Action Frames for Weak Relevant Logics

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Abstract: The article introduces extended models for the propositional dynamic logic **PDL**. In extended models, valuation assigns to every state a set of atomic formulas and a **PDL** program. The program is informally construed as an action preferred by a contextually fixed agent. **PDL** is then extended by introducing a conditional connective expressing partial correctness claims. The main contribution of the article is the observation that the partial correctness conditional is in fact a substructural implication. It is shown that a variant of the minimal distributive substructural logic **DB** is a fragment of **PDLS**, an extension of **PDL** that contains the partial correctness conditional and fusion. It is also shown that the main results can be extended to a specific class of positive substructural logics.

Keywords: Action, propositional dynamic logic, partial correctness, substructural logics

1 Introduction

Valuation in propositional modal models is usually defined as a function that assigns sets of states to atomic formulas. Equivalently, valuation can be seen as a partial description of states, since it lists the atomic formulas that 'hold' in the respective states. A natural extension of this is to re-define valuation in such a manner that it provides *more information* about states. This article introduces such *extended models* for the propositional dynamic logic **PDL**, see (Fischer & Ladner, 1979; Harel, Kozen, & Tiuryn, 2000). In these models, valuation assigns to every state a set of atomic formulas *and* a **PDL**-program. Informally, the program is seen as an action *pre-*

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ferred by an agent. The language of $\mathbf{PDL}$ is then extended by a conditional connective '$\to$', expressing global partial correctness claims. On the truth definition employed there, $\phi \to \psi$ holds in state x of a model iff the action preferred in x is partially correct with respect to input (precondition) ϕ and goal (postcondition) ψ throughout the model.

The main contribution of this article is the observation that the partial correctness conditional is a substructural implication. This provides a link between substructural logic and dynamic logic. More specifically, we show that $\mathbf{DB}$, a variant of the basic distributive substructural logic, is sound and complete with respect to extended models for $\mathbf{PDL}$. As a result, it is shown that $\mathbf{DB}$ is a fragment of $\mathbf{PDLS}$, a specific extension of $\mathbf{PDL}$ that contains the partial correctness conditional and fusion. Our proof of this result utilises a representation of the ternary Routley-Meyer accessibility relation of substructural models, see (Restall, 2000), by the binary relation corresponding to the action preferred at particular states. It is also shown that the main results can be extended to a specific class of positive substructural logics.

We note that our work is related to Dunn's investigations into the links between relevant and action logics, see (Bimbó & Dunn, 2005; Dunn, 2001, 2003) and to some informal interpretations of the ternary accessibility relation in Routley-Meyer frames, see (Mares, 1996, 2004; Restall, 1995; Slaney & Meyer, 1997) and especially (Beall et al., 2012). However, our approach is specific in that it uses action semantics to provide models for substructural logics (not the other way around) and defines the ternary relation in terms of simpler notions explicitly. Links between dynamic and substructural logics have been studied also by Kozen and Tiuryn (2003). Another related approach is epistemic logic with non-rigid agent names and varying agent domains, see (Grove, 1995; Grove & Halpern, 1993). Our framework can be seen as a fragment of the first-order version of non-rigid epistemic logic.

The article is organised as follows. Section 2 introduces the main logics to be discussed in this article, $\mathbf{PDL}$ and $\mathbf{DB}$. Extended models for $\mathbf{PDL}$ are introduced and independently motivated in Section 3. Section 4 explains that extended models allow to interpret a conditional connective that corresponds to partial correctness claims describing specific actions. Section 5 introduces the logic $\mathbf{PDLS}$ and establishes the main technical result of the article: $\mathbf{DB}$ is a fragment of $\mathbf{PDLS}$. Section 6 extends the main result to stronger substructural logics and specific extensions of $\mathbf{PDLS}$. Section 7 concludes the article and outlines one interesting topic of future research.

2 Preliminaries: $\mathbf{PDL}$ and $\mathbf{DB}$

This section briefly introduces the logics to be discussed. We build on (Harel et al., 2000) and (Restall, 2000), but our formulation of $\mathbf{DB}$ is slightly adjusted.

