Effect of hydrodynamic mixing on the photosynthetic microorganism growth: Revisited

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The problem

We want to solve

$$y_{av} = \int_0^1 y(x)\mathrm{d}x$$

where y solves the following boundary value problem

$$-[p(x)y']' + q(x)y = q(x)y_{ss}(x), \quad y'(0) = y'(1) = 0,$$

$$\begin{aligned}
p(x) &= p_0 + p_1[1 - (2x-1)^2] \\
q(x) &= (u(x) + q_2)Da_{II} \\
y_{ss}(x) &= \frac{u(x)}{q_2 + u(x)} \cdot \frac{u_{av} + q_2}{q_2 u_{av}^2 + u_{av} + q_2} \\
u(x) &= u_0\, 2^{-8x} \\
u_{av} &= \int_0^1 u(x)\mathrm{d}x = \frac{1 - 2^{-8}}{8\ln(2)} u_0
\end{aligned}$$

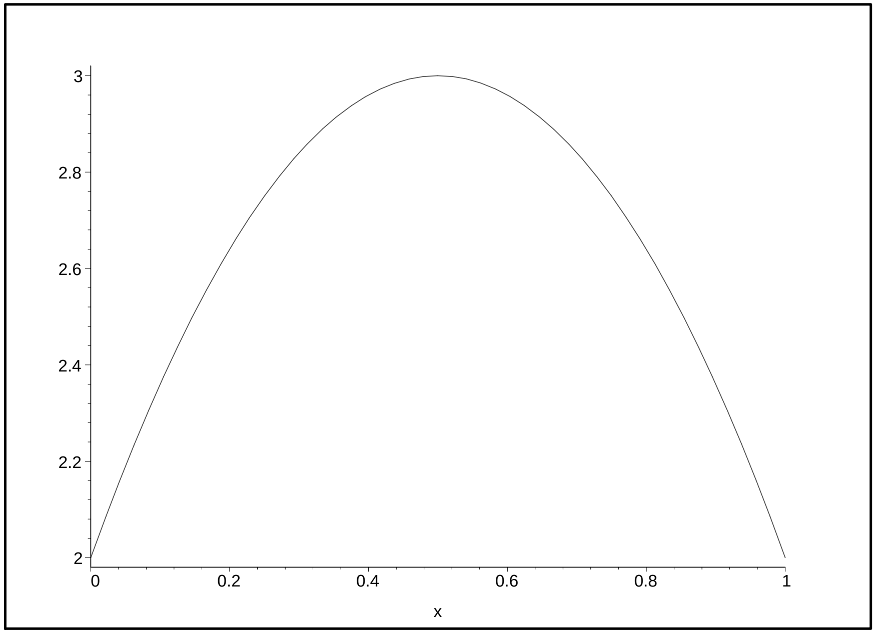
and $p_0, p_1, q_2, u_0, Da_{II} > 0$ are constants.

Function $p(x)$

$$-[p(x)y']' + q(x)y = q(x)y_{ss}(x)$$

$$p(x) = p_0 + p_1[1 - (2x-1)^2]$$

$$p_0 = 2, \quad p_1 = 1$$



Function $u(x)$

The function $u(x)$ is chosen so that:

- $u(x)$ is decreasing, $x \in \langle 0, 1 \rangle$
- $u(x) > 0 \quad \forall x \in \langle 0, 1 \rangle$
- $\int_0^1 u(x) \mathrm{d}x = 1$
- $u(0) = u_0 > 1$

