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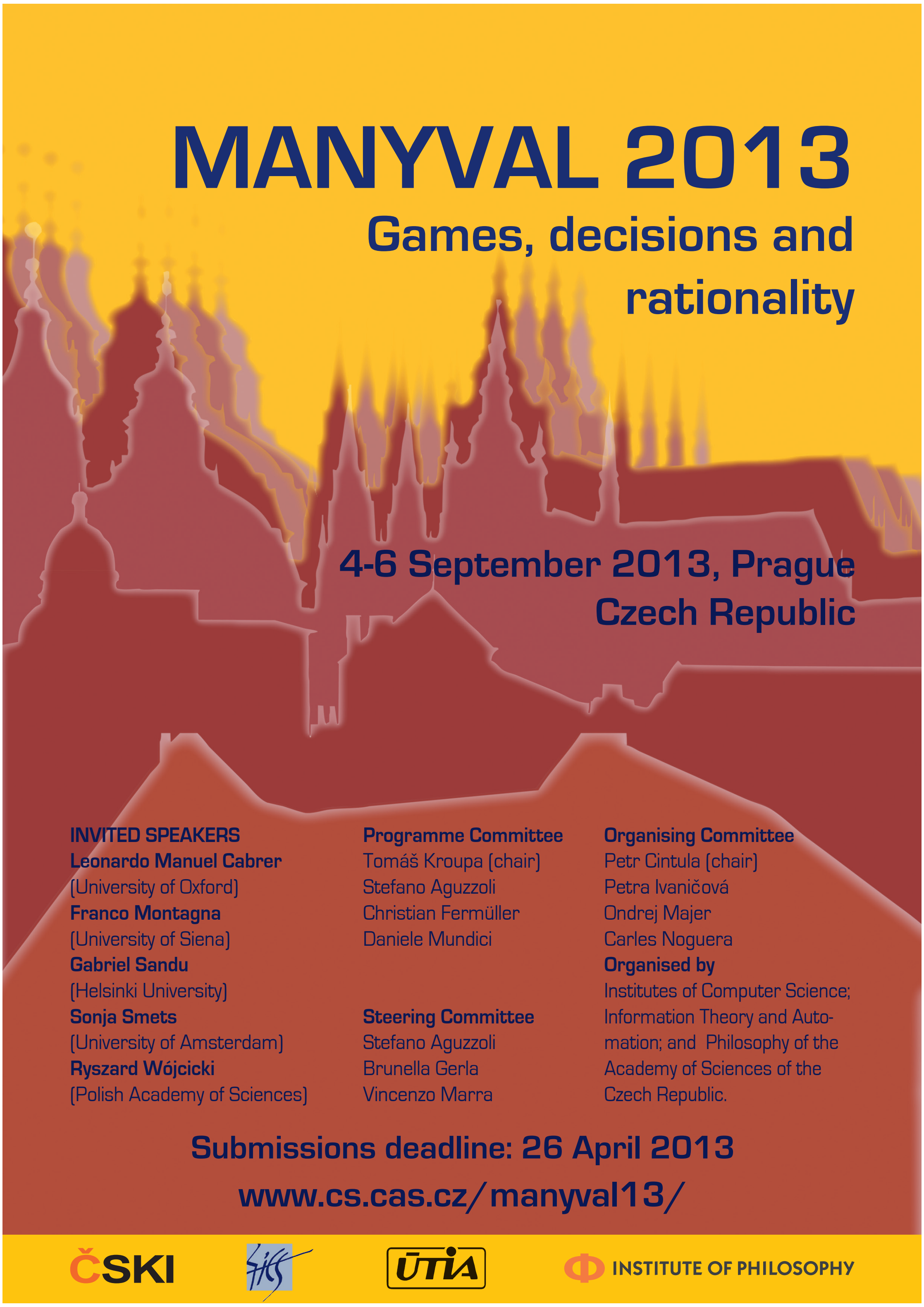
MANYVAL 2013
Games, decisions and rationality
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Czech Republic
INVITED SPEAKERS
Leonardo Manuel Cabrer
(University of Oxford)
Franco Montagna
(University of Siena)
Gabriel Sandu
(Helsinki University)
Sonja Smets
(University of Amsterdam)
Ryszard Wójcicki
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Invited talks

Strongly semisimple MV-algebras and tangents

*Leonardo Manuel Cabrer**

A formula φ is provable in Łukasiewicz propositional logic $Ł_\infty$ iff it is a tautology in $[0,1]$. More generally, a classical result by Hay [3] and Wójcicki [4] states that for every *finite* set of formulas Φ, a formula ϕ is a semantic consequence of Φ (any valuation satisfying all formulas of Φ also satisfies ϕ) iff ϕ is syntactic consequence Φ (there is a derivation of ϕ from Φ in $Ł_\infty$).

The semantic consequences of a (possibly infinite) set Θ of formulas in Łukasiewicz logic coincide with the syntactic consequences of Θ iff the Lindenbaum MV-algebra of Θ is *semisimple* (the intersection of its maximal ideals only contains 0).

Following Dubuc and Poveda [2], we say that an MV-algebra A is *strongly semisimple* if all its principal quotients are semisimple. If A is the Lindenbaum algebra of a set of $Ł_\infty$-formulas Θ, then the strong semisimplicity of A means that semantic consequences and the syntactic consequences $\Theta \cup \{\theta\}$ coincide for each formula θ.

Busaniche and Mundici [1] characterize 2-generated strongly semisimple MV-algebras using Bouligand-Severi tangents. In this paper we will present a description of strongly semisimple MV-algebras. Our result depends on a generalisation of the notion of Bouligand-Severi tangent which is reminiscent of k-dimensional tangents of k-dimensional manifolds in $\mathbb{R}^n$. Our tangents consist of tuples $(u_1, \ldots, u_k)$ of orthogonal unit vectors in $\mathbb{R}^n$. The two main differences between k-dimensional tangents and our tangents are: (a) our tangents are defined for arbitrary compact sets, and (b) the listing order of the vectors u_i is crucial.

References

[1] M. Busaniche, D. Mundici, Bouligand-Severi tangents in MV-algebras, Rev. Mat. Iberoam. (to appear, arXiv:1204.2147).

[2] E. Dubuc, Y. Poveda, Representation theory of MV-algebras, Annals of Pure and Applied Logic, 161 (2010) 1024–1046.

[3] L.S. Hay, Axiomatization of the infinite-valued predicate calculus, Journal of Symbolic Logic, 28 (1963) 77-86.

[4] R. Wójcicki, On matrix representations of consequence operations of Łukasiewicz sentential calculi, Zeit. math. Logik und Grund. Math. 19 (1973) 239-247.

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Non-standard states, conditional probability and strong coherence

Franco Montagna[*]

1 Why non-standard states?

There are several reasons for introducing non-standard states. Suppose, for instance, that we want to choose a natural number at random in such a way that all natural numbers have the same probability. Then, a natural choice would be to give each natural number an infinitesimal probability.

Moreover, non-standard probabilities allow us to treat conditional probability in terms of bets, according to Bruno de Finetti, in the case where the conditioning event has probability zero (see for instance Borel's example of the probability of chosing a point in the Western Hemisphere given that it belongs to the Equator). Our proposed solution is to replace zero-probabilities by infinitesimal, but positive probabilities.

Finally, non-standard probabilities allow us to treat the so called *strong coherence*. An assessment is said to be *strongly coherent* if not only it avoids sure loss, but also prevents us from bets which exclude any loss for the gambler and at the same time do not exclude the possibility of a win. An assessment of this kind cannot be considered completely rational, even if it avoids sure loss for the bookmaker.

2 The non-standard paradigm

Following Mundici, we treat many-valued events, represented as elements of an MV-algebra. Such elements can be regarded as $[0,1]$-valued random variables. MV-probabilities are in terms of *states*, that is, positive, normalized, homogeneous and additive maps from the given MV-algebra into $[0,1]$. But for our purposes, we need *hyperstates*, which are positive, normalized, additive and homogeneous maps into $[0,1]^*$, a non-standard extension of $[0,1]$. Note that hyperstates H satisfy $H(\alpha \cdot x) = \alpha \cdot H(x)$ even for non-standard α. Hence, hyperstates do not kill infinitesimals and allow us to treat bets in which non-standard betting odds, non-standard truth values and non-standard bets are allowed.

One first result is about faithful states and strong coherence. A (hyper)state H is *faithful* if $H(A) = 1$ implies that $A = 1$, that is, A is the certain event. An MV-algebra may fail to have faithful states. However, for every coherent assessment on an MV-algebra,

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there is a faithful hyperstate on it which differs from the given assessment by an infinitesimal. Moreover, an assessment which can be extended to a faithful hyperstate is strongly coherent. It follows that for every coherent assessment there is a hyperassessment which differs from it by an infinitesimal, which is strongly coherent.

3 Conditional probability

A typical example of coherent, but not strongly coherent assessment, is constituted by a bad assessment of conditional probability where the betting odd for the conditioning event is 0. For instance, let, in Borel's example, Eq be the event: the chosen point belongs to the Equator, let W and Ea be the events: the point belongs to the Western Hemisphere and the point belongs to the Eastern Hemisphere, respectively. Then the assessment $Eq \mapsto 0$, $W|Eq \mapsto 0$, $Ea|Eq \mapsto 0$ avoids sure loss (if the point does not belong to the Equator, there is no loss for the bookmaker). However, betting 1 Euro on each of the three events produces no loss for the gambler, because the betting odd is 0, and possibly a win (if the point beongs to the Equator).

We introduce a new rationality criterion which is based on the concept of stable coherence: an assessment Λ of conditional probability is *stably coherent* if there is a coherent non-standard assessment Λ' such that: (a) Λ and Λ' differ by an infinitesimal, and (b) every conditioning event is assessed by Λ' to a strictly positive (possibly infinitesimal) number. The main result says:

An assessment is stably coherent if it can be extended, modulo an infinitesimal, to a faithful hyperstate H^*, where $H^*(A|B) = \frac{H^*(A \cdot B)}{H^*(B)}$.

Time permitting, we will also discuss non-standard imprecise conditional probabilities.

Nash equilibrium semantics for languages of imperfect information

*Gabriel Sandu**

I will first introduce Independence-friendly logic (IF logic), a logical system which can defines two-player win-lose games of imperfect information. The syntax of the underlying language is designed to express the players knowledge in the imperfect information games which constitutes its interpretation. For sentences which are indeterminate (neither true nor false in a given model), one can implement a suggestion by Ajtai and take the value of the sentence to be given by von Neumanns Minimax theorem. We review some of the known results and analyze two proofs of the fact that IF logic realizes all rational numbers.

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Collective learning versus informational cascades: towards a logical approach to social information flow

*Sonja Smets**

In this presentation I use ideas from (Probabilistic) Dynamic Epistemic Logic, Belief Revision Theory and Formal Learning Theory to analyse examples of both successful collective learning (the "wisdom of the crowds") and its distortions (informational cascades). I argue that the standard Bayesian analysis, though useful, is insufficient for a full understanding of these phenomena. What is typically absent from the standard Bayesian models is the agents' higher-order reasoning about other agents' minds. A full understanding of these phenomena shows that even in the ideal context of unlimited higher-order reasoning or when agents adopt a qualitative heuristic method instead of probabilistic conditioning (e.g. by simply counting their evidence), cascades can still appear and examples can be given in which individual rationality may lead to group "irrationality". This presentation is based on recent joint work in [1] and provides a first step towards investigating in more generality the logical dynamics of social information.

References

[1] A. Baltag, Z. Christoff, J.U. Hansen and S. Smets, "Logical Models of Informational Cascades" in proceedings of the workshop on "Logic across the university, Foundations and Applications", Tsinghua University, to appear in Springer LNCS/Folli series 2013.

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Logics of cognitive strategies: Referentiality vs. many-valuedeness

*Ryszard Wójcicki**

1. 0-order referential languages. In the sense I am going to use this term, a standard ***0-order referential language*** is a language, $\mathbb{L}$, such that the following two conditions are satisfied

- The *algebra of formulas* of $\mathbb{L}$ consists of an infinite list of elementary formulas, and the connectives by means of which all the remaining formulas of $\mathbb{L}$ are made up.
- The semantic interpretation of $\mathbb{L}$ is provided by a *referential matrix*, $\mathbb{M}$ (see below) of the same similarity type as the *algebra of formulas* of $\mathbb{L}$.

The languages which I've selected from languages of the described kind is the language, $\mathbb{E}$, whose: (1) algebra of formulas is

$$\mathbb{E} = (E, \neg, \wedge, \vee, \rightarrow, \sim, \Box, \Diamond)$$

(2) E is the set of elementary formulas of $\mathbb{E}$, (3) the matrix interpretation of $\mathbb{E}$, to be provided, will assure that $\neg, \wedge, \vee, \vee, \rightarrow$ are the familiar connectives of the two-valued logic. The remaining three, $\sim, \Box, \Diamond$, are unary connectives will serve me to illustrate some points I am going to dicuss.