2.1 $\mathbf{PDL}$

Let π be a countable set of *atomic programs* and Φ a countable set of *atomic formulas* (disjoint from π). The sets of *formulas* and *programs* of the 'dynamic' language $\mathcal{L}(\pi, \Phi)$ are defined by mutual induction as follows:

$$\begin{array}{lll} \phi & ::= & p \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \supset \phi \mid [\alpha]\phi \\ \alpha & ::= & a \mid \alpha \cup \alpha \mid \alpha;\alpha \mid \alpha^* \mid \phi? \end{array}$$

where $p \in \Phi$ and $a \in \pi$. Φ shall be the set of atomic formulas for every language considered in this article. Accordingly, $\mathcal{L}(\pi, \Phi)$ shall be referred to as $\mathcal{L}$.

Program operators '$\cup$', ';', '$*$' and '?' stand for indeterministic choice, sequential composition, indeterministic iteration and test, respectively. Set $Prog(\pi)$ is the set of programs over π. '$[\alpha]\phi$' is read 'It is necessary that after executing α, ϕ is true'.

Formulas of the form

$$\phi \supset [\alpha]\psi \tag{1}$$

express that program α is *partially correct* with respect to input ϕ and output ψ: whenever the program is started in a state satisfying ϕ, then if it halts, it does so in a state satisfying ψ. Precursors of $\mathbf{PDL}$ such as the Floyd-Hoare Logic were devised specially for the purpose of proving partial correctness of specific programs.

A model for $\mathcal{L}$ (a 'dynamic model') is a structure

$$M = \langle S, R, v \rangle \tag{2}$$

where S is a non-empty set of 'states', v is a function from Φ to subsets of S and R is a function from $Prog(\pi)$ to binary relations on S such that

- $R(\alpha \cup \beta) = R(\alpha) \cup R(\beta)$,
- $R(\alpha;\beta) = R(\alpha) \circ R(\beta) = \{\langle x, y\rangle \mid \exists z.(R(\alpha)xz \wedge R(\beta)zy)\}$,
- $R(\alpha^*) = R(\alpha)^*$, i.e. $R(\alpha^*)$ is the reflexive transitive closure of $R(\alpha)$,

- $R(\phi?) = Id(S) \cap \{x \mid M, x \models \phi\}$, where $Id(S)$ is the identity relation on S and the truth relation $M, x \models \phi$ is defined below.

Truth conditions for Boolean ϕ in pointed models (M, x) are as usual. Moreover:

- $M, x \models [\alpha]\phi$ iff $M, y \models \phi$ for all y such that $R(\alpha)xy$.

Validity in M is defined in the usual way as truth in every state of M (Notation: $M \models \phi$). $\mathbf{PDL}$ is the set of dynamic formulas valid in every dynamic model.

2.2 DB

The sets of *formulas* of the 'positive substructural language' $\mathcal{L}_s$ and the 'positive substructural consecution language' $\mathcal{L}_s^{\supset}$ (referred to hereafter as the 'consecution language') are defined as follows:

$$\begin{array}{lcl} \phi & ::= & p \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \phi \otimes \phi \\ \phi' & ::= & \phi \mid \phi \supset \phi \end{array}$$

where $p \in \Phi$. The connective '$\to$' is the substructural implication and '$\otimes$' is fusion.

A *flat model* for $\mathcal{L}_s^{\supset}$ (a 'flat substructural model' or simply a 'flat model') is a structure

$$\mathcal{M}_s = \langle \mathcal{P}, \mathcal{R}, \nu \rangle \tag{3}$$

where $\mathcal{P}$ is a non-empty set of 'points', ν is a function from Φ to subsets of $\mathcal{P}$ and $\mathcal{R}$ is a *ternary* relation on $\mathcal{P}$. The truth conditions of substructural formulas in flat models are as follows:

- $\mathcal{M}, x \models p$ iff $x \in \nu(p)$,
- $\mathcal{M}, x \models \phi \wedge \psi$ iff $\mathcal{M}, x \models \phi$ and $\mathcal{M}, x \models \psi$,
- $\mathcal{M}, x \models \phi \vee \psi$ iff $\mathcal{M}, x \models \phi$ or $\mathcal{M}, x \models \psi$,
- $\mathcal{M}, x \models \phi \supset \psi$ iff $\mathcal{M}, x \not\models \phi$ or $\mathcal{M}, x \models \psi$,
- $\mathcal{M}, x \models \phi \to \psi$ iff for all y, z, if $\mathcal{M}, y \models \phi$ and $\mathcal{R}xyz$, then $\mathcal{M}, z \models \psi$,
- $\mathcal{M}, x \models \phi \otimes \psi$ iff there are y, z such that $\mathcal{R}yzx$ and $\mathcal{M}, y \models \phi$ and $\mathcal{M}, z \models \psi$.