Here

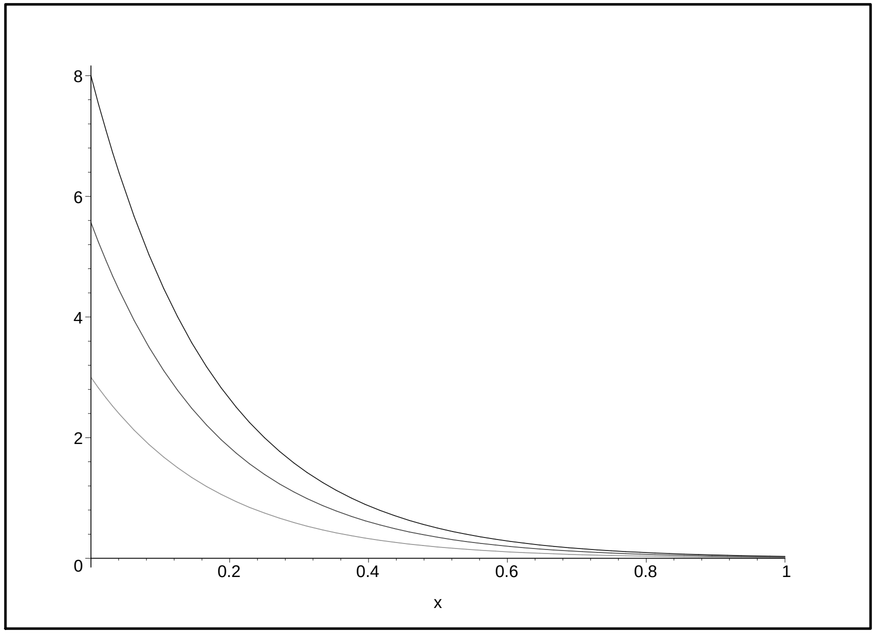
$$u(x) = u_0 \, 2^{-8x}$$

and u_0 is chosen so that

$$u_{av} \equiv \int_0^1 u(x) \mathrm{d}x = 1$$

$\Rightarrow$

$$u_0 = \frac{8 \ln(2)}{1 - 2^{-8}} \approx 5.567$$



Condition $u_{av} = 1$ is not necessary, we tried also other values u_0.