2. Referential matrices. To begin with a ***referential algebra*** $\mathbb{A}$ ***similar to*** $\mathbb{E}$ *i*s structure of the form

$$\mathbb{A} = (\mathbf{A}, \neg, \wedge, \vee, \rightarrow, \sim, \Box, \Diamond)$$

such that the following conditions are satisfied:

- All elements in the set **A** are sequences, $e = e_1, e_2, e_3, \ldots, e_i, \ldots$ of the same, either finite or infinite, length whose all elements are either 1 or 0. Under one of many possible intuitive interpretations, elements of *e* may be viewed as composed of outcomes of successive repetitions of the same experiment *e* with 1 being the outcome confirming and 0 disconfirming the verified hypothesis.
- The algebra $\mathbb{A}$ is the same similarity type as $\mathbb{E}$, i.e. each of the operations $\neg, \wedge, \vee, \rightarrow, \sim, \Box, \Diamond$ is of the same number of places as the corresponding to it connective of $\mathbb{E}$. Needless to say that even though the connectives of $\mathbb{E}$ and the operations of $\mathbb{A}$ are denoted by the same symbol, they stand for different things.
- The set of all elements of $\mathbb{A}$, is the least superset of **A** which is closed under the operations of the algebra.

A ***referential matrix*** [Wójcicki 1979] for the language $\mathbb{E}$ is a couple ($\mathbb{R}$,{**1,1,1…**}}, composed of a *referential algebra* $\mathbb{A}$ *similar to* $\mathbb{E}$ and {**1,1,1…**} being its only ***designated element***.

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NOTE 1. Multi-valued logic in the sense going back to J. Łukasiewicz are logics which are definable by means of matrices whose elements are deprived of any inner structure. The way in which referential matrices are related to many valued ones is fairly obvious. Thus e.g., the simplest one of all referential matrices is a two-element matrix ($\mathbb{A}$, **D**), i.e., one whose all elements are one-element sequences, i.e. either {1} or {0}. This matrix is the familiar matrix of two-valued classical propositional logics. A more complex is a matrix whose elements are: . <1,1>, <1,0>, <0,1>, and <0.0>, with <1,1> being the designated element..

3. Epistemic matrices. A matrix ($\mathbb{A}$,**D**) will be said to be an ***epistemic matrix*** for $\mathbb{E}$ iff the following conditions are satisfied:

- $\mathbb{A}$ is a denumerably infinite referential algebra similar to $\mathbb{E}$;
- For every e, e' in $\mathbb{A}$, $\neg e = \neg e_1, \neg e_2, \ldots$, ; $e \wedge e' = e_1 \wedge e'_1, e_2 \wedge e'_2, \ldots$, etc*;* in the discussed context $\neg, \wedge, \vee, \rightarrow$ being familiar Boolean operations on 0 and 1;
- **D** is a subset of the set, $\mathbf{D}^*$, of all elements e of $\mathbb{A}$, such that
  $$\lim_{n \to \infty} e_1 + e_2 + \ldots + e_n / n = 1$$
  (the sequence of quotients $e_1 + e_2 + \ldots + e_3 / n$ is divergent to 1).

By a ***standard epistemic matrix*** I shall mean an epistemic matrix ($\mathbb{A}$,**D**) such that **D** is the set of elements of $\mathbb{A}$, whose "almost all" (all but finely many) elements are 1. One easily see that $\mathbf{D} \subseteq \mathbf{D}^*$, as the provided definition requires.

4. Matrix interpretations. A matrix interpretation of $\mathbb{E}$ in a standard epistemic matrix ($\mathbb{A}$,**D**) for $\mathbb{E}$ (actually in any matrix ($\mathbb{A}$,**D**) such that $\mathbb{A}$, and $\mathbb{R}$, are of the same similarity type) is provided by the class of all ***valuations*** of formulas of $\mathbb{E}$ in ($\mathbb{A}$,**D**), i.e. of all homomorphisms from $\mathbb{E}$ into $\mathbb{A}$. As usual:

1. A formula α in $\mathbb{E}$ is said to be to be ***logically valid***, in symbols $\models \alpha$, iff $h(\alpha) \in \mathbf{D}$, for all valuations h;
2. is said to ***logically follow*** from $\beta_1, \beta_2, \ldots, \beta_n$, in symbols $\beta_1, \beta_2, \ldots, \beta_n \models \alpha$, iff $h(\alpha) \in \mathbf{D}$ whenever $h(\beta_i) \in \mathbf{D}$, for all I = 1,2, …, n.

One may easily verify that under the above described matrix interpretation the logic of the language $\mathbb{E}|\{\neg, \wedge, \vee, \rightarrow\}$ (the reduction of $\mathbb{E}$ to $\{\neg, \wedge, \vee, \rightarrow\}$ is the familiar two-valued classical logic.

NOTE 2. It is customary to define a ***logic*** to be the set of all logically valid formulas. In fact, logic is a tool for s deducing conclusions for available premises rather a set of formulas. In case of classical logic, either its variant is definable in terms of the other. But it need not be so in case of any logic. Moreover, even if its formulaic and deductive variants are interdefinable, they still may display some essentially different properties. Thus e.g. while one may define in the intuitionistic logic new connectives in term of old one in such a way that that the logic corresponding to them will be classical logic in the formulaic sense of the word [Gödel 1932], one cannot do this [Wójcicki 1970] if the intended result is to have classical logic in the deductive sense of the word.

5. Modal logics. Various "classical" modal logics, such as e.g. Feys-von Wright's logic **T**, Lewis' logics $\mathbf{S_4}$, $\mathbf{S_5}$, are "self-extensional" (see Wójcicki [1979]) and as such have a referential matrix interpretation. The epistemic matrix representations extend considerably the possibility of providing some intuitive interpretations of non-classical connectives. Thus e.g,, one may set an e-referential interpretation of the necessity connective $\Box$ by postulating that its algebraic counterpart is defined as follows:

$$\Box(e) = 1,1,\ldots,1,\ldots \text{ whenever } e \in \mathbf{D}, \text{ or } \Box(e) = 0,0,\ldots,0,\ldots \text{ otherwise}$$

With □ being defined, one may follow the familiar policy and put ◊ **=** ¬□¬.

6. "Constructive" negation. The idea of constructive truth is a typically formal idea. It is not clear what it may mean when truth is mean to be factual truth (one that can be established by empirical investigations. Anyway, roughly speaking, if for some formula α it is not possible to demonstrate that it is true (logically valid?) than one has to accept as valid its negation. The below idea is not intended to initiate forming another logic in the enormous abundance of already exiting, but merely to demonstrate capacities of epistemic semantic. Postulate:

$$\sim(e) = 1,1,\ldots,1,\ldots \text{ whenever } e \notin \mathbf{D}, \text{ or } \sim(e) = \neg(e) \text{ otherwise}$$

Check that (1) $|\neq p \vee \sim p$, (2) $|\neq p \rightarrow \sim\sim p$; but (3) $|= \sim\sim p \rightarrow p$.

7. Probability, possible world and other semantic ideas. It goes without saying that sequences $e_1 + e_2 + \ldots + e_n / n$ m of the kind discussed in section 3. may diverge to any x in the interval [0,1]. That opens the possibility of combining semantic analyses which are carried out in terms of confirmation, disconfirmation, logical truth with those carried out in terms of probabilities. Both attractiveness and feasibility of this approach is debatable. The capacity of epistemic semantics seems to be enormous. For the reasons I have mentioned already it is definitely much larger than that of many-valued semantics; the latter is (which, incidentally on many occasions, may be its advantage) a simplified vdersion of the former.

Annotated bibliography. To be provided.

Contributed talks

Involutive left-continuous t-norms arising from completion of MV-chains

Stefano Aguzzoli[*] *Anna Rita Ferraioli*[†] *Brunella Gerla*[‡]

Chang's MV-algebra is the prototypical example of a linearly ordered MV-algebra having infinitesimals. It can be defined as

$$\Gamma(\mathbb{Z}\,\mathrm{lex}\,\mathbb{Z}, (1,0))\,,$$

where $\mathbb{Z}\,\mathrm{lex}\,\mathbb{Z}$ is the abelian ℓ-group obtained as the lexicographic product of two copies of the ℓ-group $\mathbb{Z}$ of the integer numbers, and Γ is Mundici's functor, which implements a categorical equivalence between abelian ℓ-groups with a distinguished strong unit and MV-algebras [1].

Notice that Chang's MV-algebra is not complete as a lattice, as clearly the set $S^- = \{(0,a) \mid a \in \mathbb{Z}\,, a \geq 0\}$ is a subset of $\Gamma(\mathbb{Z}\,\mathrm{lex}\,\mathbb{Z}, (1,0))$ having no supremum, and, analogously, the set $S^+ = \{(1,a) \mid a \in \mathbb{Z}\,, a \leq 0\}$ has no infimum. However, the lattice reduct of Chang's MV-algebra can clearly be completed by adjoining just one new point, forming both the supremum of S^- and the infimum of S^+. One drawback of this construction is that the resulting structure is no more an MV-algebra.

Consider now the MV-algebra

$$\Gamma(\mathbb{Z}\,\mathrm{lex}\,\mathbb{R}, (1,0))\,.$$

Trivially, $\Gamma(\mathbb{Z}\,\mathrm{lex}\,\mathbb{Z}, (1,0))$ is a subalgebra of $\Gamma(\mathbb{Z}\,\mathrm{lex}\,\mathbb{R}, (1,0))$. Moreover, we can represent $\Gamma(\mathbb{Z}\,\mathrm{lex}\,\mathbb{R}, (1,0))$ isomorphically as an MV-algebra $\mathcal{A}$ over the universe $[0,1] \setminus \{1/2\}$, which is clearly not a subalgebra of the standard MV-algebra $[0,1]$, nor it is complete as a lattice. We can then embed $\mathcal{A}$ in a standard IMTL-algebra $\mathcal{B}$ (*i.e.*, one with universe $[0,1]$), which is a lattice completion of $[0,1] \setminus \{1/2\}$. Actually it turns out that $\mathcal{B}$ is the left-continuous, but not continuous t-norm defined as the Jenei's (connected) rotation [5] of the product t-norm [6, Example 4].

This fact throws light on the close relationship between the algebras in the variety generated by Chang's MV-algebra, and the algebras in the variety generated by the product t-norm, and hence with cancellative hoops. Moreover, it provides us with an interesting family of left-continuous involutive t-norms, arising as lattice completion of some MV-chains.

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For what concerns the relationship with the product t-norm, we discuss different characterisations of the free algebras in the variety $\mathbb{DLMV}$ generated by Chang's MV-algebra (the name reflects the fact that this variety is axiomatised by adding to MV-algebra equations the Di Nola–Lettieri axiom $2(x^2) = (2x)^2$, see [3, 2]). In particular we consider the well known representation by means of weak Boolean products of disconnected rotations of the free cancellative hoop (cfr. [2, 4, 7]), and a more concrete representation by means of a class of continuous functions (w.r.t. the usual Euclidean topology) from a power of $[0, 1] \setminus \{1/2\}$ to $[0, 1] \setminus \{1/2\}$.

We then consider the variety $\mathbb{J\Pi}$ of IMTL-algebras generated by Jenei's (connected) rotation of product t-norm. We introduce characterisations of the free algebras in this variety both as weak Boolean products and by a class of functions from a power of $[0, 1]$ to $[0, 1]$. We show that functions in the free algebras of $\mathbb{DLMV}$ are obtained by restricting the domain of the functions of free $\mathbb{J\Pi}$-algebras to the set of points having no $1/2$ components.

For what regards the t-norms arising as completion of MV-chains, let us fix some notation: we set

$$\begin{aligned} S_n^\omega &= \Gamma(\mathbb{Z} \operatorname{lex} \mathbb{Z}, (n-1, 0)) , \\ S_n^\mathfrak{c} &= \Gamma(\mathbb{Z} \operatorname{lex} \mathbb{R}, (n-1, 0)) , \\ S_n &= \Gamma(\mathbb{Z}, n-1) . \end{aligned}$$

Note that each S_n is MV-isomorphic to the n-element standard Łukasiewicz chain $L_n = \left\{0, \frac{1}{n-1}, \frac{2}{n-1} \dots, \frac{n-2}{n-1}, 1\right\}$. Analogously, for each integer $n > 1$, we can find an MV-chain $L_n^\mathfrak{c}$ with universe $[0, 1] \setminus (L_{n+1} \setminus \{0, 1\})$ such that

$$S_n^\omega \subseteq S_n^\mathfrak{c} \cong L_n^\mathfrak{c} .$$

Note $L_n^\mathfrak{c}$ is not complete, but it can be completed by adding a finite number of new points.

We can then define for each integer $n > 1$ an involutive left-continuous t-norms $\odot_n^*$ defined, for every $x, y \in [0, 1]$, by:

$$x \odot_n^* y = \begin{cases} x \odot_n^\mathfrak{c} y & \text{if } x, y \notin L_{n+1} \\ x \odot_{n+1} y & \text{if } x, y \in L_{n+1} \\ x \odot_{n+1} \lceil y \rceil_{n+1} & \text{if } x \in L_{n+1}, y \notin L_{n+1} \\ \lceil x \rceil_{n+1} \odot_{n+1} y & \text{if } x \notin L_{n+1}, y \in L_{n+1} \end{cases}$$

where $\odot_n^\mathfrak{c}$ is the monoidal conjunction of $L_n^\mathfrak{c}$, $\odot_{n+1}$ is the monoidal conjunction of L_{n+1} and for each $x \in [0, 1]$, $\lceil x \rceil_{n+1}$ is the smallest element of L_{n+1} greater or equal to x.

It turns out that $L_n^\mathfrak{c}$ is an IMTL-subalgebra of the standard algebra $([0, 1], \odot_n^*, \rightarrow_n^*, 0)$ (where $\rightarrow_n^*$ is the residuum of $\odot_n^*$).