ϕ is valid in $\mathcal{M}_s$ iff $\mathcal{M}, x \models \phi$ for all $x \in \mathcal{P}$. $\mathbf{DB}$ is the set of consecution formulas valid in every flat model.

Note that the distributive laws

$$\phi \wedge (\psi \vee \chi) \quad \supset \quad (\phi \wedge \psi) \vee (\phi \wedge \chi) \tag{4}$$

$$\phi \vee (\psi \wedge \chi) \quad \supset \quad (\phi \vee \psi) \wedge (\phi \vee \chi) \tag{5}$$

and their converses are clearly valid in $\mathbf{DB}$.

Our logic $\mathbf{DB}$ corresponds to the positive fragment of $\mathbf{DBSub}$ without truth and falsity constants and the converse implication '$\leftarrow$', see (Restall, 2000). Note that $\phi_1 \odot \phi_2$ ($\odot \in \{\supset, \rightarrow, \otimes\}$) is a consecution formula only if ϕ_i do not contain occurrences of '$\supset$'. This formulation of the language replaces 'consecutions' $X \vdash \phi$, where

$$X ::= \phi \mid X; X \mid X, X \tag{6}$$

by material implications $f(X) \supset \phi$, where $f(X)$ is the result of replacing every occurrence of ';' in X by an occurrence of '$\otimes$' and every occurrence of ',' by an occurrence of '$\wedge$' (Semantically, $\phi; \psi$ is equivalent to $\phi \otimes \psi$, ϕ, ψ to $\phi \wedge \psi$ and a consecution $\phi \vdash \psi$ is valid in a model iff there is no point x such that ϕ holds at x and ψ does not.) Hence, the original formulation of $\mathbf{DB}$ as a set of consecutions is replaced by a formulation of $\mathbf{DB}$ as a set of formulas. Note that every $\phi \in \mathbf{DB}$ is a material implication.

Our flat models are a special case of 'standard' relational models, where $\mathcal{P}$ is a poset $(P, \leq)$. The poset is 'flattened' here as $\leq$ is replaced by $=$. The use of flat models is allowed by the fact that $\mathbf{DB}$ is sound and complete with respect to flat models, see (Restall, 2000, Chapter 13.3).

Proposition 1 $\mathbf{DB}$ *has the finite model property.*

Proof. A simple corollary of Theorem 14.11 of (Restall, 2000). □

3 Extended dynamic models

This section explains the ideas motivating extended valuations and extended dynamic models. We concentrate on extending models for $\mathbf{PDL}$, but it is clear that the technique may be applied to models of any multi-modal logic.

Assume that we have a robot Rob. Let $p \in \Phi$ stand for some possible atomic features of the environments Rob is built to operate in. Let $a \in \pi$

stand for some atomic instructions Rob is designed to carry out. Now every dynamic model (2) can be seen as specifying a set of relevant possible environments (S) and transitions on these environments given by the available actions built on π (given by R). Importantly, the valuation v provides a *partial description* of the possible environments: it lists for every $x \in S$ the atomic features of the environment x.

Now assume that we have a user Ann who interacts with Rob to achieve specific goals. One interesting feature of the Rob/Ann interactions are Ann's *preferences*. In particular, we can see Ann as preferring Rob to perform specific actions $\alpha \in Prog(\pi)$. Note that Ann's preferences can be seen as part of Rob's environment. One way of formalising this is to *extend* the notion of valuation to describe Ann's preferences (in the various possible environments $x \in S$), in addition to the atomic facts represented by $p \in \Phi$.