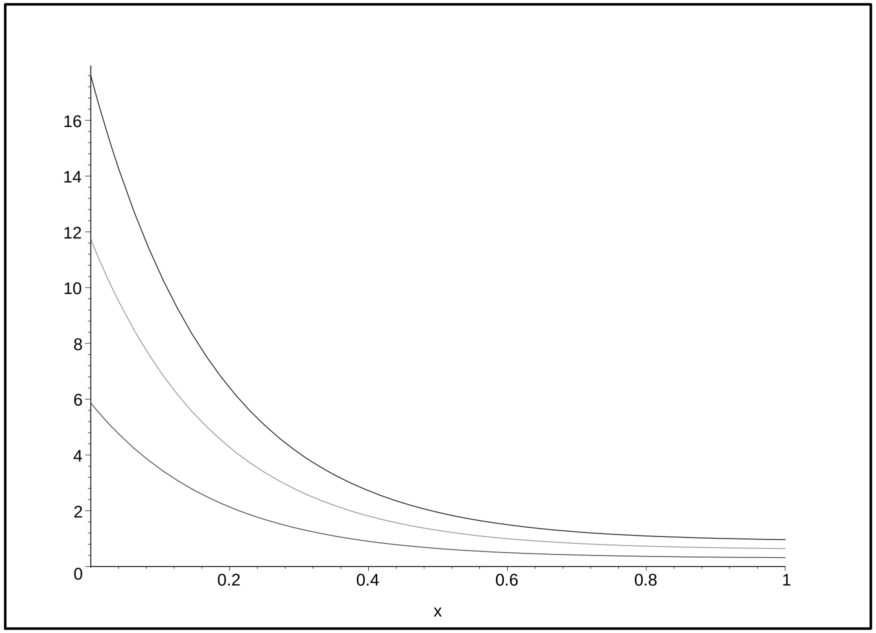
Function $q(x)$

$$-[p(x)y']' + q(x)y = q(x)y_{ss}(x)$$

$$q(x) = (u(x) + q_2)Da_{II}$$

$$q_2 = 0.3$$

$Da_{II} > 0$ ... Damköhler number of second type

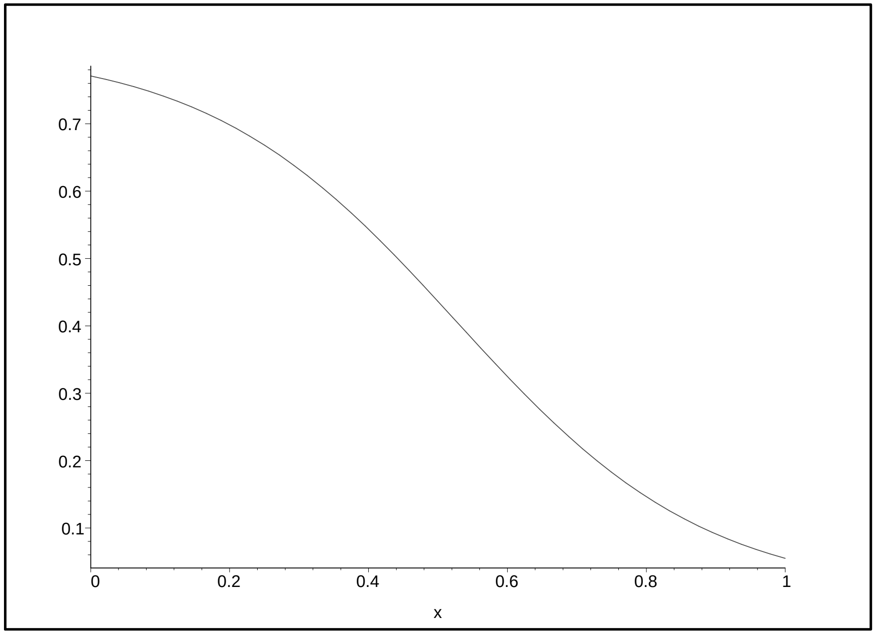


Function $y_{ss}(x)$

$$-[p(x)y']' + q(x)y = q(x)y_{ss}(x)$$

$$y_{ss}(x) = \frac{u(x)}{q_2 + u(x)} \cdot \frac{u_{av} + q_2}{q_2 u_{av}^2 + u_{av} + q_2} = \frac{u(x)}{q_2 + u(x)} \cdot \frac{1 + q_2}{1 + 2q_2}$$

