First-Order Fuzzy Modal Logics with Variable Domains

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In modal discourse of various kinds (alethic, temporal, doxastic, deontic, etc.) it is often appropriate to regard some individuals as existing only in some (or even none) of the possible worlds: for instance, the individual called Socrates existed in some of the past world-times, but not in the world-times before his birth or after his death; the golden mountain, being possible, does exist in some possible world(s), though not in the actual world; and the largest natural number, being contradictory, is commonly modeled as not existing in any world. In other words, in predicate modal logic it is quite natural to consider different domains of individuals for different possible worlds. However, in normal predicate modal logics it is easy to prove for any term t that $\vdash \Box(\exists x)(x = t)$, i.e., that the referent of t necessarily exists in every possible world, contrary to our starting point. A natural solution is to employ quantification principles of some member of the family of *free logics,* or logics 'free of existential assumptions' [Nol07, Pos07, Gar01]. Free logics permit to block some classically valid inferences that would lead to the formula above.

In recent decades, modal logics have been studied not just in classical bivalent settings, but also in gradual settings over suitable fuzzy logics; an important early contribution to fuzzy modal logic was made by Hájek in [Háj98, Sect. 8.3]. To handle non-existing individuals analogously not just in bivalent, but also in gradual contexts, free variants of predicate fuzzy logics are needed. The first variant of free fuzzy logic has recently been proposed by the present authors [BD18]. Here we focus on its application in predicate fuzzy modal logic with variable domains of individuals.

As argued in [BD18], a reasonable choice for free fuzzy logic is the fuzzification of *positive* free logic, in which empty-termed formulas can be true (to a degree) or truth-valueless. To accommodate truth value gaps, a simple system L* of partial fuzzy logic proposed in [BN15, BD16] is employed (further options are left for future work). In L* (definable over any $\triangle$-core fuzzy logic L), truth-value gaps are represented by an 'error code' $*$ for an undefined truth degree. Propositional connectives and quantifiers in formulas are marked by the manner in which the error code truth-functionally propagates from subformulas (e.g., Bochvar-style for a fatal error, Sobociński-style for an ignorable error, and Kleene-style for an overridable error); auxiliary defined connectives provide the means to explicitly express the definability preconditions of valid inference rules.

The free fuzzy logic of [BD18] employs the *dual-domain* semantics, in which each first-order model is equipped with a crisp non-empty *outer* domain D_0, and an *inner* domain D_1, which is a crisp or fuzzy subset of D_0. All terms have (possibly dummy) referents in D_0, while D_1 collects existent individuals. Predicate and function symbols are interpreted over the outer domain, with truth-value gaps allowed. The usual ('inner') quantifiers range over D_1. The 'outer' quantifiers (over D_0), useful for formalizing such propositions as "some things do not exist", behave as in the usual (non-free) fuzzy logic; the inner quantifiers can be defined by restricting the outer ones (by Kleene connectives) to the inner domain. The resulting apparatus enables making the existence preconditions of inference rules explicit.

In our talk we will present a semantics for predicate fuzzy modal logic with variable domains, based on the free fuzzy logic of [BD18]. Similarly to the latter, each possible world $w \in W$ in a fuzzy Kripke frame comes equipped with an inner domain D_1^w, comprising individuals that exist in w. Furthermore, there is a common outer domain $D_0 \supseteq \bigcup_{w \in W} D_1^w$, which enables formulating statements about objects that exist in only some (or even none) of the possible worlds. The language and evaluation of formulas are defined in a straightforward manner, combining the Tarski conditions of non-modal free fuzzy logic and modal fuzzy logic with constant domains; the modification works for a broad class of underlying fuzzy modal logics. We will give the initial observations on the proposed semantics and discuss the (in)validity of important quantified modal formulas in resulting fuzzy modal logics. Finally we will outline its further development.

Acknowledgment: The work is supported by projects 16–19170S of GA ČR and LQ1602 of MŠMT ČR (NPU II).

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