We can hence investigate the properties of the subvarieties of IMTL-algebras generated by the t-norms $\odot_n^*$, and relate them to the corresponding MV-algebras. In particular we note that, for each $n > 1$, the MV-algebras in the variety generated by $([0, 1], \odot_n^*, \rightarrow_n^*, 0)$ are exactly those in the subvariety of MV-algebras generated by $\{S_n^\omega, S_{n+1}\}$.

Some consequences and further generalisations of this construction are explored.

References

[1] R. Cignoli, I.M.L. D'Ottaviano, D. Mundici, *Algebraic Foundations of many-valued Reasoning*, Trends in Logic, Vol. 7, Kluwer Academic Publishers, Dordrecht, 2000.

[2] R. Cignoli, A. Torrens, *Free Algebras in Varieties of Glivenko MTL-algebras Satisfying the Equation* $2(x^2) = (2x)^2$. Studia Logica 83: 157-181, 2006.

[3] A. Di Nola, A. Lettieri, *Equational characterization of all varieties of MV-algebras*, J. Algebra 221: 463–474, 1999.

[4] F. Esteva, J. Gispert, C. Noguera, *Perfect and bipartite IMTL-algebras and disconnected rotations of prelinear semihoops*, Archive for Mathematical Logic 44: 869-886, 2005.

[5] S. Jenei, *On the structure of rotation-invariant semigroups.* Arch. Math. Log. 42: 489-514, 2003.

[6] S. Jenei, F. Montagna, *A general method for constructing left-continuous t-norms.* Fuzzy Sets and Systems 136: 263-282, 2003.

[7] C. Noguera, *Algebraic study of axiomatic extensions of triangular norm based fuzzy logics*, Monografies de l'Institut d'Investigació en Intelligncia Artificial vol. 27, Barcelona, 2007.

A proof theoretical approach to standard completeness

Paolo Baldi[*]

Standard completeness, that is completeness of a logic with respect to algebras based on the real interval $[0, 1]$, is one of the major issues in Mathematical Fuzzy logic. Checking or discovering whether a logic is standard complete is sometimes a challenging task which often deserves a paper on its own, see e.g., [6, 7, 12, 4, 9]. The usual approach to the problem is algebraic and consists of the following steps. Let L be a logic described in a Hilbert-style system.

1. The algebraic semantics of the logic are identified (L-algebras).
2. The completeness of the logic is shown w.r.t countable L-chain (linearly ordered L-algebra).
3. It is shown that any countable L-chain can be embedded into a countable *dense* L-chain by adding countably many new elements to the algebra and extending the operations appropriately. This establishes *rational completeness*: a formula is derivable in L iff it is valid in all countable dense L-chains.
4. Finally, a countable dense L-chain is embedded into a standard L-algebra, that is an L-algebra with lattice reduct $[0, 1]$, using a Dedekind-MacNeille-style completion.

The crucial step 3. above (rational completeness) is the most difficult to establish, as it relies on finding the right embedding, if any. A different approach to step 3. was introduced in [10] by using proof-theoretic techniques. The idea there is that the admissibility in a logic L of a particular syntactic rule (called *density*) can lead to a proof of rational completeness for L.

Introduced by Takeuti and Titani [13], the density rule formalized Hilbert-style has the following form

$$\frac{(A \to p) \vee (p \to B) \vee C}{(A \to B) \vee C}$$

where p is a propositional variable not occurring in A, B, or C. Ignoring C, this can be read contrapositively as saying (very roughly) "if $A > B$, then $A > p$ and $p > B$ for some p"; hence the name "density" and the intuitive connection with rational completeness.

The proof-theoretic method was used in [10] to establish standard completeness for various logics for which algebraic techniques do not appear to work. In this approach, to establish standard completeness for a logic L we need to:

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(a) define a suitable proof system PS_L for L extended with the density rule

(b) check that this rule is eliminable (or admissible) in PS_L, i.e. that density does not enlarge the set of provable formulas

(c) standard completeness may then be obtained in many cases (but not in general) by means of the Dedekind-MacNeille completion.

Convenient proof system for fuzzy logic are based on *hypersequents*, that are simple generalizations of Gentzen sequents whose basic objects of inference are “disjunctions" of sequents, see [1, 11] for an overview. Step (b) above (density-elimination) was established in [2, 10] for various hypersequent calculi. These proofs are however calculi-specific and use heavy combinatorial arguments, in close analogy with Gentzen style cut-elimination proofs. A different method to eliminate applications of the density rule from derivations was introduced in [5] and called *density elimination by substitutions*. In this approach, inspired to normalization for natural deduction systems, applications of the density rule are removed by making suitable substitutions for the newly introduced propositional variables. In this talk, we will show some generalization of the method of density elimination by substitutions, extending the results in [5]. We systematically investigate which classes of sequent and hypersequent structural rules allow for density elimination [3]. In particular, we will focus on axiomatic extensions of UL (*uninorm logic*), whose corresponding hypersequent calculi lack the weakening rule, but may contain other structural rules, such as *contraction*, *mingle*, *n-contraction* (see e.g. [11]). Density elimination will be proved for classes of hypersequent calculi, thus leading to standard completeness for the formalized logics. An interesting feature of this proof theoretic approach is that standard completeness can be achieved in a completely automated way. This means that, besides subsuming existing results the method allows for the automated discovery of new fuzzy logics.

References

[1] A. Avron. The method of hypersequents in the proof theory of propositional non-classical logics. In W. Hodges et al. editors, *Logic: from foundations to applications. Proc. Logic Colloquium, Keele, UK, 1993*, pages 1–32, 1996.

[2] M. Baaz, R. Zach. Hypersequents and the proof theory of intuitionistic fuzzy logic. *Proc. of Computer Science Logic (CSL'2000)*. LNCS 1862, Springer, pages 187–201. 2000.

[3] P. Baldi, A. Ciabattoni, L. Spendier. Standard completeness for extensions of MTL: an automated approach. *Proceedings of WOLLIC 2012* . LNCS 7456, pages 154–167, 2012.

[4] A. Ciabattoni and F. Esteva and L. Godo. T-norm based logics with n-contraction. *Neural Network World* 12(5):441–453, 2002.

[5] A. Ciabattoni and G. Metcalfe. Density elimination. *Theor. Comput. Sci.* 403(2-3):328–346, 2008.

[6] F. Esteva and L. Godo. Monoidal t-norm based logic: towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems* 124:271–288, 2001.

[7] S. Jenei and F. Montagna. A proof of standard completeness for Esteva and Godo's MTL logic, *Studia Logica* 70(2):183–192, 2002.

[8] P. Hájek. *Methamatematics of Fuzzy Logic*, Kluwer. 1998.

[9] R. Horčik. Alternative Proof of Standard Completeness Theorem for MTL, *Soft Computing* 11(2): 123–129, 2006.

[10] G. Metcalfe and F. Montagna. Substructural fuzzy logics. *Journal of Symbolic Logic* 7(3):834–864, 2007.

[11] G. Metcalfe and N. Olivetti and D. Gabbay. *Proof Theory for Fuzzy Logics*, Springer Series in Applied Logic Vol. 36, 2008.

[12] F. Montagna and H. Ono. Kripke semantics, undecidability and standard completeness for Esteva and Godo's logic MTL∀. *Studia Logica* 71(2):227–245, 2002.

[13] G. Takeuti and T. Titani. Intuitionistic fuzzy logic and intuitionistic fuzzy set theory. *Journal of Symbolic Logic* 49(3):851–866, 1984.

A temporal semantics for Nilpotent Minimum logic

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Abstract

Nilpotent Minimum logic NM was introduced in [EG01] as the logical calculi associated to Nilpotent Minimum t-norm [Fod95]. In this talk we present a very simple and natural temporal like semantics for NM ([Fod95, EG01]), in which the logic of every instant is given by three-valued Łukasiewicz logic $Ł_3$: a completeness theorem will be shown. This is the prosecution of the work initiated in [AGM08] and [ABM09], in which the authors construct a temporal semantics for the many-valued logics of Gödel ([God32], [Dum59]) and Basic Logic ([Haj98]).

Extended Abstract

In recent years there has been a development for alternative semantics, for many-valued logics: some examples are given by dialogue game semantics for Łukasiewicz and Gödel logics (see [Fer08] for an overview), evaluation games for first-order Łukasiewicz logic ([CM09]), and temporal like semantics for Gödel logic and BL ([AGM08, ABM09]). Every semantics (algebraic, game-theoretic, temporal like...) has its peculiarities and allows to show different aspects and peculiarities of the logical counterpart.

In this talk we focus our attention on Nilpotent Minimum logic (NM): this logic was introduced in [EG01] as the logical system associated to the variety of algebras generated by $[0,1]_{\mathrm{NM}}$, an algebraic structure induced by Nilpotent Minimum t-norm ([Fod95]).

However, the algebraic semantics is only a possible candidate for this logic. Here we present a temporal like semantics, in which the formulas are evaluated over a temporal flow of time, and the logic of every instant is given by three-valued Łukasiewicz logic $Ł_3$. In particular, a temporal flow is given by any totally ordered infinite set $\langle T, \leq\rangle$: a temporal assignment (over a temporal flow $\langle T, \leq\rangle$) is a map v that associates to every formula φ and instant of time $t \in T$ a value in $\left\{0, \frac{1}{2}, 1\right\}$, satisfying one of the following conditions, for every formula φ:

- $v(\varphi, \cdot)$ is constant (to $0, \frac{1}{2}$, or 1) for every instant of time.
- There is an instant t such that $v(\varphi, t') = 1$, for every $t' \geq t$, and $v(\varphi, t'') = \frac{1}{2}$, for every $t'' < t$.
- There is an instant t such that $v(\varphi, t') = 0$, for every $t' \geq t$, and $v(\varphi, t'') = \frac{1}{2}$, for every $t'' < t$.

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As we will see these temporal assignments will be "truth-functional" over the negation connective, whilst on the implication the truth value of a formula like $\varphi \to \psi$, at the instant t, will depends also on the truth values of φ, ψ in the instants $t' \geq t$.

We will show a completeness theorem of this form:

Theorem 1 (Completeness theorem)**.** *Let $\langle T, \leq \rangle$ be a temporal flow. Then for each formula φ and finite theory Γ.*

$$\Gamma \vdash_{NM} \varphi \quad \textit{iff} \quad \Gamma \models_T \varphi.$$

All the technical details will be developed during the talk.

We conclude by pointing out that in [AM12] the authors, by using the temporal semantics for Gödel logic introduced in [AGM08], have developed a de Finetti's like coherence criterion for events (described by formulas) whose truth value varies over a temporal flow of time. Also the semantics presented in this talk could be at the base for a similar work for Nilpotent Minimum logic.

References

[ABM09] S. Aguzzoli, M. Bianchi, and V. Marra, *A temporal semantics for basic logic*, Studia Logica **92** (2009), 147–162.

[AGM08] S. Aguzzoli, B. Gerla, and V. Marra, *Embedding Gödel propositional logic into Prior's tense logic*, Proceedings of IPMU'08 (Torremolinos (Málaga)) (L. Magdalena, M. Ojeda-Aciego, and J.L. Verdegay, eds.), June 2008, pp. 992–999.

[AM12] S. Aguzzoli and V. Marra, *Betting on events observed over time: de Finetti's Dutch-Book argument for Gödel logic*, Logic, Algebra and Truth Degrees 2012, 10-14 September 2012.

[CM09] P. Cintula and O. Majer, *Towards evaluation games for fuzzy logics*, Games: Unifying Logic, Language, and Philosophy (Ondrej Majer, Ahti-Veikko Pietarinen, and Tero Tulenheimo, eds.), Logic, Epistemology, and the Unity of Science, vol. 15, Springer Netherlands, 2009, pp. 117–138.

[Dum59] M. Dummett, *A Propositional Calculus with Denumerable Matrix*, The Journal of Symbolic Logic **24** (1959), no. 2, 97–106.

[EG01] F. Esteva and L. Godo, *Monoidal t-norm based logic: Towards a logic for left-continuous t-norms*, Fuzzy sets and Systems **124** (2001), no. 3, 271–288.

[Fer08] C. G. Fermüller, *Dialogue games for many-valued logics - an overview*, Studia Logica **90** (2008), no. 1, 43–68.

[Fod95] J. Fodor, *Nilpotent minimum and related connectives for fuzzy logic*, Fuzzy Systems, 1995. International Joint Conference of the Fourth IEEE International Conference on Fuzzy Systems and The Second International Fuzzy Engineering Symposium., Proceedings of 1995 IEEE International Conference on, IEEE, 1995, pp. 2077–2082.

[God32] K. Gödel, *Zum intuitionistischen Aussagenkalkül*, Anzeiger Akademie der Wissenschaften in Wien, Mathematisch-Naturwissenschaftliche Klasse **69** (1932), 65–66, English translation in [God01].

[God01] ______, *Collected works*, paperback ed., vol. 1 Publications: 1929-1936, Oxford University Press, 2001.

[Haj98] P. Hájek, *Metamathematics of fuzzy logic*, Trends in Logic, vol. 4, Kluwer Academic Publishers, 1998.

On logics of formal inconsistency and fuzzy logics

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Paraconsistency is the study of logics (as deductive systems) having a negation operator $\neg$ such that not every contradiction $\{\varphi, \neg\varphi\}$ trivializes or explodes. In other words, a paraconsistent logic is a logic having at least a contradictory, non-trivial theory.