Definition 1 (Extended dynamic models) *An extended dynamic model is a structure*

$$\mathfrak{M} = \langle S, R, V \rangle \tag{7}$$

where S and R are as in (2)*, and $V : S \to (Prog(\pi) \times 2^{\Phi})$. $V(x)$ will also be denoted as $\langle \alpha_x, \Phi_x \rangle$. Truth conditions for dynamic formulas and validity are defined as before, with the exception that*

- $\mathfrak{M}, x \models p$ *iff* $p \in \Phi_x$

It is clear that $\phi \in \mathbf{PDL}$ iff ϕ is valid in every $\mathfrak{M}$, i.e. extended models are 'proper' models for $\mathbf{PDL}$.

Again, α_x can be seen as the action Ann prefers Rob to perform in x.[2] It is clear that although extended models are defined as models for $\mathcal{L}$, interesting new operators can be introduced and interpreted in them. One simple example follows.

Example 1 We can extend $\mathcal{L}$ by a unary operator A and read $A\phi$ as 'the preferred action yields ϕ'. The corresponding truth condition would be

- $\mathfrak{M}, x \models A\phi$ iff for all $y \in S$, if $R(\alpha_x)xy$, then $\mathfrak{M}, y \models \phi$.

This extended language would be expressive enough to describe how Rob's actions can influence Ann's preferences. For example

$$p \supset [\alpha]Aq \tag{8}$$

[2] Or, in general, α_x can be seen as the action preferred in x by some contextually fixed agent. Our informal discussions will be confined to the Rob/Ann context throughout the article.

means that if p is the case, then Rob's performing α necessarily results in a state where Ann's preferred action (whatever it is) yields q.

Note that we could select an action $a_i \in \pi$ and read it in terms of Ann's doxastic accessibility, i.e. translate $R(a_i)xy$ as 'Everything Ann believes in x holds in y'. Hence, $[a_i]\phi$ could be read 'Ann (implicitly) believes that ϕ' and written as '$B\phi$'. A situation in which Ann falsely believes that her preferred action yields p, for example, could then be formalised by

$$\neg Ap \wedge BAp. \tag{9}$$

However, we shall not investigate this extension here.

4 The partial correctness conditional

The dynamic language with the partial correctness conditional $\mathcal{L}_c$ ('correctness language') extends $\mathcal{L}$ by a binary operator '$\rightarrow$', with the proviso that ϕ? is a program only if ϕ contains no occurrence of '$\rightarrow$'.

Definition 2 ($\rightarrow$ in extended models) *We add to the $\mathcal{L}$-truth conditions in extended models the following:*

- $\mathfrak{M}, x \models \phi \rightarrow \psi$ *iff for all* $y, z \in S$*, if* $\mathfrak{M}, y \models \phi$ *and* $R(\alpha_x)yz$*, then* $\mathfrak{M}, z \models \psi$.

Validity is defined in the usual way.

Hence, $\phi \rightarrow \psi$ holds at x iff the following holds: if the action preferred at x is performed at a state satisfying ϕ, then every possible output state satisfies ψ. This condition is familiar, as the next theorem shows.

Theorem 1 *The following are equivalent (with a slight abuse of notation):*

- $\mathfrak{M}, x \models \phi \rightarrow \psi$
- $\mathfrak{M} \models \phi \supset [\alpha_x]\psi$

Proof. Firstly, assume that (a) $\mathfrak{M}, x \models \phi \rightarrow \psi$ and (b) $\mathfrak{M} \not\models \phi \supset [\alpha_x]\psi$. (b) implies that there is y such that $\mathfrak{M}, y \models \phi$ and $\mathfrak{M}, y \not\models [\alpha_x]\psi$. The latter means that there is z such that $R(\alpha_x)yz$ and $\mathfrak{M}, z \not\models \psi$. But (a) entails that this is impossible.

Secondly, assume that (c) $\mathfrak{M}, x \not\models \phi \rightarrow \psi$ and (d) $\mathfrak{M} \models \phi \supset [\alpha_x]\psi$. (c) entails that there are y, z such that $\mathfrak{M}, y \models \phi$, $R(\alpha_x)yz$ and $\mathfrak{M}, z \not\models \psi$. But (d) entails that this is impossible. □

Theorem 1 vindicates our calling '$\to$' a partial correctness conditional: $\phi \to \psi$ holds at x iff the action preferred at x is (globally) partially correct with respect to input (precondition) ϕ and output (postcondition) ψ.

