# Substructural logics for pooling information

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**Abstract.** This paper puts forward a generalization of the account of pooling information – offered by standard epistemic logic – based on intersection of sets of possible worlds. Our account is based on information models for substructural logics and pooling is represented by fusion of information states. This approach yields a representation of pooling related to structured communication within groups of agents. It is shown that the generalized account avoids some problematic features of the intersection-based approach. Our main technical result is a sound and complete axiomatization of a substructural epistemic logic with an operator expressing pooling.

## 1 Introduction

Alice is visiting her friends, Bob and Cathy. She needs to get to the train station now, and the only option is to take a bus. Alice is not familiar with the bus routes. Bob tells her that it is best to take the bus no. 25, get off at the Main Square and change lines there. However, he does not remember the no. of the connecting line. Cathy does not know this either (she rather bikes), but she takes a look the public transport mobile application and learns that the right bus to take at the Main Square is no. 17.

The information provided by Bob and Cathy needs to be *pooled together* to be helpful for Alice. Similar situations arise on a daily basis. In order to perform even the most rudimentary tasks, agents need to pool information coming from a multitude of sources. While communicating with others, agents pool the received information with the information they already have and possibly send the results further. A good model of pooling is therefore crucial for modelling deliberations, actions and interactions of agents, be they human or artificial.

In epistemic logic the standard representation of information is a set of possible worlds [1]. Information available to an agent (her information state) is modelled as a set of possible worlds "accessible" to the agent [2]. Pooling information states or pieces of information in general is represented by *intersection* of

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the corresponding sets. This paper puts forward a generalization of the standard model. Our account, motivated by some unintuitive features of the standard framework, is based on information models for substructural logics.[1] Pooling is modelled by *fusion* of information states, a binary operation that generalizes intersection. This approach yields a more fine-grained representation of pooling; one that allows to model, for example, structured communication within groups of agents ("structured pooling"). Our main technical result is a sound and complete axiomatization of a substructural epistemic logic with an operator expressing outcomes of structured pooling.

Sect. 2 motivates our approach in more detail. Sect. 3 introduces the semantics based on information models and Sect. 4 extends the semantics by modalities representing outcomes of structured pooling; our main technical result is established in this section as well. Sect. 5 shows that the standard intersection-based model is a special case of our framework. Sect. 6 concludes the paper.

## 2 Motivation

In general, we may see information states and pieces of information as elements of a partially ordered set, with $x \leq y$ meaning that $x$ supports $y$. Equivalently, $x \leq y$ means that $x$ "extends" $y$ as $x \leq y$ iff every $z$ supported by $y$ is supported by $x$. Information pooling can be represented by a binary operation; let us denote as $x \cdot y$ the result of pooling $x$ with $y$. The representation of information by sets of possible worlds combined with the representation of information pooling in terms of intersection then corresponds to a special case of this framework where the poset at hand is a subset-ordered system of sets closed under intersection.

This special case is intuitively objectionable on several grounds. Firstly, the intersection-based model is *monotonic*; $x$ pooled with $y$, i.e. $x \cap y$, extends both $x$ and $y$. This means that support is irrevocable (every piece of information supported by $x$ or $y$ remains supported.)

*Example 1.* As a counterexample to monotonicity, let us consider the following situation. Assume that Ann had believed that her boyfriend David is an honest man (Ann's information state at some point in time, $x$, supports the information that David is honest, $h$). Then she dropped that belief as the result of the conversation with Bob who told her that David had an affair with Cathy (the result of pooling $x$ with Bob's information state, $y$, does not support $h$; we may assume for the sake of simplicity that it supports $\neg h$).

The standard model would represent the situation incorrectly; Ann would believe that David is honest and that he had an affair with Cathy at the same time.

A related feature of the standard model is that, if $x$ is inconsistent with $y$ (i.e. $x \cap y = \emptyset$), then the result of pooling $x$ with $y$ is the empty set. Hence, pooling

[1] We do not have space to provide an outline of substructural logics. We refer the reader to [7, 9, 10].

any pair of inconsistent pieces of information gives the same result, and this result supports every piece of information (this feature is known as *explosion*).

*Example 2.* Let us consider the following counterexample to explosion. Assume that Ann believes that she has free will and that free will is incompatible with physical determinism. Then she talks to Bob who persuades her that the physical world is deterministic. However, as it sometimes happens, she does not abandon the belief that she has free will. This means that the system of her beliefs is inconsistent, but not necessarily that the system supports any information whatsoever. She might hold inconsistent beliefs abut free will without being a right-wing extremist.

A feature of the standard model that comes into play here is that support is closed under classical consequence which validates *ex falso quodlibet*.

Monotonicity and explosion can be avoided by generalizing the standard model so that (i) a pooling operation is used such that $x \cdot y \leq x$ does not hold in general; (ii) mutual inconsistency of $x$ and $y$ is not modelled by $x \cdot y = 0$, where 0 is a trivially inconsistent piece of information; (iii) the support relation between pieces of information is not closed under classical consequence.

In what follows, we provide such a model. Models of this kind are offered, for instance, by various versions of *operational* semantics for substructural logics dating back to [14]. A more complete formulation was provided by Došen [3] and recently by Punčochář [8]. We build on the latter kind of model, extending it with modalities expressing structured communication within groups of agents. To motivate the introduction of such modalities, let us take a look at the connection between the standard model of pooling and communication within groups of agents.

A widespread intuitive interpretation of the standard model of pooling is that $\bigcap_{a \in G} x_a$ is the information state the members of a group $G$ (having information states $x_a$ for $a \in G$) would end up with after *communicating* with each other.[2] From the perspective of this interpretation, the standard framework represents a very special kind of communication within a group of agents, one in which agents pool *all* their information *instantenously* and every piece of information shared by each agent is *equally* considered. This is not how communication within groups usually works. Imagine an office or a research team; members of such groups may exchange partial information sequentially (e.g. Ann talks to Bob and then to Cathy) and some information may not be considered at all (e.g.

