \notin \Gamma$. Hence, $C^\vdash \Delta \Gamma$.

Secondly, we have to show that if $C^\vdash \Gamma \Delta$ and $R^\vdash \Gamma \Sigma$, then there is Ω such that $R^\vdash \Delta \Omega$ and $C^\vdash \Sigma \Omega$. Take

$$\underline{\Delta} = \{\varphi \mid \Box\varphi \in \Delta\} \qquad \underline{\Sigma} = \{\psi \mid \frown\psi \in \Sigma\}$$

We show that $\langle\underline{\Delta},\underline{\Sigma}\rangle$ is a pair. If it were not a pair, then $\bigwedge_{i\leq n}\delta_i \to \bigvee_{j\leq m}\sigma_j$ is in $\mathsf{BK}^\frown$ for some $\delta_i \in \underline{\Delta}$ and $\sigma_j \in \underline{\Sigma}$. But then, by $n+1$ applications of the monotonicity rule for $\Box$, $\bigwedge_{i\leq n}\Box\delta_i \to \Box(\bigvee_{j\leq m}\sigma_j)$ is in $\mathsf{BK}^\frown$. However, $\bigwedge_{i\leq n}\Box\delta_i \in \Delta$, so $\Box(\bigvee_{j\leq m}\sigma_j) \in \Delta$. The assumption that $C^\vdash\Gamma\Delta$ implies that $\frown\Box(\bigvee_{j\leq m}\sigma_j) \notin \Gamma$. It follows from the fact that each $\Diamond\frown\chi \to \frown\Box\chi \in \mathsf{BK}^\frown$ that $\Box\frown(\bigvee_{j\leq m}\sigma_j) \notin \Gamma$. However, $\bigwedge_{j\leq m}\frown\sigma_j \in \Sigma$, so $\frown\bigvee_{j\leq m}\sigma_j \in \Sigma$ by the fact that each $(\frown\chi_1 \wedge \frown\chi_2) \to \frown(\chi_1 \vee \chi_2)$ is in $\mathsf{BK}^\frown$. By the assumption that $R^\vdash\Gamma\Sigma$, $\Diamond\frown\bigvee_{j\leq m}\sigma_j \in \Gamma$.[5] But, using the fact that $\Diamond\frown\chi \to \frown\Box\chi \in \mathsf{BK}^\frown$ again, this entails a contradiction. Hence, by the Pair Extension Theorem, there is a prime theory Ω such that $\underline{\Delta} \subseteq \Omega$ and $\underline{\Sigma} \cap \Omega = \emptyset$. This means that $R^\vdash\Delta\Omega$ and $C^\vdash\Sigma\Omega$. ☐

The *canonical* $\mathsf{BK}^\frown$*-model* ('canonical model') $M^\vdash$ is the canonical frame together with two functions $\mathscr{P}(S^\vdash)$:

- $V^{\vdash+} : p \mapsto \{\Gamma \in S^\vdash \mid p \in \Gamma\}$
- $V^{\vdash-} : p \mapsto \{\Gamma \in S^\vdash \mid {\sim}p \in \Gamma\}$

Proposition 4.6. *The canonical model is a compatibility model.*

Proof. We have to prove that if $C^\vdash\Gamma\Delta$ and $p \in \Gamma$, then ${\sim}p \notin \Delta$. This follows from the fact that $p \to \frown{\sim}p \in \mathsf{BK}^\frown$. ☐

We define the relations $\models^+_{M^\vdash}$ and $\models^-_{M^\vdash}$ in the same way as in compatibility models.

Proposition 4.7. *For each prime theory* Γ,

$$\Gamma \models^+_{M^\vdash} \varphi \iff \varphi \in \Gamma \quad \textit{and} \quad \Gamma \models^-_{M^\vdash} \varphi \iff {\sim}\varphi \in \Gamma$$

Proof. Proof by induction on the complexity of φ. The base case $\varphi \in AT$ holds by definition. The cases where the main connective of φ is in $\{\bot, {\sim}, \wedge, \vee, \to\}$ are standard and we omit them (the proofs use the BK axioms and the substitution rule).