steady-state solution



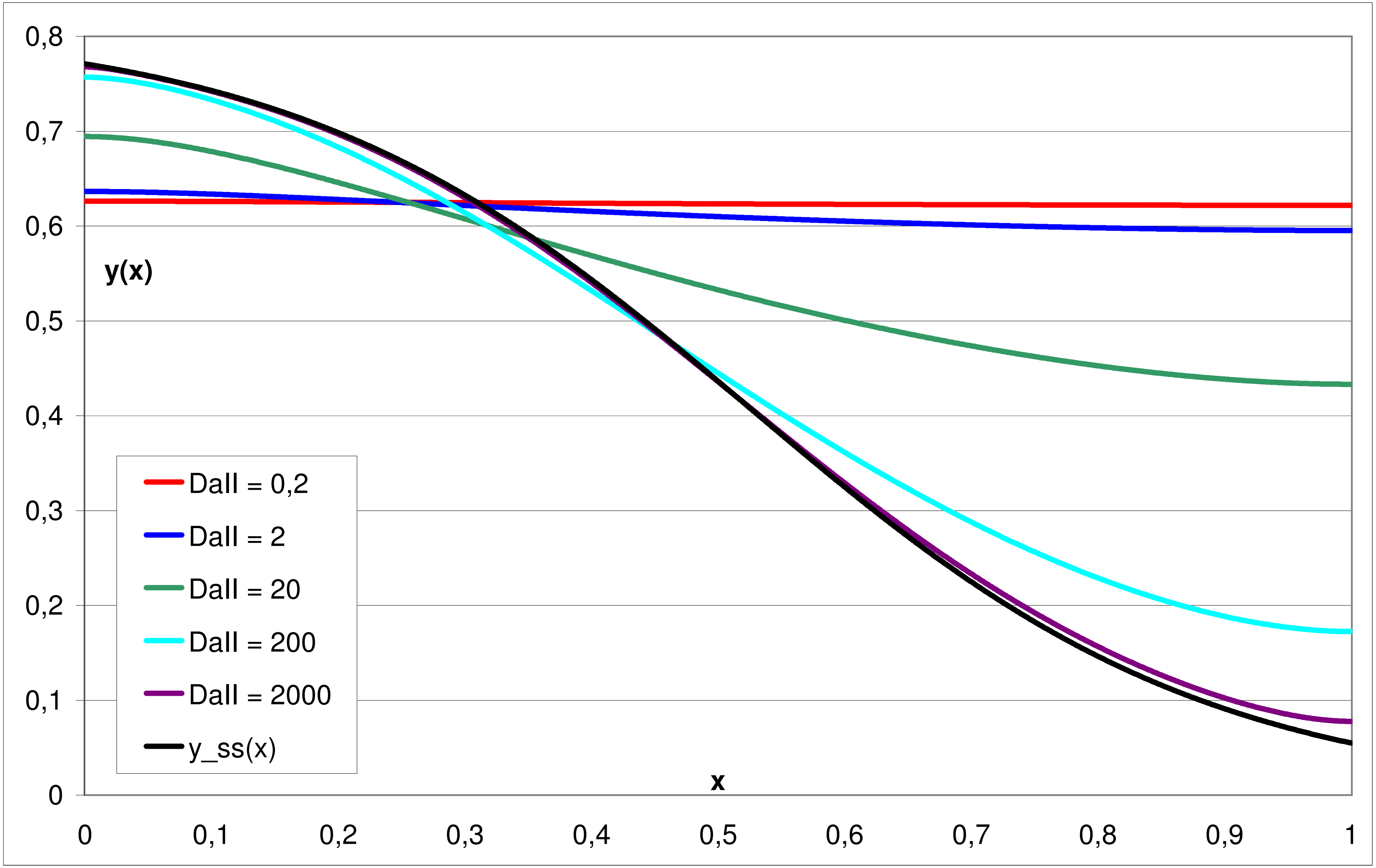
Numerical solution

$$y_{av} = \int_0^1 y(x)\mathrm{d}x, \quad -[p(x)y']' + q(x)y = q(x)y_{ss}(x), \quad y'(0) = y'(1) = 0$$

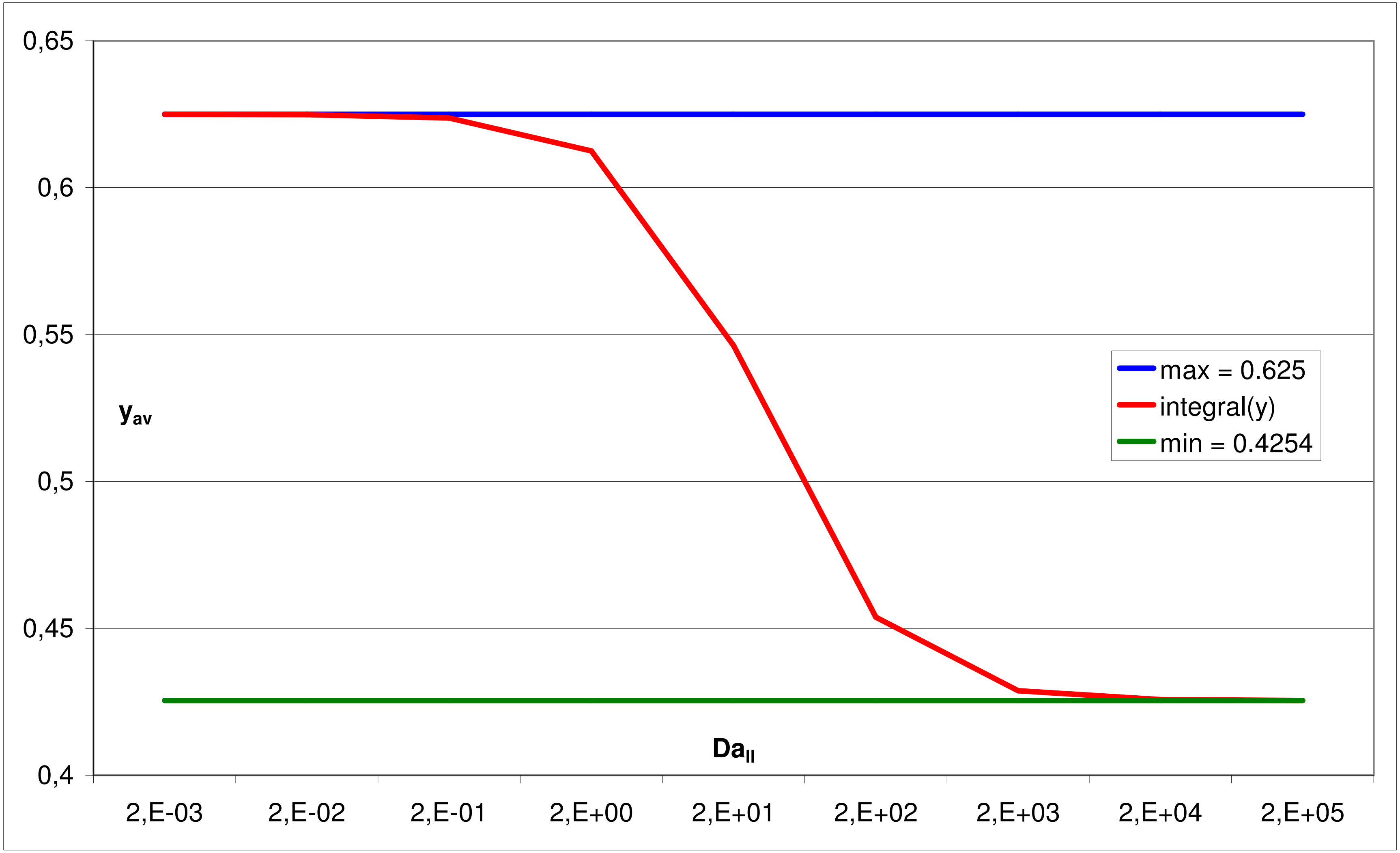
We will study the dependence of the solution on Da_{II}. Properties:

- The boundary value problem with Neumann initial conditions and inhomogeneous right-hand side.
- It is symmetric and positive and the corresponding linear differential operator of second order on the left side is self-adjoint.
- As $q(x) > 0$, this problem has a unique solution.
- It was solved numerically using a finite difference scheme with uniformly distributed nodes.
- It leads to a spd system of linear equations with tridiagonal, irreducible, diagonally dominant, monotonic, and regular matrix.
- The resulting scheme approximates the exact solution with accuracy of order h^2.
- The integral was solved using the Simpson rule.

Function $y(x)$ vs. x



Function y_{av} vs. Da_{II}



Limit values

$$-[p(x)y']' + (u(x)+q_2)Da_{II}\,y = (u(x)+q_2)Da_{II}\,y_{ss}(x)$$

- $Da_{II} \to \infty \quad \Rightarrow \quad y(x) \to y_{ss}(x) \quad \Rightarrow \quad y_{av} \to 0.4254$:

$$\int_0^1 y_{ss}(x)\mathrm{d}x = \frac{1+q_2}{1+2q_2} \cdot \frac{1}{8\ln 2} \cdot \ln \frac{q_2+u_0}{q_2+u_0 2^{-8}} \approx 0.4254$$

- $Da_{II} \to 0 \quad \Rightarrow \quad y(x) \to$ constant $= 0.625 \quad \Rightarrow \quad y_{av} \to 0.625$:
  This value corresponds to values

$$u_{av} = \int_0^1 u(x)\mathrm{d}x = 1 \quad \text{and} \quad u(x) = 1 \quad \Rightarrow \quad y_{ss}(x) = \frac{1}{1+2q_2} = 0.625$$

which means that the ODE system performs the "averaging" of $u(x)$.