[2] For example, "A group has distributed knowledge of a fact $\varphi$ if the knowledge of $\varphi$ is distributed among its members, so that by pooling their knowledge together the members of the group can deduce $\varphi$, even though it may be the case that no member of the group individually knows $\varphi$." [4, p. 3]. This interpretation suffers from well-known problems [5, 11, 16]; we point out some additional ones. Hence, our paper can be seen as providing an additional argument against considering the standard model to be a good model of communication-related pooling. In the future, we plan to study the "full communication principle" of [5, 11] and the dynamic approach of [16] in the context of our framework.

information that contradicts a belief that an agent is not willing to give up). The structure of such *communication scenarios* is often critical when it comes to the outcome of communication.

Before developing this point, we introduce some notation. We may represent the (hypothetical) communication scenario of Ann talking to Bob and then to Cathy (about some issue) by the expression $(a * b) * c$. "Communication within group $G = \{a, b, c\}$" can be seen as being ambiguous between $(a*b)*c$, $(a*c)*b$ etc. Alternatively, the different expressions are related to different ways how agents in $G$ can communicate with each other. If $x, y, z$ are information states of $a, b$ and $c$, respectively, then the outcome of $(a * b) * c$ should be related to the structured pooling resulting in $(x \cdot y) \cdot z$. When a more specific formulation is preferred, we may say that $(x \cdot y) \cdot z$ represents $a$'s information state after the scenario $(a * b) * c$ has been realized.

It turns out that, given the link between communication scenarios and pooling, some algebraic properties of intersection are problematic. Take commutativity and associativity, for example.[3]

*Example 3.* Assume that Ann has not yet formed an opinion about a new colleague, Bob. She has the tendency to accept the first strong opinion she hears from others. Cathy likes Bob very much but David does not like him at all. When it comes to her eventual opinion about Bob, it is obviously important to whom she talks first.

In general, assume that $a$'s information state is partial with respect to $p$ (it supports neither $p$ nor $\neg p$), $b$'s state supports $p$ and $c$'s state supports $\neg p$. Assume that when communicating with other agents, $a$ accepts only information that is consistent with her state. Now if $a$ communicates with $b$ and then with $c$ – that is scenario of the type $(a * b) * c$ – then her resulting state supports $p$; if $a$ communicates with $c$ and then with $b$ – that is scenario of the type $(a * c) * b$ – her state supports $\neg p$.

*Example 4.* Assume that Cathy has read recently in a newspaper that Ms. X, Bob's favourite politician, obtained some money from Mr. Y, a man involved in organized crime. Consider two communication scenarios about the credibility of Ms. X. In the first scenario, which is of the type $a * (b * c)$, Cathy first talks to Bob and Bob is subsequently discussing the same issue with Ann. Since Bob trusts Ms. X, he does not believe the information conveyed by Cathy about the problematic money from Mr. Y, and he does not pass this information to Ann. In the second scenario, which is of the type $(a * b) * c$, Ann is discussing Ms. X's credibility with Bob first and subsequently with Cathy. Unlike in the first scenario, she ends up believing that Ms. X obtained some money from Mr. Y, which she learned from Cathy.

These examples show that scenarios $(a*b)*c$ and $(a*c)*b$, and $(a*b)*c$ and $a * (b * c)$, respectively, might actually lead to different outcomes. In addition, some communication scenarios may be more effective or leading to more desirable

[3] For associativity, see also [13].

results than others. It makes sense, therefore, to extend the formal language at hand with modalities indexed by communication scenarios; e.g. $\Box_{(a*b)*c}\alpha$ meaning that, after $(a*b)*c$ is realized, $a$'s information state supports $\alpha$. This language can then be used to formalize reasoning of agents about communication scenarios. Such reasoning is, of course, a vital part of reasoning about agent interactions.

*Example 5.* Going back to Example 1, we may assume that if David's information state supports $\Box_a h$ and $\Box_{a*b}\neg h$, then he would try to prevent $a*b$ from realizing.

As another example, consider the situation where a team leader's information state supports both $\Box_{(a*b)*c}\alpha$ and $\Box_{a*c}\alpha$, where $\alpha$ is necessary for $a$ to perform some task. It is then reasonable for the team leader to suggest $a*c$ and not $(a*b)*c$ as the former requires less resources (team members, time) than the former to reach the same goal ($a$' having information $\alpha$).

Our framework, introduced in the next two sections, combines a generalization of the intersection-based model of pooling, based on operational substructural semantics, with a modal language allowing to express reasoning about hypothetical communication scenarios.

## 3 Information models

In this section, we reconstruct the semantic framework for substructural logics introduced in [8] and summarize some of the results needed in the next section. For the sake of brevity, the results are presented without proofs; the interested reader is referred to [8].

Let us fix a set of atomic formulas $At$. The variables $p, q, \dots$ range over elements of $At$. An *information model* is a structure of the following type:

$$\mathcal{M} = \langle S, +, \cdot, 0, 1, C, V \rangle.$$

$S$ is an arbitrary nonempty set, informally construed as a set of information states[4]; $+$ and $\cdot$ are binary operations on $S$ (addition and fusion of states); 0 and 1 are two distinguished elements of $S$ (the trivially inconsistent state and the logical state); and $V$ is a valuation, that is a function assigning to every atomic formula a subset of $S$. The following conditions are assumed:

1. $\langle S, +, 0\rangle$ is a join-semilattice with the least element 0, i.e. $+$ is idempotent, commutative and associative, and $x + 0 = x$ for every $x \in S$. The semilattice determines an ordering of $S$: $a \leq b$ iff $a + b = b$.
2. The operation $\cdot$ is distributive in both directions over $+$, i.e. $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ and $(y + z) \cdot x = (y \cdot x) + (z \cdot x)$

[4] By "information states" we mean bodies of information that might be "available to" agents, but we do not assume that every information state is an information state of an agent.