Among the plethora of paraconsistent logics proposed in the literature, the so-called *Logics of Formal Inconsistency* (LFIs), proposed in [3] (see also [2]), play an important role, since they internalize in the object language the very notions of consistency and inconsistency by means of specific connectives (either primitive or not). This generalizes the strategy of da Costa, which introduced in [5] the well-known hierarchy of systems C_n, for $n > 0$. Besides being able to distinguish between contradiction and inconsistency, on the one hand, and non-contradiction and consistency, on the other, LFIs are non-explosive logics, that is, paraconsistent: in general, a contradiction does not entail arbitrary statements, and so the Principle of Explosion $\varphi, \neg\varphi \vdash \psi$ does not hold. However, LFIs are *gently explosive*, in the sense that, adjoining the additional requirement of consistency, then contradictoriness does cause explosion: $\bigcirc(\varphi), \varphi, \neg\varphi \vdash \psi$ for every φ and ψ. Here, $\bigcirc(\varphi)$ denotes the consistency of φ. The general definition of LFIs we will adopt here, slightly modified from the original one proposed in [3] and [2], is the following:

Definition 1. Let $(\mathrm{L}, \vdash)$ be a logic defined in a language $\mathcal{L}$ containing a negation $\neg$, and let $\bigcirc(p)$ be a nonempty set of formulas of $\mathcal{L}$ depending exactly on the propositional variable p. Then L is an LFI (with respect to $\neg$ and $\bigcirc(p)$) if the following holds (here, $\bigcirc(\varphi) = \{\psi[p/\varphi] \ : \ \psi(p) \in \bigcirc(p)\}$):

(i) $\varphi, \neg\varphi \nvdash \psi$ for some φ and ψ, i.e. the logic is not explosive;

(ii) $\bigcirc(\varphi), \varphi \nvdash \psi$ for some φ and ψ;

(iii) $\bigcirc(\varphi), \neg\varphi \nvdash \psi$ for some φ and ψ; and

(iv) $\bigcirc(\varphi), \varphi, \neg\varphi \vdash \psi$ for every φ and ψ.

In many situations $\bigcirc(\varphi)$ is a singleton, whose element will we denoted by $\circ\varphi$, and $\circ$ is called a *consistency* operator in L with respect to $\neg$. It has to be noticed that in the frame of LFIs, the term *consistent* rather refers to formulas that basically exhibit a classical logic behaviour, so in particular an explosive behaviour. Such a consistency operator can be primitive (as in the case of most of the systems treated in [3] and [2]) or, on the contrary, it can be defined in terms of the other connectives of the language. For instance, in the

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well-known system C_1 by da Costa, consistency is defined by the formula $\circ\varphi = \neg(\varphi \wedge \neg\varphi)$ (see [5]).

Systems of mathematical fuzzy logic, understood as truth-preserving many-valued logics in the sense of [7, 4], are not paraconsistent. Indeed, in these systems, $\varphi \& \neg\varphi$ is always evaluated to 0, and hence any formula can be deduced from the set of premises $\{\varphi, \neg\varphi\}$. However, the situation is different if one considers, for each truth-preserving logic L, its companion $L^{\leq}$ that preserves degrees of truth as studied in [1]. In fact, in these systems $L^{\leq}$, a formula φ follows from a (finite) set of premises Γ when, for all evaluations e on a corresponding class of L-chains, $e(\varphi) \geq \min\{e(\psi) \mid \psi \in \Gamma\}$. Obviously, if L is not pseudo-complemented, there is always some evaluation e such that $e(\varphi \wedge \neg\varphi) > 0$. This says that $\{\varphi, \neg\varphi\}$ is not explosive in $L^{\leq}$ and thus, there are fuzzy logics preserving degrees of truth that are paraconsistent (see [6] for a preliminary study).

In this paper, given an axiomatic extension L of MTL that is not SMTL, we first study natural conditions a consistency operator $\circ$ on L-chains has to satisfy. These conditions are used then to define both a semilinear truth-preserving logic $L_\circ$, over the language of L expanded with a new unary connective $\circ$, as well as its paraconsistent companion $L_\circ^{\leq}$. Finally we consider several extensions of $L_\circ^{\leq}$, capturing several further properties one can ask to the consistency operator $\circ$. For instance, we introduce the logics $(L_\circ^{\neg\neg})^{\leq}$ (where the negation in the chains of the quasi-variety of L-algebras satisfies the condition $\neg\neg x = 1$ iff $x = 1$), the logic $(L_\circ^{c})^{\leq}$ (where the operator $\circ$ is Boolean) and the logics $(L_\circ^{\min})^{\leq}$ or $(L_\circ^{\max})^{\leq}$ (where the consistency operators are the minimum and the maximum ones respectively).

Finally we study in the above logics the problem of recovering the classical reasoning by means of the consistency connective $\circ$, a very desirable property in the context of LFIs (see [2]), called DAT (Derivability Adjustment Theorem). When the operator $\circ$ enjoys a suitable propagation property in the logic L with respect to the classical connectives, then the DAT in $L_\circ^{\leq}$ assumes the following simplified form: for every finite set of formulas $\Gamma \cup \{\varphi\}$ in the language of classical propositional logic (**CPL**),

$$\text{(PDAT)} \quad \Gamma \vdash_{\mathbf{CPL}} \varphi \;\text{ iff }\; \{\circ p_1, \ldots, \circ p_n\} \cup \Gamma \vdash_{L_\circ^{\leq}} \varphi$$

where $\{p_1, \ldots, p_n\}$ is the set of propositional variables occurring in $\Gamma \cup \{\varphi\}$.

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References

[1] F. Bou, F. Esteva, J.M. Font, A. Gil, L. Godo, A. Torrens, and V. Verdú. Logics preserving degrees of truth from varieties of residuated lattices. *Journal of Logic and Computation*, 19(6):1031–1069, 2009.

[2] W.A. Carnielli, M.E. Coniglio, and J. Marcos. Logics of Formal Inconsistency. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic (2nd. edition)*, volume 14, pages 1–93. Springer, 2007.

[3] W.A. Carnielli and J. Marcos. A taxonomy of **C**-systems. In W.A. Carnielli, M.E. Coniglio, and I.M.L. D'Ottaviano, editors, *Paraconsistency - the Logical Way to the Inconsistent*, volume 228 of *Lecture Notes in Pure and Applied Mathematics*, pages 1–94, New York, 2002. Marcel Dekker.

[4] P. Cintula, P. Hájek, and C. Noguera, editors. *Handbook of Mathematical Fuzzy Logic (in 2 volumes)*, volume 37, 38 of *Studies in Logic, Mathematical Logic and Foundations.* College Publications, London, 2011.

[5] N.C.A da Costa. Sistemas formais inconsistentes (Inconsistent formal systems, in Portuguese). Habilitation thesis, Universidade Federal do Paraná, Curitiba, Brazil, 1963. Republished by Editora UFPR, Curitiba, Brazil, 1993.

[6] R. Ertola, F. Esteva, T. Flaminio, L. Godo, and C. Noguera. Exploring paraconsistency in degree-preserving fuzzy logics. To appear in Proc. of EUSFLAT 2013, 2013.

[7] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic.* Kluwer, Dordrecht, 1998.

Lexicographic MV-algebras through a generalization of the Di Nola-Lettieri functors

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An *MV-algebra* is a structure $(A, \oplus, ^*, 0)$, where $(A, \oplus, 0)$ is an abelian monoid and the following identities hold for all $x, y \in A$: $(x^*)^* = x$, $0^* \oplus x = 0^*$, $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$. We refer to [1] for all the unexplained notions concerning the theory of MV-algebras. MV-algebras are categorically equivalent with abelian lattice-ordered groups with a strong unit (henceforth called abelian ℓu-groups). This categorical equivalence is due to Mundici [6] and the corresponding functor is denoted by Γ. As a consequence, for any MV-algebra A there exists an ℓu-group (G, u) such that $A \simeq \Gamma(G, u)$.

The subclass of *perfect MV-algebras* was introduced and studied by Di Nola and Lettieri in [3]. They proved that $\Delta(G) = \Gamma(\mathbb{Z} \times_{lex} G, (1, 0))$ is a perfect MV-algebra for any abelian ℓ-group G, where $\times_{lex}$ is the usual lexicographic product of groups. Moreover, the above construction gives rise to a categorical equivalence between the category $\mathcal{P}$ of perfect MV-algebras and the category $\mathcal{G}$ of abelian ℓ-groups. The functors $\Delta : \mathcal{G} \to \mathcal{P}$ and $D : \mathcal{P} \to \mathcal{G}$ establishing the equivalence are called *the Di Nola-Lettieri functors*.

In this paper we study the class of MV-algebras which is obtained by replacing, in the definition of the functor Δ, the ℓu-group $(\mathbb{Z}, 1)$ by an arbitrary ℓu-group (H, u) and we aim to prove a generalization of the Di Nola-Lettieri categorical equivalence.

We say that an MV-algebra M is *lexicographic* if

$$M \simeq \Gamma(H \times_{lex} G, (u, 0)),$$

where (H, u) is an abelian ℓu-group and G is an abelian ℓ-group.

There are two important examples of lexicographic MV-algebras:

- *Perfect MV-algebras* [3], which are obtained by taking the abelian ℓu-group (H, u) to be the group $(\mathbb{Z}, 1)$ of integers,
- *Local MV-algebras with retractive radical* [4], which are obtained by taking the abelian ℓu-group (H, u) to be a, ℓu-subgroup of $(\mathbb{R}, 1)$.

The lexicographic MV-algebras are characterized as follows:

Theorem 1. *An MV-algebra M is lexicographic iff there there exists a retractive ideal $I \neq \{0\}$ of M such that*

$$\epsilon \leq x \leq \epsilon^*, \text{ for any } \epsilon \in I \text{ and } x \in M \setminus (I \cup I^*), \text{ where } I^* = \{x^* | x \in I\}.$$

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Let A be an arbitrary, but fixed, MV-algebra. If M is a lexicographic MV-algebra such that $M/I \simeq A$ where $I \subseteq M$ is the corresponding retractive ideal, then M contains a subalgebra that is isomorphic with A. We denote by $\iota_M : A \to M$ the embedding of A in M. We can now define the category $Lex\mathcal{MV}_A$ whose objects are the lexicographic MV-algebras as above and whose morphisms are MV-algebra morphisms $f : M_1 \to M_2$ such that $f(\iota_{M_1}(a)) = \iota_{M_2}(a)$ for any $a \in A$. The following result generalizes the Di Nola-Lettieri equivalence:

Theorem 2. *The categories $Lex\mathcal{MV}_A$ and $\mathcal{G}$ are equivalent.*

We also study the theory of states defined on lexicographic MV-algebras. *States* on MV-algebras were introduced by Mundici in [7] as $[0,1]$-valued, additive and normalized mappings. The categorical equivalence between MV-algebras and abelian ℓu-groups has a counterpart from the point of view of states. Indeed, following [5], a state on an abelian ℓu-group (H, u) is a positive homomorphism h from (H, u) into the additive group of reals $(\mathbb{R}, 1)$ such that $h(u) = 1$, and the following holds:

Theorem 3. [7] *If $A = \Gamma(H, u)$ then any state on A can be uniquely extended to a state on (H, u). The states on A are in bijective correspondence with the states on (H, u).*

Following Theorem 2, we introduce *the lexicographic states* defined on lexicographic MV-algebras and we prove an analogue of Theorem 3. If M is a lexicographic MV-algebra then a *lexicographic state* is a hyperreal-valued additive and normalized functions, whose codomain is isomorphic with $\Delta(\mathbb{R})$.

For an abelian ℓ-group G we call *state* any positive morphism of groups from G to $\mathbb{R}$. Our main theorem is the following:

Theorem 4. *Assume (H, u) is an abelian ℓu-group, G is an abelian ℓ-group and $A = \Gamma(H, u)$ and $M = \Gamma(H \times_{lex} G, (u, 0))$. Then any lexicographic state on M is uniquely determined by a state on A and a state on G.*

As a consequence of our definition, we get a notion of lexicographic state for the particular cases of perfect and local MV-algebras. It is worth noticing that our notion of lexicographic state, when applied to perfect MV-algebras, gives a definition of state which is analogous to that of *local state* that was already introduced by Di Nola, Georgescu and Leuştean [2], with the unique difference that the codomain of a lexicographic state on a perfect MV-algebra is the perfect MV-algebra described as $\Delta(\mathbb{R})$, while every local state in the sense of [2] ranges over the positive cone of $\mathbb{R}$. Moreover, while any perfect MV-algebra A has just one state in the sense of Mundici [7], which coincides with that unique homomorphism from A into the two-valued Boolean algebra $\{0, 1\}$, lexicographic states for perfect MV-algebras offer several non-trivial examples of monotone and normalized functions for this particular case of MV-algebras. In particular, for every perfect MV-algebra A, each positive morphism of groups from $D(A)$ into the group of reals, induces a lexicographic state on A.

References

[1] R. Cignoli, I.M.L. D'Ottaviano and D. Mundici, Algebraic Foundations of many-valued Reasoning, Kluwer, Dordrecht, 2000.

[2] A. Di Nola, G. Georgescu, I. Leuştean, States on Perfect MV-algebras. V. Novak, I. Perfilieva (eds.), *Discovering the World With Fuzzy Logic*, Studies in Fuzziness and Soft Computing 57, Physica, Heidelberg, 105–125, 2000.