Providing an axiomatisation of $\mathbf{PDLC}$, the set of correctness formulas valid in every extended dynamic model, is a natural open problem. Instead of solving the problem, we turn to investigating the links between extended dynamic models and substructural logic.

5 DB in extended models

The correctness language with fusion $\mathcal{L}_{ds}$ (the 'dynamic substructural language') extends $\mathcal{L}_c$ by the fusion connective '$\otimes$', with the proviso that $\phi?$ is a program only if ϕ contains no occurrence of '$\to$' or '$\otimes$'.

Definition 3 ($\otimes$ in extended models) *Formulas in the $\mathcal{L}_c$-fragment of $\mathcal{L}_{ds}$ are interpreted in extended models as usual. Moreover,*

- $\mathfrak{M}, x \models \phi \otimes \psi$ *iff there are $y, z \in S$ such that $R(\alpha_y)zx$ and $\mathfrak{M}, y \models \phi$ and $\mathfrak{M}, z \models \psi$.*

Validity is defined in the usual way. $\mathbf{PDLS}$ *is the set of dynamic substructural formulas valid in every extended dynamic model.*

$\phi \otimes \psi$ holds at x iff x can be obtained by performing an action that is preferred at some state that satisfies ϕ in a state that satisfies ψ. Note that the consecution language $\mathcal{L}_s^{\supset}$ is a fragment of $\mathcal{L}_{ds}$. The *$\mathcal{L}_s^{\supset}$-fragment of* $\mathbf{PDLS}$ is the set of consecution formulas valid in every extended dynamic model.

Theorem 2 *If $\phi \in \mathbf{DB}$, then $\phi \in \mathbf{PDLS}$.*

Proof. We show that if $\phi \in \mathcal{L}_s^{\supset}$ and $\phi \notin \mathbf{PDLS}$, then $\phi \notin \mathbf{DB}$. Assume that $\mathfrak{M} \not\models \phi$ for some $\mathfrak{M} = \langle S, R, V\rangle$. Define $\mathfrak{M}_s = \langle S, \mathcal{R}^V, \nu^V\rangle$ as follows:

- $\mathcal{R}^V$ is a ternary relation on S such that $\mathcal{R}^V xyz$ iff $R(\alpha_x)yz$ (in $\mathfrak{M}$).
- ν^V is a function from Φ to subsets of S such that $x \in \nu^V(p)$ iff $p \in \Phi_x$ (in $\mathfrak{M}$).

It is plain that $\mathfrak{M}_s$ is a flat model. It suffices to show that for all $x \in S$ and $\phi \in \mathcal{L}_s^{\supset}$, $\mathfrak{M}, x \models \phi$ iff $\mathfrak{M}_s, x \models \phi$. But this is established easily by induction on the complexity of ϕ. □

Theorem 3 $\phi \in \mathcal{L}_s^{\supset}$ *and* $\phi \in \mathbf{PDLS}$*, then* $\phi \in \mathbf{DB}$.

Proof. We show that if $\phi \in \mathcal{L}_s^{\supset}$ and $\phi \notin \mathbf{DB}$, then $\phi \notin \mathbf{PDLS}$. Assume that $\phi \notin \mathbf{DB}$. By Proposition 1, there is a finite $\mathcal{M} = \langle \mathcal{P}, \mathcal{R}, \nu \rangle$, such that $\mathcal{M} \not\models \phi$. Hence, there is an injective $\tau : \mathcal{P} \to \pi$. Let us have any $\mathcal{M}^\tau = \langle \mathcal{P}, R^\tau, V^\tau \rangle$ that satisfies the following:

- R^τ is a function from $Prog(\pi)$ to binary relations on $\mathcal{P}$ such that for all $a \in \pi$: if $a = \tau(x)$ for some $x \in \mathcal{P}$, then $R^\tau(a)yz$ iff $\mathcal{R}xyz$. Moreover, R^τ complies with the requirements concerning '$\cup$', ';', '$*$' and '?', explained in Section 2.1 (Note that it is 'always' possible to define R^τ so that the requirements are satisfied).

- $V^\tau(x) = \langle \tau(x), \{p \mid x \in \nu(p)\} \rangle$.