3. $1 \cdot x = x$ and $0 \cdot x = 0$
4. $C$ satisfies the following conditions: (a) there is no $x$ such that $0Cx$, (b) if $xCy$, then $yCx$, (c) $(x+y)Cz$ iff $xCz$ or $yCz$
5. $V$ assigns to every atomic formula an ideal in $\mathcal{M}$, that is a subset $I \subseteq S$ satisfying: (a) $0 \in I$, (b) $x + y \in I$ iff $x \in I$ and $y \in I$

Information models derive from Došen's grupoid models for substructural logics [3]. We extend Došen's models with 0, allowing us to have a simpler semantic clause for disjunction. Moreover, these structures are enriched with the compatibility relation $C$ that allows us to introduce a paraconsistent negation avoiding the principle of explosion (*ex falso quotlibet*).[5]

Informally, information states $x \in S$ represent *bodies of information* that can be said to *support* specific pieces of information. For example, the beliefs of an agent or the evidence produced during a criminal trial can be seen as bodies of information supporting information that is not explicitly part of the respective body. The state 1 represents the "logical" state supporting all the logically valid formulas and 0 represents the trivially inconsistent state supporting every formula. The relation $C$ represents compatibility between information states. Informally, $xCy$ means that $y$ does not support any information that contradicts the information supported by $x$; for more details, see [6]. The operation $+$ yields the *common content* of the states $x, y$. The state $x + y$ supports any piece of information supported by both $x$ and $y$. The operation $+$ will correspond to intersection of the sets of supported formulas (see the construction of canonical models in this section). Dually, in the specific models in which the states $x$ and $y$ are represented as sets of possible worlds, $+$ corresponds to union (see Section 5). The operation $\cdot$ yields a *fusion* $x \cdot y$ of information states $x, y$. Importantly, a fusion of two information states may involve far more (or less) than the intersection of sets of possible worlds. None of the following are assumed:

– $x \cdot y \leq x$ (monotonicity)
– if not $xCy$, then $x \cdot y = 0$ (explosion)
– $(x \cdot y) \cdot z = (x \cdot z) \cdot y$ (commutativity)
– $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associativity)

Formulas of the language $L$ are defined as follows:

$$\alpha ::= p \mid \bot \mid t \mid \neg\alpha \mid \alpha \to \alpha \mid \alpha \wedge \alpha \mid \alpha \otimes \alpha \mid \alpha \vee \alpha.$$

With respect to a given information model $\mathcal{M}$ a relation of support $\Vdash_{\mathcal{M}}$ between information states from $S$ and $L$-formulas is defined recursively by the following clauses (we drop the subscript):

– $x \Vdash p$ iff $p \in V(x)$.
– $x \Vdash \bot$ iff $x = 0$.
– $x \Vdash t$ iff $x \leq 1$.

[5] An alternative extension of Došen's semantics is due to Wansing [15] who adds to Došen's models a constructive negation based on positive and negative valuation.

- $x \Vdash \neg\alpha$ iff for any $y$, if $yCx$ then $y \nVdash \alpha$.
- $x \Vdash \alpha \to \beta$ iff for any $y$, if $y \Vdash \alpha$, then $x \cdot y \Vdash \beta$.
- $x \Vdash \alpha \wedge \beta$ iff $x \Vdash \alpha$ and $y \Vdash \beta$.
- $x \Vdash \alpha \otimes \beta$ iff there are $y, z$ such that $y \Vdash \alpha$, $z \Vdash \beta$, and $x \leq y \cdot z$.
- $x \Vdash \alpha \vee \beta$ iff there are $y, z$ such that $y \Vdash \alpha$, $z \Vdash \beta$, and $x \leq y + z$.

If $x \Vdash \alpha$, we say that $x$ supports $\alpha$. The proposition $||\alpha||_{\mathcal{M}}$ expressed by $\alpha$ in $\mathcal{M}$ is the set of states of $\mathcal{M}$ that support $\alpha$.

**Theorem 1.** *For any information model $\mathcal{M}$ and any $L$-formula $\alpha$, $||\alpha||_{\mathcal{M}}$ is an ideal in $\mathcal{M}$.*

Accordingly, $y \leq x$ only if every $\alpha$ supported by $x$ is supported by $y$ ("information state $y$ *extends* $x$"). We say that an $L$-formula $\alpha$ is valid in an information model $\mathcal{M}$ if the logical state 1 supports $\alpha$ in $\mathcal{M}$. An $L$-formula is valid in a class of information models if it is valid in every model in the class. Let $\alpha$ be an $L$-formula and $\Delta$ a nonempty set of $L$-formulas. We say that $\alpha$ is semantically $FL$-valid ($\vDash_{FL} \alpha$) if $\alpha$ is valid in every information model. $\alpha$ is a semantic $FL$-consequence of $\Delta$ ($\Delta \vDash_{FL} \alpha$) if for any state $x$ of any information model, if $x$ supports every formula from $\Delta$, then $x$ supports $\alpha$.