[3] A. Di Nola, A. Lettieri. Perfect MV-algebras are categorically equivalent to Abelian ℓ-groups. Studia Logica, 53: 417–432, 1994.

[4] A. Di Nola, A. Lettieri. Coproduct MV-Algebras, Nonstandard Reals, and Riesz Spaces. Journal of Algebra 185(3): 605–620, 1996.

[5] K.R. Goodearl, *Partially Ordered Abelian Group with Interpolation.* AMS Math. Survey and Monographs, vol 20, 1986.

[6] D. Mundici, Interpretation of AF C*-algebras in Łukasiewicz sentential calculus, Journal of Functional Analysis, 65, 15–63, 1986.

[7] D. Mundici, Averaging the Truth-Value in Łukasiewicz Logic. Studia Logica 55: 113–127, 1995.

Games, equilibrium semantics, and many-valued connectives

*Christian G. Fermüller**

Jaakko Hintikka [7] famously introduced a game based characterization of Tarski's central semantic notion of 'truth in a model'. It can be presented as follows: A *Proponent* **P** defends the claim that a formula F is true in a given model $\mathcal{M}$ by engaging in a game against an *Opponent* **O** who aims at refuting the claim. At each stage of the game one of the two players asserts a subformula of F. The game is initiated by **P**'s assertion of F and proceeds in accordance with the following rules (here stated for **P**, but analogously for **O**):

($R_\wedge$) If **P** asserts $G \wedge H$ then **O** may choose between G and H as the formula to be asserted by **P** at the next stage.

($R_\vee$) If **P** asserts $G \vee H$ then **P**has a choice between asserting G or H at the next stage.

($R_\neg$) If **P** asserts $\neg G$ then the roles of the two players are switched and the game continues with **O** asserting G.

($R_\forall$) If **P** asserts $\forall x F(x)$ then **O** may choose any constant[1] c and **P** has to assert $F(c)$ at the next stage.

($R_\exists$) If **P** asserts $\exists x F(x)$ then **P** has to chose some constant c and as $F(c)$ at the next stage.

When the players arrive at an atomic formula A, **P** wins and **O** looses if A is true in $\mathcal{M}$; otherwise **O** wins an **P** looses. **P** has a winning strategy in this game if and only if F is true in $\mathcal{M}$ according to classical logic. A whole new branch of logic, called IF-logic (Independency Friendly logic, see [9]) arises by investigating the consequences of imperfect knowledge in the above game. Not knowing all previous choices of the other player at a stage of the game implies that in general neither **P** nor **O** has a winning strategy for the formula under consideration. In this manner IF-logic goes beyond bivalent (classical) logic and achieves a higher level of expressibility. *Equilibrium semantics* [11, 9], considers mixed strategies in the strategic game that corresponds to the extensive (imperfect information) game just presented. At least when restricting attention to finite models, one may compute a unique Nash equilibrium for any given initial formula of IF-logic. By identifying **P**'s winning of a game run with the pay off value 1 and losing with 0, a link with many-valued logic emerges: the Nash equilibrium of the game for a formula F can be interpreted as (in general intermediate) truth value of F in the given model. As shown in [11, 9] this results

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[1] We assume that there is a constant for every element of the domain of the model in question.

in a justification of standard many-valued truth functions: the values for $F \vee G$ and for $F \wedge G$ equal the minimum and maximum, respectively, of the values for F and G. Likewise, negation corresponds to $\lambda x.1 - x$, existential quantification to supremum, and universal quantification to infimum. In other words, one recovers the truth functional semantics of the so-called weak fragment of Łukasiewicz logic.

The main purpose of this contribution is to connect this result with another game based interpretation of Łukasiewicz logic that, at least at a first glimpse, seems to be incompatible with the principles of IF-logic. Already in the 1970s, Robin Giles, in an attempt to model logical reasoning in physics from constructivist point of view, introduced a game in which, like in Hintikka's game, two players systematically reduce logical complex assertions to atomic assertions. Although Giles referred to Lorenzen, rather than to Hintikka, his rules for disjunction, conjunction, and the two quantifiers are virtually identical to those stated above. An essential difference between the two games only emerges with Giles' rule for implication, which may be presented as follows.

($R_{\rightarrow}$) If **P** asserts $G \rightarrow H$ then, if **O** asserts G, **P** has to assert H.

This rule[2], if invoked by **O**, leads to *two* assertions, one of **O** and one of **P**, that have to be considered in the continuation of the game. Consequently, Giles' game calls for a more general notion of the state of the game. A state can now be specified by $[F_1, \ldots, F_n \,\|\, G_1, \ldots, G_m]$ where $\{F_1, \ldots, F_n\}$ denotes the multiset of formulas currently asserted by **O** and $\{G_1, \ldots, G_m\}$ denotes the multiset of of formulas currently asserted by **P**. Giles' procedural constraint for admissible runs of the game is quite liberal: at every stage any player can pick any occurrence of non-atomic formula asserted by the other player for attack or to declare that the chosen formula will not be attacked at all and thus is to be discarded. Once a state is reached, where all formulas are atomic, an experiment is performed for each atomic assertion to check whether it is true or false. These experiments are dispersive, i.e., if a player asserts the same atomic formula twice, it may happen that one of the assertions is evaluated as false and the other is true. Only a certain probability of a positive result is associated with each experiment. The players have to pay a unit of money to the other player for each of their atomic assertions that is evaluated as false and seek to minimize the total of money that they are *expected* to pay to the other player. In this setting, a many-valued model is given by identifying the success probability of an experiment with the truth of the corresponding atomic formula. Giles proved that a formula F evaluates to truth value v in a given many-valued model according to the standard truth functions of Łukasiewicz logic if and only if, for every $\epsilon > 0$, **P** has a strategy in the above game that limits her expected (average) loss by $1 - v + \epsilon$ units of money, while **O** has a strategy that guarantees an average gain of $1 - v - \epsilon$ units.

Note that Giles' game semantics goes beyond equilibrium semantics in also justifying the following truth function i for implication: $i(x, y) = 1 - x + y$ if $x \geq y$ and $i(x, y) = 1$ otherwise. Moreover, as shown in [3] and [4] one can extend Giles' game by a simple rule for so-called *strong conjunction* &, that is characterized by taking the Łukasiewicz t-norm $\lambda x, y. \max(0, x+y-1)$ as truth function. In other words, we obtain a game that is adequate for full (standard) first-order Łukasiewicz logic.

[2] A rule for negation arises by identifying $\neg A$ with $A \rightarrow \bot$, where $\bot$ is a constantly false statement.

To relate equilibrium semantics to the seemingly quite different set-up of Giles' game, we consider a richer language that consists of an inner level, corresponding to the syntax of IF-logic, and an outer one that employs the propositional connectives and quantifiers of full Łukasiewicz logic. The levels remain separated in the sense Łukasiewicz connectives can only be used to join pure IF-formulas to yield new formulas at the outer level. (This kind of two-tiered language is familiar from the literature on combining fuzzy logic with probability theory in the manner initiated by [6].) For this compound language we propose a matching game semantics with the following features:

- On the outer level, where the leading connective of a formula refers to Łukasiewicz logic, the game proceeds exactly as specified in [5, 3, 4]: the attacked formula F is replaced by (some of) the subformulas of F in the corresponding multiset of currently asserted formulas.
- As soon as a pure IF-formula F is attacked the game switches to a 'Hintikka-style' mode for the corresponding formula occurrence of current state. In other words, F is iteratively replaced by one of its subformulas in accordance with the rules of Hintikka's game until an atomic formulas is reached, for which the given model determines whether **P** or **O** wins this sub-game of the overall game.
- The overall pay off for the game is specified in analogy to Giles' game: at the end of each Hintikka-style sub-game the loosing player has to pay one unit of money to the other player. The players seek to minimize the total amount of money that they owe according to this arrangement.

From the point of view of Giles' analysis of reasoning under uncertainty, one may understand the compound game as a straightforward extension of Giles' original game, where IF-formulas take the place of atomic Łukasiewicz formulas. In this manner results of dispersive experiments are replaced by results of runs of a Hintikka-style game for an IF-formula. This amounts to an interpretation of intermediate truth values as equilibria in games of imperfect information that involve only ordinary bi-valued atomic predicates. In the context of mathematical fuzzy logics, of which Łukasiewicz logic is a particularly prominent example, this is a non-trivial achievement that should be compared to other attempts to justify truth values and truth functions with respect to principles that do not simply take the notion of (real-valued) graded truth for granted (cf. the overview paper [10]).

On the other hand, from the viewpoint of equilibrium semantics, new modes of combining results of individual sub-games emerge that can only be modeled in the more general framework of Giles-style games, where for each player may assert more than one formula at any given state of the game. We emphasize that we here focus on just those forms of combining games that preserve truth-functionality. This is achieved by stipulating that no information is passed between individual runs of sub-games for IF-formulas. As soon as one lifts this restriction, a plethora of non-truth-functional connectives emerge as seen in game semantics for linear logics [2] or in Japaridze's Computability Logic [8].

References

[1] A.D.C. Bennett, J.B. Paris, and A. Vencovská. A new criterion for comparing fuzzy logics for uncertain reasoning. *Journal of Logic, Language and Information*, 9(1):31–63, 2000.

[2] A. Blass. A Game Semantics for Linear Logic. *Annals of Pure and Applied Logic*, 56(1992), 183–220.

[3] C.G. Fermüller. Revisiting Giles - connecting bets, dialogue games, and fuzzy logics. In O. Majer et.al. (eds.), *Games: Unifying Logic, Language, and Philosophy*, volume 15 of *Logic, Epistemology, and the Unity of Science*, 209–227. Springer, 2009.

[4] C.G. Fermüller and G. Metcalfe. Giles's game and the proof theory of Łukasiewicz logic. *Studia Logica*, 92:27–61, 2009.

[5] R. Giles. A non-classical logic for physics. *Studia Logica*, 4(33):399–417, 1974.

[6] P. Hájek, L. Godo, and F. Esteva. Fuzzy logic and probability. Proceedings of the 11th conference on Uncertainty in Artificial Intelligence. Morgan Kaufmann Publ., 237–244, 1995.

[7] J. Hintikka. *Logic, language-games and information: Kantian themes in the philosophy of logic.* Clarendon Press Oxford, 1973.

[8] G. Japaridze. Introduction to computability logic. Annals of Pure and Applied Logic 123(1), 1–99, 2003.

[9] A.L. Mann, G. Sandu, M. Sevenster. *Independence-friendly logic: A game-theoretic approach.* Cambridge University Press, 2011.

[10] J.B. Paris. Semantics for fuzzy logic supporting truth functionality. In V. Novák and I. Perfilieva, editors, *Discovering the world with fuzzy logic*, pages 82–104. Physica-Verlag, 2000.

[11] M. Sevenster, G. Sandu. Equilibrium semantics of languages of imperfect information. Annals of Pure and Applied Logic 161(5):618–631, 2010.

A betting metaphor for belief functions on MV-algebras and fuzzy epistemic states

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Consider the following game. Two players, Bookmaker (**B**) and Gambler (**G**) agree in betting on a finite set of events described by functions $e_1, \ldots, e_n$ from a set of possible worlds $X = \{w_1, \ldots, w_k\}$ into $[0,1]$ whose realizations are unknown now and which, in the future, will be evaluated in the possible worlds of X.

The events will be evaluated by the following stipulation: the two players will share a *common epistemic state* about the whole class of possible worlds which is represented by a map $\pi : X \to [0,1]$ such that, for each $w_i \in X$, $\pi(w_i)$ represents the *feasibility degree* of w_i for both **B** and **G**. Therefore, given any epistemic state π and an event e_i, the aggregated value of e_i from π is computed by the following formula:

$$N_\pi(e_i) = \min\{\pi(w_j) \Rightarrow w_j(e_i) : j = 1, \ldots, k\}.$$

The game can hence be described by the following steps:

Stage 1 **G** fixes finitely many events $e_1, \ldots, e_n \in [0,1]^X$ and publishes her book $\beta : e_i \mapsto \beta_i$ (for $i = 1, \ldots, n$).

Stage 2 **B** chooses stakes $\sigma_1, \ldots, \sigma_n$ each for each event in the book β and pays $\sum_{i=1}^n \sigma_i \cdot \beta_i$ to **G**.

Now, assume that an *epistemic state* $\pi : X \to [0,1]$ is reached by both players as according to the rule previously described. Then the game proceeds in the following way:

Stage 3 Both players **B** and **G**, evaluate N_π of each event e_i of β in π. In other words they calculate $N_\pi(e_i)$ for each $i = 1, \ldots, n$.

Stage 4 **B** pays to **G** the amount $\sum_{i=1}^n \sigma_i \cdot N^\pi(e_i)$.