Optimization problem

Now we can formulate the optimization problem to maximize the integral

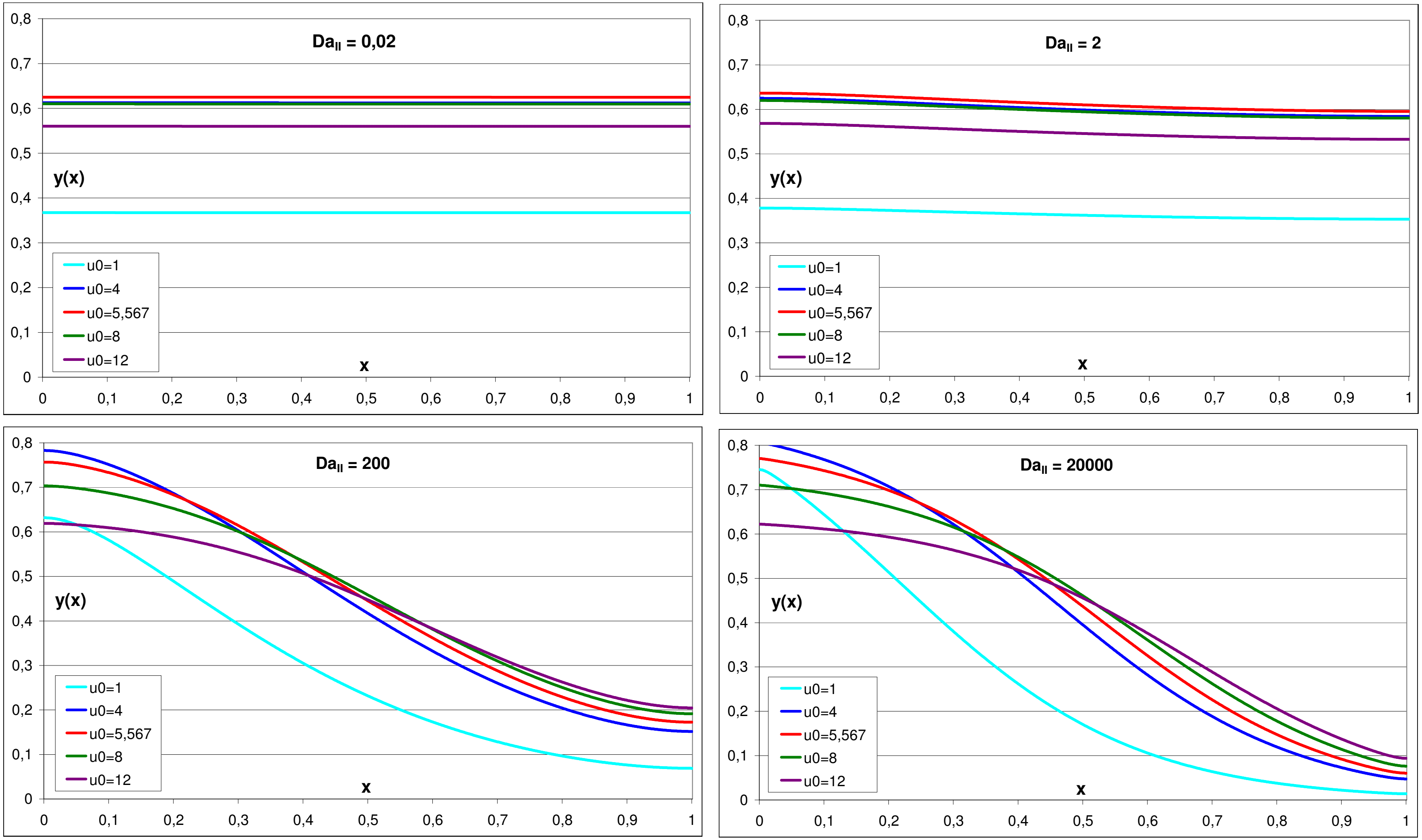
$$y_{av}(u_0) = \int_0^1 y(x)\mathrm{d}x.$$

We have made several simulations for various u_0 with $Da_{II} \to 0$ and the values of y_{av} were always smaller than that for u_0 satisfying the condition