**Lemma 1.** *An implication $\alpha \to \beta$ is valid in $\mathcal{M}$ iff, for all $x \in S$, $x \Vdash \alpha$ only if $x \Vdash \beta$.*

The logic of all information models is a non-distributive, non-associative, and non-commutative version of Full Lambek calculus with a paraconsistent negation. The logic can be axiomatized by a Hilbert-style axiomatic system (that we call $FL$) containing the following axiom schemata and inference rules:

| | |
|---|---|
| $A1$ | $\alpha \to \alpha$ |
| $A2$ | $\bot \to \alpha$ |
| $A3$ | $(\alpha \wedge \beta) \to \alpha$ |
| $A4$ | $(\alpha \wedge \beta) \to \beta$ |
| $A5$ | $\alpha \to (\alpha \vee \beta)$ |
| $A6$ | $\beta \to (\alpha \vee \beta)$ |
| $A7$ | $(\alpha \otimes (\beta \vee \gamma)) \to ((\alpha \otimes \beta) \vee (\alpha \otimes \gamma))$ |

| | |
|---|---|
| $R1$ | $\alpha, \alpha \to \beta / \beta$ |
| $R2$ | $\alpha \to \beta / (\beta \to \gamma) \to (\alpha \to \gamma)$ |
| $R3$ | $\gamma \to \alpha, \gamma \to \beta / \gamma \to (\alpha \wedge \beta)$ |
| $R4$ | $\alpha \to \gamma, \beta \to \gamma / (\alpha \vee \beta) \to \gamma$ |
| $R5$ | $\alpha \to \neg\beta / \beta \to \neg\alpha$ |
| $R6$ | $\alpha \to \beta / (\gamma \otimes \alpha) \to (\gamma \otimes \beta)$ |
| $R7$ | $\alpha \to (\beta \to \gamma) / (\alpha \otimes \beta) \to \gamma$ |
| $R8$ | $(\alpha \otimes \beta) \to \gamma / \alpha \to (\beta \to \gamma)$ |
| $R9$ | $t \to \alpha / \alpha$ |
| $R10$ | $\alpha / t \to \alpha$ |

**Lemma 2.** *Every axiom of $FL$ is semantically $FL$-valid and all the rules preserve semantic $FL$-validity in all information models.*

A proof in the system $FL$ is defined in the standard way as a finite sequence of $L$-formulas such that every formula in the sequence is either an instance of an axiom schema, or a formula that is derived by applying an inference rule to formulas that occur earlier in the sequence. We say that $\alpha$ is $FL$-provable ($\vdash_{FL} \alpha$), if there is a proof $\beta_1, \ldots, \beta_n$ such that $\alpha = \beta_n$. The expression $\alpha_1, \ldots, \alpha_n \vdash_{FL} \beta$ is an abbreviation for $\vdash_{FL} (\alpha_1 \wedge \ldots \wedge \alpha_n) \rightarrow \beta$, and if $\Delta$ is a set of $L$-formulas, $\Delta \vdash_{FL} \beta$ means that there are $\alpha_1, \ldots, \alpha_n \in \Delta$ such that $\alpha_1, \ldots, \alpha_n \vdash_{FL} \beta$.

**Definition 1.** *A set of $L$-formulas $\lambda$ is a logic over $FL$ iff (a) $\lambda$ contains all the axioms of $FL$, (b) $\lambda$ is closed under the rules of $FL$, and (c) $\lambda$ is closed under uniform substitutions of $L$-formulas.*

For any logic over $FL$ we construct a canonical model. For a given logic $\lambda$ the canonical model of $\lambda$ is constructed out of $\lambda$-theories.

**Definition 2.** *Let $\lambda$ be a logic over $FL$. A nonempty set of $L$-formulas $\Delta$ is an $\lambda$-theory if it satisfies the following two conditions:*

*(a) if $\alpha \in \Delta$ and $\beta \in \Delta$, then $\alpha \wedge \beta \in \Delta$,*
*(b) if $\alpha \in \Delta$ and $\alpha \rightarrow \beta \in \lambda$, then $\beta \in \Delta$.*

**Definition 3.** *Let $\lambda$ be a logic over $FL$. The canonical model of $\lambda$ is the structure $\mathcal{M}^\lambda = \langle S^\lambda, +^\lambda, \cdot^\lambda, 0^\lambda, 1^\lambda, C^\lambda, V^\lambda \rangle$, where*

- *$S^\lambda$ is the set of all $\lambda$-theories,*
- *$\Gamma +^\lambda \Delta = \Gamma \cap \Delta$,*
- *$\Gamma \cdot^\lambda \Delta = \{\alpha;$ for some $\gamma \in \Gamma$ and $\delta \in \Delta, (\gamma \otimes \delta) \rightarrow \alpha \in \lambda\}$,*
- *$0^\lambda$ is the set of all $L$-formulas,*
- *$1^\lambda = \lambda$,*
- *$\Gamma C^\lambda \Delta$ iff for all $\alpha$, if $\neg\alpha \in \Gamma$, then $\alpha \notin \Delta$,*
- *$\Gamma \in V^\lambda(p)$ iff $p \in \Gamma$.*

**Theorem 2.** *$\mathcal{M}^\lambda$ is an information model.*

**Theorem 3.** *For any $L$-formula $\alpha$ and $\lambda$-theory $\Gamma$, the following holds:*

*$\Gamma \Vdash \alpha$ in $\mathcal{M}^\lambda$ iff $\alpha \in \Gamma$.*

Assume that $\lambda$ is given by an axiomatic system that is sound with respect to a class of information models that contains the canonical model of $\lambda$. The following direct corollary of Theorem 3 guarantees that the system must be also complete with respect to the class.

**Corollary 1.** *$\alpha \in \lambda$ iff $\alpha$ is valid in $\mathcal{M}^\lambda$.*

In particular, using Lemma 2 and Corollary 1 we obtain completenes of $FL$.

**Corollary 2.** *$\Delta \vDash_{FL} \beta$ iff $\Delta \vdash_{FL} \beta$.*

Since the construction leads to strong completeness of $FL$, we obtain compactness immediately.

**Corollary 3.** *If $\Delta \vDash_{FL} \beta$, then there is a finite $\Gamma \subseteq \Delta$ such that $\Gamma \vDash_{FL} \beta$.*

## 4 Communication models

This section extends information models by a representation of information about the results of communications scenarios.