Definition 1. *According with the previous game, a book $\beta : e_i \mapsto \beta_i$ is called* B-coherent *iff there is no possible choice of stakes $\sigma_1, \ldots, \sigma_n$ ensuring* **G** *a sure win in every epistemic state π.*

In [8, 5] a generalization of belief function theory in the frame of MV-algebras has been proposed. The main idea of this approach is to define a belief function **b** over an MV-algebra of fuzzy sets $M = [0,1]^X$ (where X is a finite set of cardinality k that represents the set of possible worlds we will take in consideration) as a state [12] over a separable

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MV-subalgebra $\mathcal{R}$ of $[0,1]^M$, that strictly contains the free MV-algebra over k generators $Free(k)$ (see [4]). More precisely, we call a mapping $\mathbf{b} : M \to [0,1]$ a *generalized belief function* if there is an state $\mathbf{s} : \mathcal{R} \to [0,1]$ such that, for every $f \in M$,

$$\mathbf{b}(f) = \mathbf{s}(\rho_f),$$

where $\rho_f : M \to [0,1]$ is defined as

$$\rho_f(g) = \inf_{x \in X} g(x) \Rightarrow f(x),$$

with $\Rightarrow$ being Łukasiewicz implication function in the standard MV-algebra $[0,1]_{MV}$.

The following result shows that B-coherence is a characterization of belief functions on MV-algebras in the same way as de Finetti's coherence [2] is a characterization for probability measures on Boolean algebras. It is worth recalling that a generalization of de Finetti's coherence criterion to the case of MV-algebras has been proved by Mundici [12] and Kühr and Mundici [9]. Moreover the following theorem generalizes classical results by Jaffray [6] and Paris [13].

Theorem 2. *Let X be a finite set of possible worlds, let $e_1, \ldots, e_n \in [0,1]^X$ be events and let $\beta : e_i \mapsto \beta_i$ be a book. Then the following are equivalent:*

- *β is B-coherent;*
- *There exists a belief function $\mathbf{b}$ on $[0,1]^X$ extending β.*

It is worth noticing that, if we restrict our attention to those particular possibility distributions like $\pi_w : X \to [0,1]$ such that $\pi_w(w') = 0$ if $w \neq w'$ and $\pi_w(w) = 1$, then $N^{\pi_w}(\cdot) = w(\cdot)$ and hence the resulting betting game coincides with the usual betting game for states. On the other hand, in the general case, a natural notion of *indeterminacy* of an event e in an epistemic state defined by a possibility distribution π, is given by the value $I^\pi(e) = \Pi^\pi(e) - N^\pi(e)$, where $\Pi^\pi(e) = 1 - N^\pi(\neg e)$.

In this setting, following [3, 10], we can consider a variant of the above discussed betting game in which, for every event e_i, the Bookmaker is obliged to give back to the Gambler a proportional amount of the balance regarding e_i according to $I_\pi(e_i)$. In particular, when $I_\pi(e_i) = 1$ (i.e. when there is total indeterminacy about e_i) the Bookmaker is obliged to call off the bet on e_i. The resulting game is hence a conditional game in which the realization of each event e_i is conditioned by its determinacy whose total balance is given by the expression

$$\sum_{i=1}^{n} (1 - I^\pi(e_i)) \cdot (\sigma_i \cdot (\alpha_i - N^\pi(e_i))).$$

The measure which characterizes the coherence of this variant of the game we have discussed can be regarded as a conditional probability on modal formulas. In particular a book $\beta : e_i \mapsto \beta_i$ is coherent iff there exists a conditional state $\mathbf{s}(\cdot \mid \cdot)$ in the sense of [7] on a suitably defined MV-algebra such that, for every $i = 1, \ldots, n$, $\beta_i = \mathbf{s}(\Box e_i \mid \Diamond e_i \to \Box e_i)$, where $\Box$ is the many-valued modal operator defined on Łukasiewicz logic as in [1] and as usual $\Diamond e = \neg\Box\neg e$.

References

[1] F. Bou, F. Esteva, L. Godo, R. O. Rodríguez, *On the Minimum Many-Valued Modal Logic over a Finite Residuated Lattice*. Journal of Logic and Computation, 21(5): 739-790, 2011.

[2] B. de Finetti. *Sul significato soggettivo della probabilità*, Fundamenta Mathematicae 17, 298–329, 1931. Translated into English as "On the subjective meaning of probability", in: Paola Monari and Daniela Cocchi (Eds.), Probabilità e Induzione, Clueb, Bologna, pp. 291–321, 1993.

[3] D. Dubois, H. Prade. Conditional Objects as Nonmonotonic Consequence Relationships. IEEE Transactions on Systems, Man, and Cybernetics, Vol 24 (12), 1724–1740, 1994.

[4] T. Flaminio, L. Godo, T. Kroupa. *Combination and Soft-Normalization of Belief Functions on MV-Algebras*. Proc. of Modeling Decisions for Artificial Intelligence, MDAI 2012. Lecture Notes in Artificial Intelligence, vol. 7647, pp. 23-34, 2012.

[5] T. Flaminio, L. Godo, E. Marchioni. *Logics for Belief Functions on MV-algebras*. International Journal of Approximate Reasoning, in print.

[6] J.Y. Jaffray. *Coherent bets under partially resolving uncertainty and belief functions*. Theory and Decision, 26, 99-105, 1989.

[7] T. Kroupa. *Conditional probability on MV-algebras*. Fuzzy Sets and Systems 149(2): 369–381, 2005.

[8] T. Kroupa. *Extension of Belief Functions to Infinite-valued Events*. Soft Computing 16(11): 1851–1861, 2012.

[9] J. Kühr, D. Mundici. *De Finetti theorem and Borel states in* $[0,1]$*-valued algebraic logic*. International Journal of Approximate Reasoning, 46(3), 605–616, 2007.

[10] F. Montagna. Partially Undetermined Many-Valued Events and Their Conditional Probability. *Journal of Philosophical Logic* 41:563–593, 2012.

[11] D. Mundici. *Averaging the truth-value in Łukasiewicz logic*. Studia Logica 55(1) (1995) 113–127.

[12] D. Mundici. *Bookmaking over Infinite-valued Events*. International Journal of Approximate Reasoning 43 (2006) 223–240.

[13] J. B. Paris. *A note on the Dutch Book method*. Revised version of a paper of the same title which appeared in The Proceedings of the Second Internat. Symp. on Imprecise Probabilities and their Applications, ISIPTA01, Ithaca, New York, 2001.

New involutive FL_e-algebra constructions

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FL_e-algebras are algebraic counterparts of substructural logics with exchange. A structural description and also a classification of certain subclasses of involutive FL_e-algebras have been obtained in [4] and [3], respectively, as follows:
If an involutive FL_e-monoid $\langle X, \circledast, \leq, t, f\rangle$ is conic then $\circledast$ is the twin-rotation of the two cone operations $\otimes$ and $\oplus$, that is,

$$x \circledast y = \begin{cases} x \otimes y & \text{if } x, y \in X^- \\ x \oplus y & \text{if } x, y \in X^+ \\ \neg(\oplus \to_\circledast x \neg y) & \text{if } x \in X^+,\ y \in X^-, \text{ and } x \leq \neg y \\ \neg(\oplus \to_\circledast y \neg x) & \text{if } x \in X^-,\ y \in X^+, \text{ and } x \leq \neg y \\ \neg(\otimes \to_\circledast y(\neg x \wedge t)) & \text{if } x \in X^+,\ y \in X^-, \text{ and } x \not\leq \neg y \\ \neg(\otimes \to_\circledast x(\neg y \wedge t)) & \text{if } x \in X^-,\ y \in X^+, \text{ and } x \not\leq \neg y \end{cases}.$$

If U is an absorbent-continuous group-like FL_e-algebra on a subreal chain then its negative cone is a BL-algebra with components (see [1]) which are either cancellative or MV-algebras with two elements, and with no two consecutive cancellative components, its positive cone is the dual of its negative cone with respect to $\neg$, and its monoidal operation is given by the twin-rotation of its cones.

In this talk we introduce several new construction methods resulting in involutive (but not group-like) FL_e-algebras along with some related characterizations.

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References

[1] P. Aglianó, F. Montagna, "Varieties of BL-algebras I: general properties", *Journal of Pure and Applied Algebra*, **181** (2-3), 2003, 105–129.

[2] S. Jenei, F. Montagna, "Strongly Involutive Uninorm Algebras", *Journal of Logic and Computation*, doi:10.1093/logcom/exs019.

[3] S. Jenei, F. Montagna, "A classification of certain group-like FL_e-chains", (submitted).

[4] S. Jenei, H. Ono, "On involutive FL_e-monoids", *Archive for Mathematical Logic*, **51** (7), 2012, 719–738.

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f-MV-algebras and piecewise polynomial functions

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MV-algebras are the structures of Łukasiewicz ∞-valued logic. An *MV-algebra* is a structure $(A, \oplus, ^*, 0)$, where $(A, \oplus, 0)$ is an abelian monoid and the following identities hold for all $x, y \in A$:

$(x^*)^* = x$, $0^* \oplus x = 0^*$, $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

A fundamental result in the theory of MV-algebras is their categorical equivalence with the category of abelian lattice-ordered groups with strong unit, proved by Mundici in [15].

Di Nola and Dvurečenskij introduced in [3] the *PMV-algebras*, which are MV-algebras endowed with a product operation $\cdot : A \times A \to A$, satisfying some particular identities. The category of PMV-algebras is equivalent with the category of lattice-ordered rings with strong unit. In [4] the internal product is replaced by a scalar multiplication with scalars from $[0, 1]$, so the MV-algebra is endowed with a map $\star : [0, 1] \times A \to A$. The structures obtained in this way are called *Riesz MV-algebras* and they are categorically equivalent with Riesz spaces with strong unit.

We consider in the following MV-algebras endowed with both internal and external product. They are connected, by extensions of Mundici's categorical equivalence, with *unital f-algebras.* s

Definition 1. An f-MV-module A is an algebraic structure $(A, \star, \cdot, \oplus, ^*, 0)$ where $\cdot$ and $\oplus$ are binary operations, * is unary, 0 is a constant and $\star : [0, 1] \times A \to A$ is such that:
(F1) $(A, \cdot, \oplus, ^*, 0)$ is a unital PMV-algebra,
(F2) $(A, \star, \oplus, ^*, 0)$ is a Riesz MV-algebra,
(F3) $r \star (x \cdot y) = (r \star x) \cdot y = x \cdot (r \star y)$ for any $r \in [0, 1]$ and $x, y \in A$,
(F4) $(z \cdot (x \odot y^*)) \wedge (y \odot x^*) = 0$ for any $x, y, z \in A$.

In the following we simply write rx for $r \star x$ for any $r \in [0, 1]$ and $x \in A$.

Note that one can assume that the PMV-algebra reduct is not unital, but the present approach is more suitable for our purposes.

Recall that an f-algebra V is an f-ring endowed with a structure of Riesz space such that $\alpha(x \cdot y) = (\alpha x) \cdot y = x \cdot (\alpha y)$ for any $\alpha \in \mathbb{R}$ and $x, y \in V$. They were introduced by Birkhoff and Pierce in [1] and we refer to [2] for a survey on this topic. A *unital* f-algebra is an f-algebra with strong unit such that the strong unit is also unit for the product.

Theorem 2. *The category of f-MV-algebras is equivalent with the category of f-algebras with strong unit with unit-preserving morphisms.*

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The real interval $[0,1]$ is obviously an f-MV-algebra if the both products coincide with the real one. By a result of Hion [6], one can prove that this is the only structure of f-MV-algebra that can be defined on $[0,1]$. Following [10], we say that a f-MV-algebra is *formally real* if it belongs to $\mathrm{HSP}([0,1])$.

We denote by $\mathbb{FR}$ the class of formally real f-MV-algebras and we note that it is a proper subvariety of f-MV-algebras. By well-known results of universal algebra [7], the free f-MV-algebra in $\mathbb{FR}$ exists an its elements are term functions defined on $[0,1]$. More precisely, the language of f-MV-algebras is $\mathcal{L}_f = \{\oplus, \cdot, ^*, 0, \} \cup \{\delta_r \mid r \in [0,1]\}$, where δ_r is a unary operation that is interpreted by $x \mapsto rx$ for any $r \in [0,1]$. For any $n \geq 1$, let $Term_n$ be the set of $\mathcal{L}_f$-terms with n variables and let us denote by FR_n the free f-MV-algebra in $\mathbb{FR}$ with n free generators. It follows that

$$FR_n = \{\widetilde{t} \mid t \in Term_n,\ \widetilde{t} : [0,1]^n \to [0,1] \text{ is the term function of } t\}.$$

In order to characterize FR_n we give the following definition.

Definition 3. A *piecewise polynomial function* defined on the n-cube is a function

$$f : [0,1]^n \to [0,1]$$

such that there exists a finite number of polynomials $f_1, \ldots, f_k \in \mathbb{R}[x_1, \ldots, x_n]$ with the property that $f(a_1, \ldots, a_n) = f_i(a_1, \ldots, a_n)$ for any $(a_1, \ldots, a_n) \in [0,1]^n$ and for some $i \in \{1, \ldots, k\}$.

Proposition 4. *The elements of FR_n are continuous piecewise polynomial functions defined on the n-cube.*

The converse of the above proposition is related to the Birkhoff-Pierce conjecture [1, 10]. Since the conjecture is true for $n \leq 2$ [12], we get the following.

Theorem 5. *For $n \leq 2$, the f-MV-algebra FR_n is the set of all continuous piecewise polynomial functions defined on the n-cube, i.e any continuous piecewise polynomial function defined on the n-cube is a term function from FR_n.*

The notion of *state* has been introduced by Mundici in [16], as MV-algebraic counterpart of the boolean probability theory. A state on an f-MV-algebra is just a state on its MV-algebra reduct.