We discuss the modal cases in more detail. The cases for $\models^+$ are established by standard arguments; see [13, 116–119] for the case that $\Box$ is the main connective of φ and [6, 196–199] for the case where the main connective is $\Diamond$. Now the cases for $\models^-$. Firstly, assume that the main connective is $\Box$. Assume that ${\sim}\Box\varphi \in \Gamma$. To prove that $\Gamma \models^- \Box\varphi$, we have to produce a prime theory Δ such that $R^\vdash\Gamma\Delta$ and ${\sim}\varphi \in \Delta$. Using the fact that $\Diamond\varphi \Leftrightarrow {\sim}\Box{\sim}\varphi$ is in $\mathsf{BK}^\frown$, we infer that ${\sim}\Box\varphi \leftrightarrow \Diamond{\sim}\varphi$ is in $\mathsf{BK}^\frown$, so $\Diamond{\sim}\varphi \in \Gamma$. Then we reason as in the standard $\models^+$-case for $\Diamond$. Conversely, assume that $R^\vdash\Gamma\Delta$ and ${\sim}\varphi \in \Delta$. Reasoning as in footnote 4, we conclude that $\Diamond{\sim}\varphi \in \Gamma$. Since ${\sim}\Box\varphi \leftrightarrow \Diamond{\sim}\varphi$ is in $\mathsf{BK}^\frown$, ${\sim}\Box\varphi \in \Gamma$. The arguments are similar if the main connective is $\Diamond$.

To conclude the proof, we need to check the cases where $\frown$ is the main connective. If $\frown\varphi \in \Gamma$ and $C^\vdash\Gamma\Delta$, then $\varphi \notin \Delta$ by the definition of $C^\vdash$. Conversely, assume that $\frown\varphi \notin \Gamma$. We have to produce a Δ such that $C^\vdash\Gamma\Delta$ and $\varphi \in \Delta$. The existence of such a Δ follows from the fact that $\langle\{\varphi\}, \{\psi \mid \frown\psi \in \Gamma\}\rangle$ is a

[5] If $R^\vdash\Gamma\Sigma$ and $\chi \in \Sigma$, then $\Diamond\chi \in \Gamma$. If not, then $\neg\Diamond\chi \in \Gamma$ by the fact that each $\chi \vee \neg\chi \in \mathsf{BK}^\frown$. But this means that $\Box\neg\chi \in \Gamma$ as $\neg\Diamond\chi \to \Box\neg\chi \in \mathsf{BK}^\frown$. Using the definition of $R^\vdash$ we may infer that $\neg\chi \in \Sigma$ and so $\chi \notin \Sigma$.

pair.[6] The cases for $\models^-$ are established in a similar fashion, using the fact that $\neg\frown\varphi \to \sim\frown\varphi$ and $\sim\frown\varphi \to \neg\frown\varphi$ are in $\mathsf{BK}^\frown$. □

Theorem 4.8. *$\Gamma \vdash_{\mathsf{BK}^\frown} \varphi$ iff $\Gamma \models_F \varphi$ for all compatibility frames F.*

Proof. Soundness is established by induction on the length of derivations. It is a manner of easy checking that all $\mathsf{BK}^\frown$-axioms are verified in each state of each compatibility model and that this property is preserved by modus ponens. Completeness is also established by a standard argument. Assume that $\Gamma \nvdash_{\mathsf{BK}^\frown} \varphi$. The assumption implies that $\langle \Gamma, \{\varphi\}\rangle$ is a $\mathsf{BK}^\frown$-pair. Hence, by the Pair Extension Theorem, there is a prime theory $\Delta \supseteq \Gamma$ such that $\varphi \notin \Delta$. By Propositions 4.5, 4.6 and 4.7, we have a compatibility model $M^\vdash$ based on a compatibility frame where $\Gamma \not\models_{M^\vdash} \varphi$. Hence, $\Gamma \not\models_F \varphi$ for each compatibility frame F. □

We have seen in Example 4.1 that (NC), our necessary condition for mutual compatibility of states, does not hold in $\mathcal{L}^\frown$-models. Our next observation specifies the price of repairing this. Let us call a compatibility frame a *45-frame* iff C is transitive ($xCyCz \to xCz$). Note that in every 45-frame, C is also euclidean ($(xCy \wedge xCz) \to yCz$).[7] 45-models are defined as expected.