Let us fix a set of expressions $Ag$, representing a set of agents. We will define inductively a set of expressions $CS(Ag)$ (structured communication scenarios over $Ag$), or just $CS$, in the following way: 1. every $a \in Ag$ is in $CS$. 2. if $G$ and $H$ are in $CS$, then the expression $(G * H)$ is also in $CS$. 3. Nothing else is in $CS$. A communication scenario can be viewed as a binary tree whose leaves represent agents. The *principal agent of* $G \in CS$ is the agent denoted by the leftmost occurrence of an agent variable in $G$. For example, the principal agent of both $a * (b * c)$ and $(a * b) * c$ is $a$.

Variables $G, H, \ldots$ range over elements of $CS$. A *communication model* is any tuple

$$\mathcal{M} = \langle S, +, \cdot, 0, 1, C, \{f_G\}_{G \in CS}, V\rangle,$$

where $\mathcal{M} = \langle S, +, \cdot, 0, 1, C, V\rangle$ is an information model and $\{f_G\}_{G \in CS}$ is a collection of unary functions on $S$ satisfying for every $G, H \in CS$:

$$\begin{array}{ll} (f0) & f_G(0) = 0 \\ (f+) & f_G(x + y) = f_G(x) + f_G(y) \\ (f\cdot) & f_{G*H}(x) \leq f_G(x) \cdot f_H(x) \end{array}$$

Informally, $f_a(x)$ is the information state of agent $a$, *according to* the body of information $x$; see [12]. $f_{a*b}(x)$ is the information state of $a$, according to $x$, after the communication scenario $a * b$ is realized.[6] In general, $f_G(x)$ is the information state of the principal agent of scenario $G$ after $G$ has been carried out, according to $x$. The result of pooling the information of the principal agent of $G$ after realizing $G$ with the information state of the principal agent of $H$ after realizing $H$ is represented by $f_{G*H}$. We call this the information state of the scenario $G * H$.

Our three "frame conditions" represent the following informal assumptions about communication scenarios. First, every $G$ has an inconsistent information state according to 0, $(f0)$. This is straightforward as 0 supports every piece of information, i.e. it supports every piece of information about every $G$. Second, the information state of $G$ according to the common content of $x$ and $y$ is the common content of $f_G(x)$ and $f_G(y)$, $(f+)$. This is a consequence of the interpretation of $x + y$ as the intersection of the information provided by $x$ and $y$. Third, the information state of $G * H$ according to $x$ extends the fusion of $f_G(x)$ and $f_H(x)$, $(f\cdot)$. This represents the fact that structured pooling is based on fusion of information states.

[6] The body of information $x$ consists of information on a number of topics, including agents $a$ and $b$, and how they react to receiving specific information in communication. $f_{a*b}(x)$ is the information constituting $a$'s information state after receiving information from $b$, according to what $x$ says about $a$ and $b$.

Note, however, that on the level of information states we do not define pooling *as* fusion, that is we do not require

$$(f{=}) \quad f_{G*H}(x) = f_G(x) \cdot f_H(x)$$

The reason will become clear in the next section where we explain how our framework generalizes the standard intersection-based model of pooling (i.e. epistemic logic with distributed knowledge). A spoiler: in the particular cases where our models correspond to standard models, the information states are sets of possible worlds, $*$ is union and $\cdot$ is intersection. In these cases $(f{=})$ naturally fails; it is only the case that $f_{G\cup H}(x) \subseteq f_G(x) \cap f_H(x)$.

The language $L_\Box$ is obtained by adding to $L$ a modality $\Box_G$ for every communication scenario $G$:

$$\alpha ::= p \mid \bot \mid t \mid \neg\alpha \mid \alpha \to \alpha \mid \alpha \wedge \alpha \mid \alpha \otimes \alpha \mid \alpha \vee \alpha \mid \Box_G\alpha$$

The semantic clauses for the language $L_\Box$ extend the semantic clauses for $L$ with the following clause for the group modalities:

$x \Vdash \Box_G\alpha$ iff $f_G(x) \Vdash \alpha$.

The following result extends Theorem 1. The result shows that complex formulas and atomic formulas express propositions of the same kind. This will guarantee that the logic of all communication models is closed under uniform substitution.

**Theorem 4.** *For any communication model $\mathcal{M}$ and any $L_\Box$-formula $\alpha$, $||\alpha||_\mathcal{M}$ is an ideal in $\mathcal{M}$.*

*Proof.* Let $\mathcal{M}$ be a communication model and $\alpha$ an $L_\Box$-formula. We have to show that $||\alpha||_\mathcal{M}$ is an ideal in $\mathcal{M}$. This can be proved by induction. We will show the inductive step for $\Box_G$. The inductive assumption is that for a given $L_\Box$-formula $\beta$, $||\beta||_\mathcal{M}$ is an ideal in $\mathcal{M}$. We will show that $||\Box_G\beta||_\mathcal{M}$ is also an ideal. First, since $0 \Vdash \beta$ and $f_G(0) = 0$, we have $f_G(0) \Vdash \beta$, i.e. $0 \Vdash \Box_G\beta$. Second, $x+y \Vdash \Box_G\beta$ iff $f_G(x+y) \Vdash \beta$ iff $f_G(x)+f_G(y) \Vdash \beta$ iff $f_G(x) \Vdash \beta$ and $f_G(y) \Vdash \beta$ iff $x \Vdash \Box_G\beta$ and $y \Vdash \Box_G\beta$.

Let $\alpha$ be an $L_\Box$-formula and $\Delta$ a nonempty set of $L_\Box$-formulas. We say that $\alpha$ is semantically $PFL$-valid ($\vDash_{PFL} \alpha$) if $\alpha$ is valid in every communication model; $\alpha$ is a semantic $PFL$-consequence of $\Delta$ ($\Delta \vDash_{PFL} \alpha$) if for any state $x$ of any communication model, $x$ supports every formula from $\Delta$ only if $x$ supports $\alpha$.