The Hausdorff moment problem [8, 9] gives the necessary and sufficient conditions for a sequence $\{m_k | k \geq 0\} \subseteq [0,1]$ to be the sequence of moments of a probability measure μ on $[0,1]$, i.e. $m_k = \int_0^1 x^k d\mu$ for any $k \geq 0$. We prove a similar result within the theory of PMV-algebras and f-MV-algebras. Our main ingredient is the integral representation for states on MV-algebras, proved independently by Kroupa [11] and Panti [17].

For any $k \geq 1$ we define $p_k : [0,1] \to [0,1]$ by $p_k(x) = x^k$ for any $x \in [0,1]$. We also set $p_0(x) = 1$ for any $x \in [0,1]$. Note that $p_k \in FR_1$ for any $k \geq 0$.

If $\{m_k | k \geq 0\}$ a sequence of real numbers from $[0,1]$ then we define:

$$\Delta^0 m_k = m_k, \quad \Delta^r m_k = \Delta^{r-1} m_{k+1} - \Delta^{r-1} m_k \text{ for any } r,\ k \geq 0.$$

The sequence $\{m_k\}_k$ satisfies the *Hausdorff moment condition* whenever $m_0 = 1$ and $(-1)^r \Delta^r m_k \geq 0$ for any r, $k \geq 0$ [5].

Theorem 6. Let P be any archimedean PMV-subalgebra of $C([0,1])$ such that $p_1 \in P$. For a sequence $\{m_k | k \geq 0\} \subseteq [0,1]$, the following are equivalent:
(i) $\{m_k\}_k$ satisfies the Hausdorff moment condition,
(ii) there exists a state $s : P \to [0,1]$ such that $s(p_k) = m_k$ for any $k \geq 0$.

In particular, the theorem applies to f-MV-subalgebras of $C([0,1])$ which contains p_1 and, consequently, to FR_1.

References

[1] Birkhoff, G., Pierce, R., *Lattice-ordered rings*, Anais da Academia Brasileira de Cienc. 28 (1956), 41-69.

[2] Boulabiar, K., Buskes, G., and Triki, A., *Some recent trends and advances in certain lattice ordered algebras* (English. English summary), Function spaces (Edwardsville, IL, 2002), 99-133, Contemp. Math., 328, Amer. Math. Soc., Providence, RI, 2003.

[3] Di Nola, A., Dvurečenskij, A., *Product MV-algebras*, Multiple -Valued Logics 6 (2001), 193-215.

[4] Di Nola, A., Leuştean, I., *Łukasiewicz logic and Riesz spaces*, Soft Computing, accepted.

[5] Feller, W., An Introduction to Probability Theory and Its Applications, Vol. II, Wiley, New York, 1971.

[6] Fuchs, L., Partially Ordered Algebraic Systems, Pergamon Press , Oxford-London-New York-Paris, 1963.

[7] Grätzer, G., Universal Algebra. Springer-Verlag, Berlin- Heilderberg-New York, 1968.

[8] Hausdorff, F., *Summationmethoden und Momentfolgen I*, Math. Z. 9 (1921), 74–109.

[9] Hausdorff, F., *Summationmethoden und Momentfolgen II*, Math. Z. 9 (1921), 280–299.

[10] Henriksen, M., Isbell, J.R., *Lattice-ordered rings and function rings*, Pacif. J. Math. 12(2) (1962), 533–565.

[11] Kroupa, T., *Every state on semisimple MV-algebra is integral*, Fuzzy Sets ad Systems, 157 (2006), 2771–2782.

[12] Mahé L., *On the Birkhoff-Pierce conjecture*, Rocky M. J. 14(4) (1984) 983–985.

[13] Mahé L., *On the Birkhoff-Pierce Conjecture in three variables*, J. Pure Appl. Algebra 211 (2007), 459–470.

[14] Miranda, E., de Cooman, G., Quaeghebeur, E., *The Hausdorff moment problem under finite additivity*, Journal of Theoretical Probability 20(3) 2007, 663–693.

[15] Mundici, D., *Interpretation of AF C^*-algebras in Łukasiewicz sentential calculus*, Journal of Functional Analysis 65 (1986), 15–63.

[16] Mundici, D., *Averaging the Truth-Value in Łukasiewicz Logic*, Studia Logica 55 (1995), 113–127.

[17] Panti, G. *Invariant measures in free MV-algebras*, Communications in Algebra 36 (2008), 2849–2861.

On discrete Łukasiewicz games

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We introduce a compact representation of non-cooperative games based on finite-valued Łukasiewicz logics [2], called *Discrete Łukasiewicz Games*. Łukasiewicz Games are inspired by, and greatly extend, a class of logic-based games known as Boolean games [4, 1]. Boolean games are games in which each individual player strives for the satisfaction of a goal, represented as a classical Boolean formula; the actions available to players correspond to valuations that can be made to variables under their control. Łukasiewicz games extend this idea by considering games played with goals represented as formulae of Łukasiewicz logic. The key advantage of this approach over conventional Boolean games is that is the use of Łukasiewicz logic makes it possible to more naturally represent much richer utility functions for players.

Let $\mathsf{V} = \{x, y, \dots\}$ be a finite set of propositional variables. The types of games with deal with involve a finite set of players $\mathsf{P} = \{P_1, \dots, P_n\}$ (also referred to as "agents"). Each player P_i is in control of a subset of propositional variables $\mathsf{V}_i \subseteq \mathsf{V}$, so that the sets V_i are mutually disjoint and their union covers V. Being in control of a set V_i of propositional variables means that P_i assigns to the variables in V_i values from

$$L_k = \left\{0, \frac{1}{k}, \dots, \frac{k-1}{k}, 1\right\}.$$

A strategy for an agent P_i is a function $s_i : \mathsf{V}_i \to L_k$ that corresponds to a valuation of the variables controlled by P_i. A strategy profile is a collection of strategy choices $(s_1, \dots, s_n)$, one by each player. Strategies can be interpreted as efforts or costs, and each player's strategic choice can be seen as an assignment to each controlled variable carrying an intrinsic cost.

Each agent has a goal, represented as an $Ł_k$-formula ϕ_i, with propositional variables from V. The evaluation of this goal formula is interpreted as the payoff function for P_i. Notice that in general, not all variables in ϕ_i will be under P_i's control and, consequently, the utility that P_i obtains from playing a certain strategy (i.e., choosing a certain variable assignment) will in part depend on the choices made by other players.

The overall goal of each player P_i is twofold:

1. First and foremost, P_i aims to maximizing its payoff.

2. Second, P_i wants to minimize efforts/costs.

Notice that payoff maximization is the primary consideration for a player; cost minimization is a secondary concern.

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Discrete Łukasiewicz Games can be seen as a generalization of Boolean Games [4, 1], since the latter obviously are a special case of the former. However, Discrete Łukasiewicz Games also incorporate the notion of cost/effort, which makes it possible to formalize situations in which agents aim at a better tradeoff between the costs of making certain choices and the resulting payoff.

In this work:

1. We provide a formal definition of *Discrete Łukasiewicz Games* and show how certain strategic interactions can be compactly formalized within this framework.

2. We introduce a notion of dominance, best response and Pure Strategy Nash Equilibrium for Discrete Łukasiewicz Games.

3. We begin to study the complexity of deciding whether a strategy is dominated and of determining the existence of a Pure Strategy Nash Equilibrium.

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References

[1] E. Bonzon, M.C. Lagasquie, J. Lang, and B. Zanuttini. Boolean games revisited. In *Proceedings of the Seventeenth European Conference on Artificial Intelligence* (ECAI-2006), Riva del Garda, Italy, 2006.

[2] R. Cignoli, I.M.L. D'Ottaviano, D. Mundici. *Algebraic Foundations of Many-Valued Reasoning*, Volume 7 of *Trends in Logic—Studia Logica Library.* Kluwer Academic Publishers, Dordrecht, 2000.

[3] J. Grant, S. Kraus, M. Wooldridge, and I. Zuckerman. Stabilizing Boolean games by sharing information. *Studia Logica*, to appear.

[4] P. Harrenstein, W. van der Hoek, J.J.Ch. Meyer, and C. Witteveen. Boolean games. In *Proceeding of the Eighth Conference on Theoretical Aspects of Rationality and Knowledge* (TARK VIII), J. van Benthem (Ed.), 287–298, Siena, Italy, 2001.

Decidability for Gödel modal logics*

George Metcalfe†

Joint work with Xavier Caicedo, Ricardo Rodríguez, and Jonas Rogger

The Gödel modal logics GK and $\mathrm{GK^C}$ combine the Kripke frames of the modal logic K with the standard semantics of the many-valued Gödel logic G. More precisely, let $\mathrm{Fml}_{\Box\Diamond}$ be the set of formulas for a language with connectives $\wedge$, $\vee$, $\to$, $\bot$, $\top$, $\Box$, and $\Diamond$, over a countably infinite set of variables Var. Let us also fix $\ell(\varphi)$ to be the *length* of a formula φ. A *fuzzy Kripke frame* is a pair $\mathfrak{F} = \langle W, R \rangle$ consisting of a non-empty set W and a fuzzy binary accessibility relation $R\colon W \times W \to [0,1]$. If $Rxy \in \{0,1\}$ for all $x, y \in W$, then R is called *crisp* and $\mathfrak{F}$ is a *crisp Kripke frame*, writing $R \subseteq W \times W$ and Rxy to mean $Rxy = 1$.

A GK-*model* is a triple $\mathfrak{M} = \langle W, R, V \rangle$ where $\langle W, R \rangle$ is a fuzzy Kripke frame and $V\colon \mathrm{Var} \times \mathrm{W} \to [0,1]$ is a mapping, called a *valuation*, extended to $V\colon \mathrm{Fml}_{\Box\Diamond} \times \mathrm{W} \to [0,1]$ by (where $x \to_{\mathrm{G}} y$ is y if $x > y$, and 1 if $x \leq y$)

$$\begin{aligned}
V(\bot, x) &= 0\\
V(\top, x) &= 1\\
V(\varphi \to \psi, x) &= V(\varphi, x) \to_{\mathrm{G}} V(\psi, x)\\
V(\varphi \wedge \psi, x) &= \min(V(\varphi, x), V(\psi, x))\\
V(\varphi \vee \psi, x) &= \max(V(\varphi, x), V(\psi, x))\\
V(\Box\varphi, x) &= \inf\{Rxy \to_{\mathrm{G}} V(\varphi, y) : y \in W\}\\
V(\Diamond\varphi, x) &= \sup\{\min(Rxy, V(\varphi, y)) : y \in W\}.
\end{aligned}$$

A $\mathrm{GK^C}$-*model* satisfies the extra condition that $\langle W, R \rangle$ is a crisp Kripke frame. In this case, the conditions for $\Box$ and $\Diamond$ may also be read as:

$$\begin{aligned}
V(\Box\varphi, x) &= \inf(\{1\} \cup \{V(\varphi, y) : Rxy\})\\
V(\Diamond\varphi, x) &= \sup(\{0\} \cup \{V(\varphi, y) : Rxy\}).
\end{aligned}$$

A formula $\varphi \in \mathrm{Fml}_{\Box\Diamond}$ is *valid* in a GK-model $\mathfrak{M} = \langle W, R, V \rangle$ if $V(\varphi, x) = 1$ for all $x \in W$. If φ is valid in all L-models for some logic L (in particular GK or $\mathrm{GK^C}$), then φ is said to be L-*valid*, written $\models_{\mathrm{L}} \varphi$.

Axiomatizations were obtained for the box and diamond fragments of GK (where the box fragments of GK and $\mathrm{GK^C}$ coincide) in [4] and for the diamond fragment of $\mathrm{GK^C}$ in [9]. It was subsequently shown in [3] that the full logic GK is axiomatized by adding

*A full paper with proofs may be downloaded from `http://www.philosophie.ch/297`.

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the Fischer Servi axioms for intuitionistic modal logic IK to the union of the axioms for both fragments, or by adding the prelinearity axiom for Gödel logic to IK. Decidability of the diamond fragment of GK was established in [4] using the fact that the fragment has the finite model property with respect to its Kripke semantics. This finite model property fails for the box fragment of GK and GK^C and the diamond fragment of GK^C; however, decidability and PSPACE-completeness for these fragments was established in [9] via analytic Gentzen-style proof systems. We note also that multimodal variants of GK have also been proposed as the basis for fuzzy description logics in [8, 1]. More general approaches to many-valued modal logics, focussing mainly on the finite-valued case, have been developed by Fitting [5, 6] and Bou et al. [2].

Our goal in this work is to establish decidability for GK and GK^C by showing that these logics have a finite model property with respect to a slightly different Kripke semantics. Let us define a GFK-*model* as a quadruple $\mathfrak{M} = \langle W, R, T, V\rangle$, where $\langle W, R, V\rangle$ is a GK-model and $T\colon W \to \mathcal{P}_{<\omega}([0,1])$ is a function from worlds to finite sets of truth values satisfying $\{0,1\} \subseteq T(x) \subseteq [0,1]$ for all $x \in W$. If $\langle W, R, V\rangle$ is also a GK^C-model, then $\mathfrak{M}$ will be called a GFK^C-*model*. The GFK-valuation V is extended using the same clauses for non-modal connectives as for GK-valuations, and

$$\begin{aligned} V(\Box\varphi, x) &= \max\{r \in T(x) : r \le \inf\{Rxy \to_{\mathrm{G}} V(\varphi, y) : y \in W\}\} \\ V(\Diamond\varphi, x) &= \min\{r \in T(x) : r \ge \sup\{\min(Rxy, V(\varphi, y)) : y \in W\}\}. \end{aligned}$$

As before, a formula $\varphi \in \mathrm{Fml}_{\Box\Diamond}$ is *valid* in a GFK-model $\mathfrak{M} = \langle W, R, T, V\rangle$ if $V(\varphi, x) = 1$ for all $x \in W$.