Proposition 4.9. *In 45-models, $[\![x]\!]^+ \cap [\![y]\!]^- = \emptyset$ and $[\![x]\!]^- \cap [\![y]\!]^+ = \emptyset$, where $[\![\cdot]\!]^{+(-)}$ are functions from S to $\mathcal{L}^\frown$.*

Proof. It is sufficient to add the induction step for $\frown$ to the proof of Prop. 3.2. Assume that the claim holds for φ. First, suppose that $x \models \frown\varphi$ and $y \models \sim\frown\varphi$. Consequently, there is $z \in C(y)$ such that $z \models \varphi$. But C is transitive, so Cxz in case Cxy. In that case, however, $x \not\models \frown\varphi$; a contradiction. The second claim follows from the fact that C is euclidean. □

From an intuitive viewpoint, however, 45-frames might seem to be too strong. Assume, for example, that x does not support any information about p, y supports p and z supports $\sim p$. Moreover, let x, z, y agree on all other atomic propositions. So xCy and xCz. By euclideanity of C, yCz as well. But this is intuitively incorrect, as y and z support mutually contradictory information. Moreover, transitivity of C implies that every state x compatible with some state, $\exists y(xCy)$, is self-compatible ($xCy \to yCx \to xCx$) and so is every such y ($xCy \to (xCy \wedge xCy) \to yCy$). Again, this might seem too strong.

This is an interesting conclusion. On the one hand, (NC) is a natural necessary condition for two states being compatible. However, (NC) is falsified once we add to the language a negation connective related to the compatibility relation in a usual way. The failure of (NC) in this case is caused by our falsification condition for the compatibility negation; yet, no other plausible candidate for the falsification condition seems available. Moreover, the class of compatibility frames that avoid the failure of (NC) embodies assumptions concerning compatibility that are obviously too strong. Does this mean that Belnapian negation $\sim$ and the compatibility negation $\frown$ are somehow "intrinsically incompatible"? We leave this issue open.

[6] If this were not a pair, then we could derive a contradiction as follows. If $\varphi \to \bigvee_{i\leq n} \psi_i$ is in $\mathsf{BK}^\frown$, then so is $\frown(\bigvee_{i\leq n} \psi_i) \to \frown\varphi$ (use the transposition rule) and also $\bigwedge_{i\leq n} \frown\psi_i \to \frown\varphi$ (use the De Morgan axiom). But this would mean that $\frown\varphi \in \Gamma$.

[7] Assume xCy and xCz. By symmetry yCx and by transitivity yCz.

5 Labeling sources

When dealing with sources of information, it is natural to *categorize* them. For example, it may be useful to divide sources into witness testimonies, physical evidence, scientific theories etc. We can incorporate this idea into our framework in several ways. The present section deals with one, and perhaps the simplest, of them.

Let us from now on distinguish a finite set $L \subseteq AT$ of atomic formulas, intuitively seen as the set of *labels*. Labels represent different categories of sources. These categories might be overlapping and they might not cover the whole set S. The former case corresponds to the fact that

$$\bigwedge_{l \neq l'} (l \to \neg l')$$

is not valid (call this formula the "No Overlap Axiom"), the latter case corresponds to the fact that

$$\bigvee_{l_i \in L} l_i$$

is not valid (call this formula the "Exhaustion Axiom").

Labels permit us to express more fine-grained statements about *what kind* of sources support a specific piece of information. For instance, consider

$$\Diamond(l \wedge \varphi), \text{ and} \tag{6}$$

$$\Box(l \to \varphi) \tag{7}$$

(sometimes written also as $\Diamond_l\varphi$ and $\Box_l\varphi$, respectively). Formulas (6) say that some source labeled with l (of category l) supports φ; (7) says that all sources labeled with l (of category l) support φ. The second construction represents a notion of *strong support.* For example, let l be the category "witness reports concerning the whereabouts of John Smith at the time the crime was committed". If only some witnesses place John at the scene of the crime (6), then there might be room for reasonable doubt concerning his whereabouts. If *all* witnesses do so (7), however, then the room is considerably smaller. (Of course, it might be the case that $x \models \Box(l \to \varphi)$ but also $x \models \Diamond(l \wedge {\sim}\varphi)$; in such case there is a witness that contradicts herself.)

Labels can also be used to express *relative reliability* (trustworthiness, priority) of sources. Assume $L = \{l_1, \ldots, l_n\}$ where l_1 denotes the most reliable sources and l_n the least reliable ones. (Let $O_k\varphi$ be a shorthand for $O_{l_k}\varphi$ for $O \in \{\Diamond, \Box\}$.) We may call k the 'reliability degree' of sources labeled with l_k. This machinery allows us to express various natural notions.