The axiomatic system $PFL$ is given by axioms and rules of $FL$ plus the axiom $A8$, and the rules $R11$ and $R12$.

| | |
|---|---|
| $A8$ | $(\Box_G\alpha \wedge \Box_G\beta) \to \Box_G(\alpha \wedge \beta)$ |
| $R11$ | $\alpha \to \beta / \Box_G\alpha \to \Box_G\beta$ |
| $R12$ | $(\alpha \otimes \beta) \to \gamma / (\Box_G\alpha \wedge \Box_H\beta) \to \Box_{G*H}\gamma$. |

The following claim extends Lemma 2.

**Lemma 3.** *Every axiom of $PFL$ is semantically $PFL$-valid and all the rules preserve semantic $PFL$-validity in all communication models.*

*Proof.* (a) Let $x$ be an arbitrary state of a communication model such that $x \Vdash \Box_G\alpha \wedge \Box_G\beta$. Then $f_G(x) \Vdash \alpha$ and $f_G(x) \Vdash \beta$, i.e. $f_G(x) \Vdash \alpha \wedge \beta$. So $x \Vdash \Box_G(\alpha \wedge \beta)$. It follows that $1 \Vdash (\Box_G\alpha \wedge \Box_G\beta) \to \Box_G(\alpha \wedge \beta)$, so we have proved that $A8$ is semantically $PFL$-valid.

(b) Assume that $1 \Vdash \alpha \to \beta$ in an arbitrary communication model. Let $x$ be a state of that model such that $x \Vdash \Box_G\alpha$. Then $f_G(x) \Vdash \alpha$ and it follows from our assumption that $f_G(x) \Vdash \beta$. So $x \Vdash \Box_G\beta$. It follows that $1 \Vdash \Box_G\alpha \to \Box_G\beta$, so we have proved that $R11$ preserves semantic $PFL$-validity.

(c) Assume that in a communication model $\mathcal{M}$, $1 \Vdash (\alpha \otimes \beta) \to \gamma$. We will prove that $1 \Vdash (\Box_G\alpha \wedge \Box_H\beta) \to \Box_{G*H}\gamma$. Assume $x \Vdash \Box_G\alpha \wedge \Box_H\beta$. Then $f_G(x) \Vdash \alpha$ and $f_H(x) \Vdash \beta$. It follows that $f_G(x) \cdot f_H(x) \Vdash \alpha \otimes \beta$, and so $f_G(x) \cdot f_H(x) \Vdash \gamma$. Since $f_{G*H}(x) \leq f_G(x) \cdot f_H(x)$, it holds $f_{G*H}(x) \Vdash \gamma$, due to Theorem 4. As a consequence, $x \Vdash \Box_{G*H}\gamma$.

**Definition 4.** *A set of $L_\Box$-formulas $\lambda$ is called a logic over $PFL$ if the following three conditions are satisfied: (a) $\lambda$ contains all the axioms of $PFL$, (b) $\lambda$ is closed under the rules of $PFL$, (c) $\lambda$ is closed under uniform substitutions of $L_\Box$-formulas.*

Theories related to logics over $PFL$ are defined in the same way as theories related to logics over $FL$ (see Definition 2) with the difference that if $\lambda$ is a logic over $PFL$ then $\lambda$-theories are sets of $L_\Box$-formulas. The construction of the canonical model for a given logic over $PFL$ extends the construction from the previous section.

**Definition 5.** *Let $\lambda$ be a logic over $PFL$. The canonical model of $\lambda$ is the structure $\mathcal{M}^\lambda = \langle S^\lambda, +^\lambda, \cdot^\lambda, 0^\lambda, 1^\lambda, C^\lambda, \{f_G^\lambda\}_{G \in CS}, V^\lambda\rangle$, where $S^\lambda, +^\lambda, \cdot^\lambda, 0^\lambda, 1^\lambda, C^\lambda$ and $V^\lambda$ are defined as in Definition 3 and for any $G \in CS$ and any $\lambda$-theory $\Gamma$ we define:*

$$f_G^\lambda(\Gamma) = \{\alpha \in L_\Box; \Box_G\alpha \in \Gamma\}.$$

Let us fix a logic $\lambda$ over $PFL$. We will write just $S$, $+$, $\cdot$, $0$, $1$, $C$, $f_G$, $V$ instead of $S^\lambda$, $+^\lambda$, $\cdot^\lambda$, $0^\lambda$, $1^\lambda$, $C^\lambda$, $f_G^\lambda$, $V^\lambda$.

**Lemma 4.** *If $\Gamma$ is a $\lambda$-theory, then $f_G(\Gamma)$ is also a $\lambda$-theory.*

*Proof.* (a) Assume that $\alpha \in f_G(\Gamma)$ and $\beta \in f_G(\Gamma)$, i.e. $\Box_G(\alpha) \in \Gamma$ and $\Box_G(\beta) \in \Gamma$. Since $\Gamma$ is a $\lambda$-theory, $\Box_G\alpha \wedge \Box_G\beta \in \Gamma$. Since $\lambda$ contains all the axioms of $PFL$, $\Box_G(\alpha \wedge \beta) \in \Gamma$, due to $A8$. It follows that $\alpha \wedge \beta \in f_G(\Gamma)$.

(b) Assume that $\alpha \in f_G(\Gamma)$ and $\alpha \to \beta \in \lambda$. Then $\Box_G\alpha \in \Gamma$ and since $\lambda$ is closed under the rules of $PFL$, $\Box_G\alpha \to \Box_G\beta \in \lambda$, due to $R11$. It follows that $\Box_G\beta \in \Gamma$. So, $\beta \in f_G(\Gamma)$.