It is plain that every such $\mathcal{M}^\tau$ is an extended model and that at least one such $\mathfrak{M}^\tau$ exists. It suffices to show that for every $\phi \in \mathcal{L}_s^{\supset}$ and $x \in \mathcal{P}$, $\mathcal{M}, x \models \phi$ iff $\mathcal{M}^\tau, x \models \phi$. But this is established easily by induction on the complexity of ϕ. □

Corollary 1 $\mathbf{DB}$ *is the* $\mathcal{L}_s^{\supset}$*-fragment of* $\mathbf{PDLS}$.

The results of this section establish a link between dynamic logic and substructural logic: a variant of the basic distributive substructural logic is a fragment of a dynamic logic where the substructural conditional '$\to$' corresponds to partial correctness claims describing specific actions.

6 Extensions of the main result

It is natural to ask if similar results can be obtained for other substructural logics. This section provides some observations.

An *extension* of $\mathbf{PDLS}$ is a set of $\mathcal{L}_{ds}$-formulas valid in a specific class of extended dynamic models. We will show that every substructural logic over the consecution language $\mathcal{L}_s^{\supset}$ that satisfies certain conditions is a $\mathcal{L}_s^{\supset}$-fragment of some extension of $\mathbf{PDLS}$.

Firstly, let us reiterate that R and V in every $\mathfrak{M}$ induce a ternary relation $\mathcal{R}^V$ such that

$$\mathcal{R}^V xyz \iff R(\alpha_x)yz. \tag{10}$$

(This observation was used in the proof of Theorem 2.) Secondly, we list several familiar 'frame-conditions' that are used in defining models for specific extensions of $\mathbf{DB}$, see (Restall, 2000, Ch. 11) (for all $x, y, z, w \in \mathcal{P}$):

$$\exists v.(\mathcal{R}xyv \wedge \mathcal{R}vzw) \quad \Rightarrow \quad \exists v.(\mathcal{R}yzv \wedge \mathcal{R}xvw) \tag{11}$$

$$\exists v.(Ryzv \wedge Rxvw) \quad \Rightarrow \quad \exists v.(\mathcal{R}xyv \wedge \mathcal{R}vzw) \tag{12}$$

$$\mathcal{R}xyz \quad \Rightarrow \quad \mathcal{R}yxz \tag{13}$$

$$\mathcal{R}xxx \tag{14}$$

(11) is usually called the 'associativity condition', (12) the 'converse associativity condition', (13) the 'weak commutativity condition' and (14) the 'weak contraction condition'.

Lemma 1 *Given* (10)*, the following claims hold:*

1. *$\mathcal{R}^V$ satisfies* (11) *iff (for all $x, y, z \in S$) $R(\alpha_x)yz$ implies $R(\alpha_z) \subseteq R(\alpha_y; \alpha_x)$ ($\mathfrak{M}$ that comply with this condition will be called 'associative')*

2. *$\mathcal{R}^V$ satisfies* (12) *iff (for all $x, y, z, w \in S$) $R(\alpha_y; \alpha_x)zw$ implies that there is u such that $R(\alpha_x)yu$ and $R(\alpha_u)zw$ (conversely associative models)*

3. *$\mathcal{R}^V$ satisfies* (13) *iff (for all $x, y, z \in S$) $R(\alpha_x)yz$ implies $R(\alpha_y)xz$ (weakly commutative models)*

4. *$\mathcal{R}^V$ satisfies* (14) *iff (for all $x \in S$) $R(\alpha_x)xx$ (weakly contractive models)*

Proof. We shall prove the right-to-left direction of the first claim. Proofs of the other claims are similar. Given (10), (11) entails that if there is v such that $R(\alpha_x)yv$ and $R(\alpha_v)zw$, then there is u such that $R(\alpha_y)zu$ and $R(\alpha_x)uw$. In other words, then $R(\alpha_y; \alpha_x)zw$. But then, by first-order logic,

$$\forall v. \left(R(\alpha_x)yv \supset (R(\alpha_v)zw \supset R(\alpha_y; \alpha_x)zw)\right). \tag{15}$$

□

Let the following extensions of $\mathbf{PDLS}$ be defined as sets of dynamic substructural formulas valid in the set of extended dynamic models indicated in the brackets: $\mathbf{PDLS}_B$ (associative models); $\mathbf{PDLS}_{B^c}$ (conversely associative models); $\mathbf{PDLS}_{CI}$ (weakly commutative models); $\mathbf{PDLS}_{WI}$ (weakly contractive models).