Example 5.1 (Support by the best sources)**.** The formula

$$\dot{\Diamond}_k\varphi := \varphi \wedge \Diamond_k\varphi$$

says that φ is supported and also justified by some source of reliability degree k. Naturally, then, $\dot{\Diamond}_1\varphi$ says that φ is is supported and justified by a source of the highest reliability degree.

The formula

$$\dot{\Diamond}_1\varphi \wedge \Box_1\varphi$$

expresses a stronger notion, namely, that φ is supported and justified by *all* sources of the highest reliability degree.[8] The $\Diamond_1$ conjunct makes sure that this statement is not vacuous, i.e. that there is at least one source of the highest reliability degree justifying φ.

Example 5.2 (Non-overruled justification)**.** It is natural to assume that if $\Diamond_k\varphi$ and $\Diamond_m{\sim}\varphi$ for some $m < k$, then the justification for φ is "overruled" by the justification for ${\sim}\varphi$. Consider

$$\varphi \wedge \bigvee_{1\leq k\leq n} \left(\Diamond_k\varphi \wedge \neg \bigvee_{m\leq k} \Diamond_m{\sim}\varphi \right)$$

This formula says that φ has a non-overruled justification: there is a justification for φ and there is no justification for ${\sim}\varphi$ that is at least as good as the former justification.

This "hierarchical" interpretation of labels seems to require the validity of the No Overlap Axiom—reliability degrees need to be unique. The status of the Exhaustion Axiom is not so clear; there may be sources that are 'incomparable' as to their reliability to others.

Completeness results are easy to obtain here, but we will not go into details. Consider a language with a finite set of labels $L_m = \{l_1, \dots, l_m\}$. Consequence in the class of models where $V(l_k) \cap V(l_{k'}) = \emptyset$ for all $k \neq k'$ is axiomatized by adding the No Overlap Axiom schema (for L_m) to BK (the canonical model obviously satisfies the condition at hand). Similarly, consequence in the class of models where, in addition, $\bigcup_{k\leq m} V(l_k) = S$ holds is axiomatized by adding also the Exhaustion axiom (for L_m).

The only thing that requires care is that the use of label axioms such as the No Overlap Axiom or the Exhaustion Axiom forbids closure under uniform substitution. We may use *sorted substitution* instead, i.e. mappings $\mathcal{L} \to \mathcal{L}$ induced by any $AT \setminus L_m \to \mathcal{L}$.

Note that if the No Overlap Axiom holds in M, then the labels induce a preference ordering on $\bigcup_{l\in L} V(l)$ given by $x \leq^L y$ iff $l_x \leq l_y$ (where l_x is *the* label of x). It is not hard to see that $\leq^L$ is a pre-order, not a partial order. If the Exhaustion Axiom holds as well, then the whole S is ordered and $\leq^L$ is a total pre-order.

6 Conclusion

We have shown that the paraconsistent modal logic BK, equipped with a source-related interpretation of modal accessibility, is suitable for articulating the different ways inconsistency in a body of information may be related to the sources for that body—we may have (i) states that support φ and ${\sim}\varphi$ without there being a source supporting either φ or ${\sim}\varphi$; (ii) states that support φ and ${\sim}\varphi$ while some source also supports both φ and ${\sim}\varphi$; and (iii) states that support φ and ${\sim}\varphi$ with sources for both φ and ${\sim}\varphi$.

This interpretation of BK naturally motivates some of its extensions. We have studied extensions of BK with compatibility (both semantically, where

[8] Strictly speaking, justified by all "accessible" sources of the highest reliability degree.

a compatibility relation was added to BK models and some interesting frame classes were studied, and syntactically, where a modal compatibility negation was added to the language) and with source labels, propositional variables used to capture the idea of a reliability ordering among the sources.

Of course, many other extensions remain to be explored. For instance, we will look in the future on a generalization of the labeling idea using a primitive preference preorder on the set of states in the style of [25, 23, 7]. Adding preorder modalities and conditional belief modalities to our language will enable us to express the notions discussed in Examples 5.1 and 5.2 in a more general setting. It will be also interesting to look at dynamic extensions of our setting, formalizing reasoning about changes in the source relation or the source preference ordering.

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