**Lemma 5.** *For any $\lambda$-theories $\Gamma, \Delta$ the following conditions are satisfied:*

*(a)* $f_G(0) = 0$,
*(b)* $f_G(\Delta + \Gamma) = f_G(\Delta) + f_G(\Gamma)$,
*(c)* $f_{G*H}(\Delta) \leq f_G(\Delta) \cdot f_H(\Delta)$.

*Proof.* The case (a) is immediate. (b) We are proving $f_G(\Delta \cap \Gamma) = f_G(\Delta) \cap f_G(\Gamma)$. It holds that $\alpha \in f_G(\Delta \cap \Gamma)$ iff $\Box_G \alpha \in \Delta \cap \Gamma$ iff $\Box_G \alpha \in \Delta$ and $\Box_G \alpha \in \Gamma$ iff $\alpha \in f_G(\Delta)$ and $\alpha \in f_G(\Gamma)$ iff $\alpha \in f_G(\Delta) \cap f_G(\Gamma)$.

(c) We are proving $f_G(\Delta) \cdot f_H(\Delta) \subseteq f_{G*H}(\Delta)$. Assume $\alpha \in f_G(\Delta) \cdot f_H(\Delta)$. That means that there are $\beta \in f_G(\Delta)$ and $\gamma \in f_H(\Delta)$ such that $(\beta \otimes \gamma) \to \alpha \in \lambda$. The rule $R12$ gives us $(\Box_G \beta \wedge \Box_H \gamma) \to \Box_{G*H} \alpha \in \lambda$. Moreover, $\Box_G \beta \in \Delta$ and $\Box_H \gamma \in \Delta$, so $\Box_G \beta \wedge \Box_H \gamma \in \Delta$. It follows that $\Box_{G*H} \alpha \in \Delta$, and, consequently, $\alpha \in f_{G*H}(\Delta)$.

Lemmas 4 and 5 lead to the following strengthening of Theorem 2.

**Theorem 5.** *$\mathcal{M}^\lambda$ is a communication model.*

Theorem 5 allows us to express the following "truth-lemma" as a meningful statement. In addition, we will show that the statement is true.

**Theorem 6.** *For any $L_\Box$-formula $\alpha$ and any $\lambda$-theory $\Gamma$:*

$\Gamma \Vdash \alpha$ *in* $\mathcal{M}^\lambda$ *iff* $\alpha \in \Gamma$.

*Proof.* We can proceed by induction on the complexity of $\alpha$. The base of the induction and the inductive steps for $\neg$, $\to$, $\wedge$, $\otimes$, and $\vee$ are the same as in the proof of Theorem 3. Let us consider the case of $\Box_G$. The induction hypothesis is that the claim holds for an $L_\Box$-formula $\beta$. To see that then the claim holds also for $\Box_G \beta$, we can observe that the following equivalences hold: $\Gamma \Vdash \Box_G \beta$ iff $f_G(\Gamma) \Vdash \beta$ iff $\beta \in f_G(\Gamma)$ iff $\Box_G \beta \in \Gamma$.

**Corollary 4.** *$\alpha \in \lambda$ iff $\alpha$ is valid in $\mathcal{M}^\lambda$.*

**Corollary 5.** *$\Delta \vDash_{PFL} \beta$ iff $\Delta \vdash_{PFL} \beta$.*

**Corollary 6.** *If $\Delta \vDash_{PFL} \beta$, then there is a finite $\Gamma \subseteq \Delta$ such that $\Gamma \vDash_{PFL} \beta$.*

## 5 The standard framework as a special case

We show in this section that every model of standard epistemic logic with distributed knowledge (i.e. every standard intersection-based model) corresponds to a particular communication model. The modality $\Box_G$ will boil down to standard distributed knowledge in these specific cases. The language $L_D$ is a basic language of classical propositional logic enriched with a modality of distributed knowledge for every set of agents $A \subseteq Ag$:

$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid D_A \alpha.$

A *standard model* is a tuple $\mathfrak{M} = \langle W, \{R_a\}_{a\in Ag}, V\rangle$, where $W$ is a non-empty set (of possible worlds), $R_a : W \to \mathcal{P}(W)$, for every $a \in Ag$, and $V : At \to \mathcal{P}(W)$. Moreover, for every set of agents $A \subseteq Ag$, we define a function $R_A : W \to \mathcal{P}(W)$ in the following way:

$$R_A(w) = \bigcap_{a\in A} R_a(w).$$

In a given standard model $\langle W, R, V\rangle$, a relation of truth between worlds and $L_D$-formulas is defined in the following way:

- $w \vDash p$ iff $w \in V(p)$,
- $w \vDash \neg\alpha$ iff $w \nvDash \alpha$,
- $w \vDash \alpha \wedge \beta$ iff $w \vDash \alpha$ and $w \vDash \beta$,
- $w \vDash D_A\alpha$ iff for every $v \in R_A(w)$, $v \vDash \alpha$.

An $L_D$-formula is valid in a standard model iff it is true in every world of that model. We can assign to every communication scenario $G$ a set of agents $s(G)$ by the following recursive equations:

$$s(a) = \{a\}, \text{ for every } a \in Ag, \text{ and } s(G * H) = s(G) \cup s(H).$$

So, $s(G)$ is the set of agents occurring in $G$. Now we will construct for any given standard model $\mathfrak{M} = \langle W, \{R_a\}_{a\in Ag}, V\rangle$ a communication model $\mathfrak{M}^i = \langle S, +, \cdot, 0, 1, C, \{f_G\}_{G\in CS}, V^\dagger\rangle$ in the following way:

- $S = \mathcal{P}(W)$,
- $x + y = x \cup y$ and $x \cdot y = x \cap y$,
- $0 = \emptyset$ and $1 = W$,
- $xCy$ iff $x \cap y \neq \emptyset$,
- $f_G(x) = \bigcup_{w\in x} R_{s(G)}(w)$,
- $x \in V^\dagger(p)$ iff $x \subseteq V(p)$.