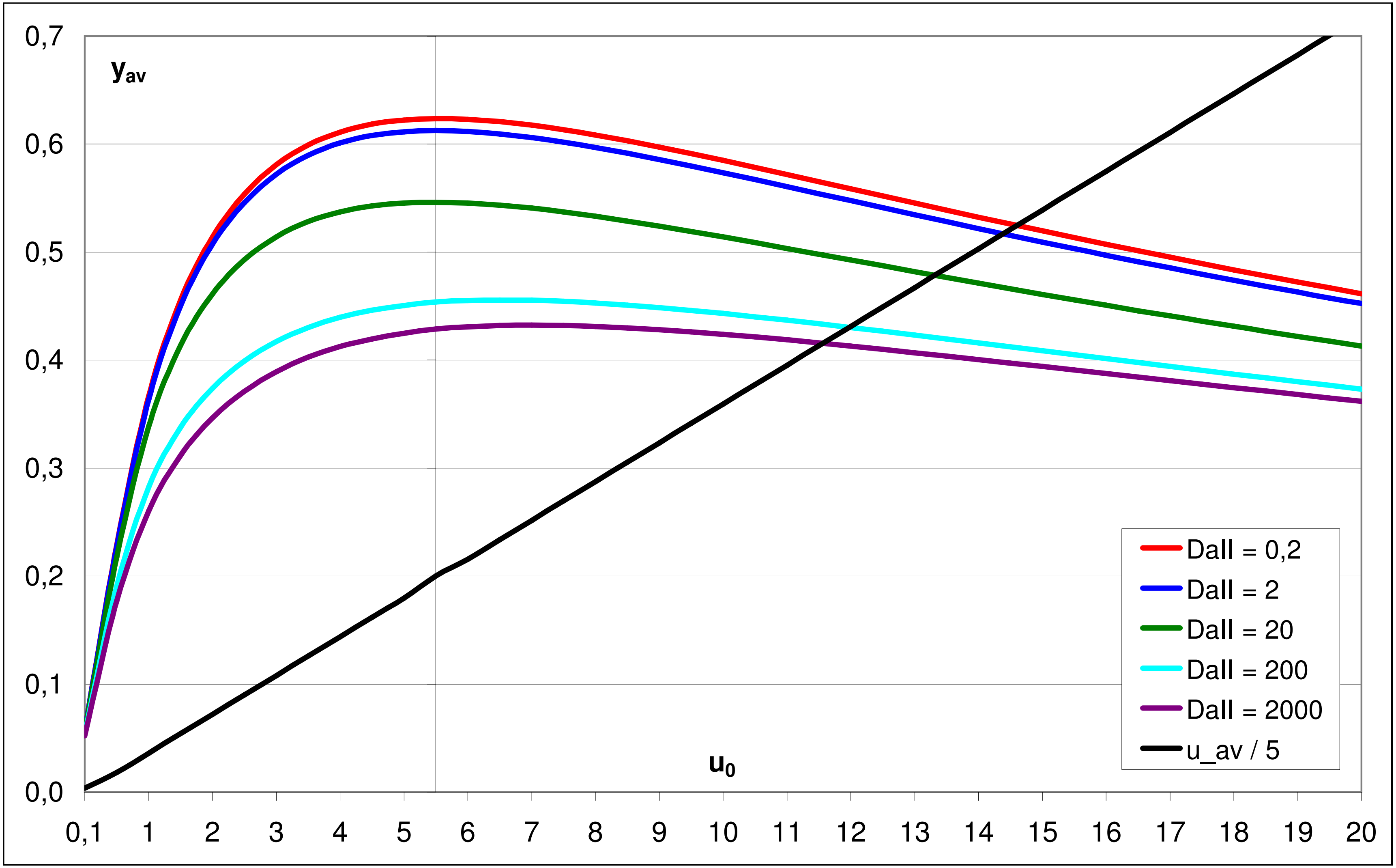
$$u_{av} = \int_0^1 u(x)\mathrm{d}x = 1.$$

This is a numerical confirmation of the hypothesis often mentioned in biotechnological literature, i.e. that for sufficiently good mixing, all the relevant quantities are homogeneously distributed. It indicates that the value $u_{av} = 1$ maximizes y_{av}.

Function $y(x)$ vs. x for various u_0



Function y_{av} vs. u_0



Conclusion

In the sequel we aim to announce this statement more rigorously as Theorem, which formulates the existence of an optimal u_0 and the simple condition to achieve an extreme level of a performance index or cost functional y_{av}.

Theorem: Let system

$$-[p(x)y']' + q(x)y = q(x)y_{ss}(x), \quad y'(0) = y'(1) = 0,$$

one input signal u_0, and one parameter Da_{II} be given. Then for any Da_{II} there exists u_0^* such that for any $u_0,\ u_0 \neq u_0^*$, we have

$$y_{av}(u_0) < y_{av}(u_0^*).$$

Moreover, the corresponding integral average of $u(x)$ maximizing y_{av} goes to the unity, i.e.

$$u_{av}^* := \int_0^1 u(x)\mathrm{d}x \to 1,$$

for $Da_{II} \to 0$.

Thank you for your attention!