Theorem 4 *For all $\phi \in \mathcal{L}_s^{\supset}$:*

1. *If ϕ is valid in every flat substructural frame that satisfies* (11)*, then $\phi \in \mathbf{PDLS}_B$.*
2. *If ϕ is valid in every flat substructural frame that satisfies* (12)*, then $\phi \in \mathbf{PDLS}_{B^c}$.*
3. *If ϕ is valid in every flat substructural frame that satisfies* (13)*, then $\phi \in \mathbf{PDLS}_{CI}$.*
4. *If ϕ is valid in every flat substructural frame that satisfies* (14)*, then $\phi \in \mathbf{PDLS}_{WI}$.*

Proof. Assume that $\phi \notin \mathbf{PDLS}_B$. Then there is an associative model $\mathfrak{M} \not\models \phi$. Define $\mathfrak{M}_s$ as in the proof of Theorem 2. By Lemma 1, $\mathfrak{M}_s$ is a flat substructural model that satisfies (11). It is easy to show by induction on the complexity of $\phi \in \mathcal{L}_s^{\supset}$ that (for all $x \in S$) $\mathfrak{M}, x \models \phi$ iff $\mathfrak{M}_s, x \models \phi$. Proofs of the other claims are similar. □

We shall say that a substructural logic $\mathbf{Log}$ (over $\mathcal{L}_s^{\supset}$) has the *countable flat model property* iff $\phi \notin \mathbf{Log}$ implies that there is a *countable* flat model $\mathcal{M} \not\models \phi$.

Theorem 5 *Any substructural logic (over $\mathcal{L}_s^{\supset}$) that is sound and complete over a class of frames specified by (some) of the frame properties* (11)–(14) *and has the countable flat model property is a $\mathcal{L}_s^{\supset}$-fragment of some extension of* $\mathbf{PDLS}$.

Proof. Given Theorem 4, it is sufficient to prove that if $\mathcal{M} \not\models \phi$ for a countable flat model $\mathcal{M} = \langle \mathcal{P}, \mathcal{R}, \nu \rangle$ that satisfies some of the conditions (11)–(14), then there is a corresponding[3] extended dynamic model $\mathfrak{M} \not\models \phi$. If $\mathcal{M}$ is countable, then there is an injection $\tau : \mathcal{P} \to \pi$ and we may proceed as in the proof of Theorem 3. □

7 Conclusion

We have shown that an independently motivated extension $\mathbf{PDLS}$ of $\mathbf{PDL}$ is interestingly linked to substructural logic. The substructural conditional

[3] Associative extended dynamic models correspond to (11), associative and (at the same time) weakly commutative models correspond to the conjunction of (11) and (13), etc.

'$\to$' within the logic $\mathbf{DB}$ can be seen as expressing partial correctness claims concerning specific 'preferred' actions. The main technical result is a soundness and completeness proof for $\mathbf{DB}$ with respect to extended dynamic models, which entails that $\mathbf{DB}$ is a specific language-fragment of the dynamic logic $\mathbf{PDLS}$. We have shown that our results extend to a specific class of stronger substructural logics.

Perhaps the most interesting topic for further research is to extend the results of this article to logics with negation. Substructural negation is a *modal* operator, usually interpreted in terms of a 'compatibility' relation C in substructural models: $\sim\phi$ holds in x iff ϕ is false in every y compatible with x, that is Cxy, see (Restall, 2000). Technically, the compatibility relation could be 'incorporated' into extended models quite easily: just pick any $a_i \in \pi$ and read $R(a_i)$ as the compatibility relation. In other words, one could define '$\sim$' thus:

$$\sim\phi ::= [a_i]\neg\phi. \tag{16}$$

As shown in Example 1, such an operator could be read informally in terms of Anne's implicit belief. Hence, the substructural $\sim\phi$ could be read as 'Anne believes that ϕ is false', or, equivalently, 'Anne believes that $\neg\phi$'. Further investigations into this are left for another occasion.

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