**Lemma 6.** *For all $\mathfrak{M}$, $\mathfrak{M}^i$ is a communication model.*

*Proof.* We will verify only that the three conditions for the group functions are satisfied. In $\mathfrak{M}^i$ these conditions boil down to the following claims:

- $\bigcup_{w\in\emptyset} R_A(w) = \emptyset$,
- $\bigcup_{w\in x\cup y} R_A(w) = (\bigcup_{w\in x} R_A(w)) \cup (\bigcup_{w\in y} R_A(w))$,
- $\bigcup_{w\in x} R_{A\cup B}(w) \subseteq (\bigcup_{w\in x} R_A(w)) \cap (\bigcup_{w\in x} R_B(w))$.

The first two claims are obvious. We will prove the third one. Assume that $v \in \bigcup_{w\in x} R_{A\cup B}(w)$. Then there is $w \in x$ such that $v \in R_{A\cup B}(w)$, i.e. for all $a \in A \cup B$, $v \in R_a(w)$. It follows that there is $w \in x$ such that for all $a \in A$, $v \in R_a(w)$, and there is $w \in x$ such that for all $a \in B$, $v \in R_a(w)$. In other words, there is $w \in x$ such that $v \in R_A(w)$, and there is $w \in x$ such that $v \in R_B(w)$, i.e. $v \in (\bigcup_{w\in x} R_A(w)) \cap (\bigcup_{w\in x} R_B(w))$.

Now it can be explained why we did not require $f_{G*H}(x) = f_G(x) \cdot f_H(x)$. This equation does not hold even in the most simple models generated by standard epistemic models. In particular, it does not generally hold that

$(\bigcup_{w\in x} R_A(w)) \cap (\bigcup_{w\in x} R_B(w)) \subseteq \bigcup_{w\in x} R_{A\cup B}(w)$.

For the sake of simplicity, assume that $A = \{a\}$, $B = \{b\}$, and $x = \{w_1, w_2\}$. Consider for example the situation described by this table:

| | $R_a$ | $R_b$ | $R_{\{a,b\}}$ |
|---|---|---|---|
| $w_1$ | $\{w_1, v\}$ | $\{w_1\}$ | $\{w_1\}$ |
| $w_2$ | $\{w_2\}$ | $\{w_2, v\}$ | $\{w_2\}$ |

In this situation, $v \in (\bigcup_{w\in x} R_A(w)) \cap (\bigcup_{w\in x} R_B(w))$ but $v \notin \bigcup_{w\in x} R_{A\cup B}(w)$. (Nevertheless, $(f{=})$ holds in $\mathfrak{M}^i$ for singleton states $x$.) Dually speaking, suppose that $v$ is the only world in which $p$ is false. Then, the state $f_{A\cup B}(x) = \{w_1, w_2\}$ supports the information that $p$ but the state $f_A(x) \cap f_B(x) = \{v, w_1, w_2\}$ does not support $p$.

As the last step, let us introduce a recursive translation $tr$ of $L_\Box$ into $L_D$. For every atomic formula $p$, we define $tr(p) = p$. Moreover, $tr(\bot) = q \wedge \neg q$ and $tr(t) = \neg(q \wedge \neg q)$, for a selected atomic formula $q$. The translation operates on complex formulas according to these equations:

$$tr(\neg\alpha) = \neg tr(\alpha) \qquad tr(\Box_G \alpha) = D_{s(G)} tr(\alpha)$$
$$tr(\alpha \wedge \beta) = tr(\alpha) \wedge tr(\beta) \qquad tr(\alpha \otimes \beta) = tr(\alpha) \wedge tr(\beta)$$
$$tr(\alpha \to \beta) = \neg(tr(\alpha) \wedge \neg tr(\beta)) \qquad tr(\alpha \vee \beta) = \neg(\neg tr(\alpha) \wedge \neg tr(\beta))$$

**Theorem 7.** *For every $\mathfrak{M}$, every set $x$ of its worlds, and every $L_\Box$-formula $\alpha$, the following holds:*

*$x \Vdash \alpha$ in $\mathfrak{M}^i$ iff for all $w \in x$, $w \vDash tr(\alpha)$ in $\mathfrak{M}$.*

*Proof.* Induction on the complexity of $\alpha$. We will show just the case of $\Box_G$. As the induction hypothesis we assume that our claim holds for an $L_\Box$-formula $\beta$. The following equivalences show that then it must hold also for $\Box_G\beta$.

$x \Vdash \Box_G\beta$ iff $f_G(x) \Vdash \beta$
iff $\bigcup_{w\in x} R_{s(G)}(w) \Vdash \beta$
iff for all $v \in \bigcup_{w\in x} R_{s(G)}(w)$, $v \vDash tr(\beta)$
iff for all $w \in x$ and for all $v \in R_{s(G)}(w)$, $v \vDash tr(\beta)$
iff for all $w \in x$, $w \vDash D_{s(G)} tr(\beta)$
iff for all $w \in x$, $w \vDash tr(\Box_G\beta)$.

**Corollary 7.** *For every $L_\Box$-formula $\alpha$, $tr(\alpha)$ is valid in $\mathfrak{M}$ iff $\alpha$ is valid in $\mathfrak{M}^i$.*

## 6 Conclusion

In this paper, we have formulated a generalization of the standard semantics for distributed knowledge. The standard modality of distributed knowledge/belief that is indexed by sets of agents has been generalized to an epistemic modality which is relative to structured communication scenarios. Our general framework allows to add this modality to a large class of non-classical logics extending a weak, non-associative, non-distributive and non-commutative Full Lambek Calculus with a paraconsistent negation.

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