Theorem 1. *For each* $\varphi \in \mathrm{Fml}_{\Box\Diamond}$*:*

(a) $\models_{\mathrm{GK}} \varphi$ *iff* $\models_{\mathrm{GFK}} \varphi$ *iff* φ *is valid in all* GFK*-models* $\mathfrak{M} = \langle W, R, T, V\rangle$ *satisfying* $|W| \le (\ell(\varphi) + 2)^{\ell(\varphi)}$ *and* $|T(x)| \le \ell(\varphi) + 2$ *for all* $x \in W$.

(b) $\models_{\mathrm{GK^C}} \varphi$ *iff* $\models_{\mathrm{GFK^C}} \varphi$ *iff* φ *is valid in all* GFK^C*-models* $\mathfrak{M}$ *satisfying* $|W| \le (\ell(\varphi) + 2)^{\ell(\varphi)}$ *and* $|T(x)| \le \ell(\varphi) + 2$ *for all* $x \in W$.

Moreover, validity in GK *and* GK^C *is decidable.*

A similar methodology may be applied to the crisp Gödel modal logic $GS5^C$, characterized by validity in GK^C-models where R is an equivalence relation, or equivalently, validity in *universal* $GS5^C$-*models* where all worlds are related. Such models may be written as $\mathfrak{M} = \langle W, V\rangle$ with

$$V(\Box\varphi, x) = \inf\{V(\varphi, y) : y \in W\} \quad \text{and} \quad V(\Diamond\varphi, x) = \sup\{V(\varphi, y) : y \in W\}.$$

$GS5^C$ can be axiomatized by extending the intuitionistic modal logic MIPC with prelinearity and $\Box(\Box\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$ [3]. It may also be viewed as the one-variable fragment of first-order Gödel logic $G\forall$ (see [7]). Making use of $GFS5^C$-models, defined similarly to GFK^C-models above, we obtain:

Theorem 2. *Validity in* $GS5^C$ *and the one-variable fragment of first-order Gödel logic is decidable and indeed co-NP-complete.*

References

[1] F. Bobillo, M. Delgado, J. Gómez-Romero, and U. Straccia. Fuzzy description logics under Gödel semantics. *International Journal of Approximate Reasoning*, 50(3):494–514, 2009.

[2] F. Bou, F. Esteva, L. Godo, and R. Rodríguez. On the minimum many-valued logic over a finite residuated lattice. *Journal of Logic and Computation*, 21(5):739–790, 2011.

[3] X. Caicedo and R. Rodríguez. Bi-modal Gödel logic over [0,1]-valued Kripke frames. To appear in *Journal of Logic and Computation*.

[4] X. Caicedo and R. Rodríguez. Standard Gödel modal logics. *Studia Logica*, 94(2):189–214, 2010.

[5] M. C. Fitting. Many-valued modal logics. *Fundamenta Informaticae*, 15(3-4):235–254, 1991.

[6] M. C. Fitting. Many-valued modal logics II. *Fundamenta Informaticae*, 17:55–73, 1992.

[7] P. Hájek. *Metamathematics of Fuzzy Logic*. Kluwer, Dordrecht, 1998.

[8] P. Hájek. Making fuzzy description logic more general. *Fuzzy Sets and Systems*, 154(1):1–15, 2005.

[9] G. Metcalfe and N. Olivetti. Towards a proof theory of Gödel modal logics. *Logical Methods in Computer Science*, 7(2):1–27, 2011.

An algebraic study of partial predicates

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Partial predicates were introduced by the neokantian philosopher Körner and studied from a logical point of view in [7] and [12]. Let X be a set, as classical sets over X are subsets of X, a **partial set** over X is a pair $\langle A, B\rangle$ such that $A, B \subseteq X$ and $A \cap B = \emptyset$, where A represents the set of individuals of X which surely belong to $\langle A, B\rangle$ and B the set of individuals which surely does not. In this sense partial sets provide a good framework to model exactness in partial contexts. We let $\mathcal{D}(X) = \{\langle A, B\rangle \mid \langle A, B\rangle \text{ is a partial set over } X\}$. Conjunctions, disjunctions and complements of partial sets are defined in a very nice way: for pick $\langle A, B\rangle, \langle C, D\rangle \in \mathcal{D}(X)$, we let $\langle A, B\rangle \cap \langle C, D\rangle = \langle A \cap C, B \cup D\rangle$, $\langle A, B\rangle \cup \langle C, D\rangle = \langle A \cup C, B \cap D\rangle$ and $\neg\langle A, B\rangle = \langle B, A\rangle$. We say that a subset $\mathcal{S} \subseteq \mathcal{D}(X)$ is a **field of partial sets** over X if $\mathcal{S}$ is closed under partial conjunctions, disjunctions, complements and contains $\langle \emptyset, \emptyset\rangle$.

Negri proved that the algebraic structure of fields of partial sets can be abstracted, yielding a class of algebras which enjoys a strong connection with the original intuitive framework of partial sets, providing a representation theorem which says that they are essentially the same (Theorem 3.1 [12]). More precisely we say that $\mathbf{A} = \langle \mathbf{A}, \wedge, \vee, \neg, n\rangle$ is a **DMF lattice** if it is a normal De Morgan lattice with one fixed point n for the negation, i.e. a distributive lattice that satisfies the following equations:

$$x \vee y = \neg(\neg x \wedge \neg y) \qquad x \wedge y = \neg(\neg x \vee \neg y)$$

$$\neg\neg x = x \qquad \neg n = n$$

$$x \wedge \neg x \leqslant y \vee \neg y.$$

We will denote by DMF the class of DMF lattices. Examples of DMF's are easy to construct: for every set X, the field of partial sets over it, $\mathcal{D}(X)$, is a DMF. We list below some basic properties of $\mathbf{A} \in \mathsf{DMF}$. As we mentioned above, it turns out that every DMF's is isomorphic to a field of partial sets. This representation can be converted into a subdirect one if we reason as follows. Given any $\boldsymbol{A} \in \mathsf{DMF}$, we say that $F \subseteq \boldsymbol{A}$ is a **partial filter** if it is a prime lattice filter such that $n \notin F$. Then let $\mathcal{F}(\boldsymbol{A}) = \{F \subseteq \boldsymbol{A} : F \text{ is a partial filter}\}$. Given any non-trivial $\boldsymbol{A} \in \mathsf{DMF}$ we defined a function $\alpha\colon \boldsymbol{A} \to \prod_{F \in \mathcal{F}(\mathbf{A})} \boldsymbol{Z}_{3_F}$ as

$$\alpha(a)(F) = \begin{cases} 1 & \text{if } a \in F \\ n & a \notin F \text{ and } \neg a \notin F \\ -1 & \text{if } \neg a \in F. \end{cases}$$

for every $a \in \boldsymbol{A}$ and $F \in \mathcal{F}(\boldsymbol{A})$. It is easy to prove that α is in fact an homomorphism.

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Lemma 1. *Let $\boldsymbol{A} \in \mathsf{DMF}$ non-trivial. $\alpha\colon \boldsymbol{A} \to \prod_{F \in \mathcal{F}(\mathbf{A})} \boldsymbol{Z}_{3_F}$ is a subdirect embedding.*

As a consequence we get that $\mathbb{V}(\boldsymbol{Z}_3) = \mathsf{DMF}$ and that $\boldsymbol{Z}_3$ is the only subdirectly irreducible member of DMF.

In order to gain more informations about the structure of DMF's, we need to introduce a new concept: given $\boldsymbol{A} \in \mathsf{DMF}$, we let $\uparrow n = \{a \in \boldsymbol{A} \,|\, n \leqslant a\}$ be its **positive cone**. We denote by $\mathsf{DL}_\perp$ the category of distributive lattices with minimum $\perp$ and lattice homomorphisms which preserve $\perp$ as arrows. It is possible to prove that there is an adjunction between DMF and $\mathsf{DL}_\perp$, which is a special case of an adjunction presented in [6]. In order to do this, we construct a functor $\pi\colon \mathsf{DL}_\perp \to \mathsf{DMF}$. Pick $\boldsymbol{L} \in \mathsf{DL}_\perp$, we let

$$\pi(\boldsymbol{L}) = \langle \{\langle a, b\rangle \in \boldsymbol{L}^2 : a \wedge b = \perp\}, \wedge, \vee, \neg, n\rangle$$

where $\langle a, b\rangle \wedge \langle c, d\rangle = \langle a \wedge c, b \vee d\rangle$, $\langle a, b\rangle \vee \langle c, d\rangle = \langle a \vee c, b \wedge d\rangle$, $\neg\langle a, b\rangle = \langle b, a\rangle$, $n = \langle \perp, \perp\rangle$ for every $\langle a, b\rangle, \langle c, d\rangle \in \pi(\boldsymbol{A})$. For every arrow $f\colon \boldsymbol{L} \to \boldsymbol{M}$ in $\mathsf{DL}_\perp$ we let $\pi(f)\colon \pi(\boldsymbol{L}) \to \pi(\boldsymbol{M})$ be defined as

$$\pi(f)\langle a, b\rangle = \langle f(a), f(b)\rangle$$

for every $\langle a, b\rangle \in \pi(\boldsymbol{L})$. It is easy to prove π is indeed a functor. The way back from DMF to $\mathsf{DL}_\perp$ is pretty natural: we pick positive cones and restrictions of homomorphisms to them. More precisely, given $\boldsymbol{A} \in \mathsf{DMF}$, we let $\uparrow(\boldsymbol{A}) = \langle \uparrow n, \wedge, \vee, n\rangle$ and, given a DMF arrow $f\colon \boldsymbol{A} \to \boldsymbol{B}$, we let $\uparrow(f)\colon \uparrow(\boldsymbol{A}) \to \uparrow(\boldsymbol{B})$ be the restriction of f to $\uparrow(\boldsymbol{A})$. Clearly $\uparrow\colon \mathsf{DL}_\perp \to \mathsf{DMF}$ is a functor.

Theorem 2. *$\uparrow \dashv \pi$ is an adjunction.*

The relation between DMF's and partial sets reflects also in the fact that free algebras enjoy a nice partial behaviour. Let $\mathbb{X}$ be an arbitrary set of variables, for every $x \in \mathbb{X}$ we let

$$\overline{x} = \langle \{\langle A, B\rangle \in \mathcal{D}(\mathbb{X}) \,|\, x \in A\}, \{\langle A, B\rangle \in \mathcal{D}(\mathbb{X}) \,|\, x \in B\}\rangle.$$

Then we let $\mathbf{F}_{\mathsf{DMF}}(\mathbb{X})$ be the field of partial sets over $\mathcal{D}(\mathbb{X})$ generated by $\{\overline{x}\}_{x \in \mathbb{X}}$.

Theorem 3. *Let $\mathbb{X}$ be a set of variables. $\mathbf{F}_{\mathsf{DMF}}(\mathbb{X})$ is the free algebra over DMF with free generators $\{\overline{x}\}_{x \in \mathbb{X}}$, where $\overline{x} \neq \overline{y}$ for every $x, y \in \mathbb{X}$ such that $x \neq y$.*

As a consequence we get that, up to equivalence in DMF, terms in just one variable x are $\mathcal{T}(x) = \{n, x \vee \neg x, x \vee n, \neg x \vee n, x \wedge \neg x, x \wedge n, \neg x \wedge n, x, \neg x\}$. This fact allows us to prove the first one of two strong negative results about logics defined over DMF's, i.e., that DMF is not the class of Leibniz algebras of any logic which defines truth with equations in just one variable x. The second one tells us that this is not the case also for every logic which defines equivalence in terms of a set of formulas in two variables (possibly with parameters). See respectively [13] and [2] for a precise definition of these two classes of logics.

Corollary 4. *If $\mathcal{L}$ be truth-equational or protoalgebraic, then $\mathsf{DMF} \nsubseteq \mathsf{Alg}^*\mathcal{L}$.*

Now we turn to define a logic for partial predicates which preserves exact truth. Since we have seen, both in the intuitive explaination and in the representation theorems, that partial sets are intrinsically three-valued, the natural choice is to define a logic with the matrix $\langle \boldsymbol{Z}_3, \{1\}\rangle$. More precisely we let

$$\Gamma \vdash \varphi \iff \text{ if } h[\Gamma] \subseteq \{1\}, \text{ then } h(\varphi) = 1$$

for every $\Gamma \cup \{\varphi\} \subseteq \boldsymbol{Fm}$ and every homomorphism $H \colon \boldsymbol{Fm} \to \boldsymbol{A}$. We will call this logic $\mathcal{L}_{\{1\}}$ since it is intended to represent exact truth in partial predicates. It is worth to observe that $\mathcal{L}_{\{1\}}$ is finitary, since it is defined through a finite matrix and that is has no theorems since $\{n\}$ is a subalgebra of $\boldsymbol{Z}_3$. The first problem we would like to solve is to individuate the algebraic cuonterpart of this logic (see [10]), which turns out to coincide with DMF.

Lemma 5. $\mathsf{Alg}\mathcal{L}_{\{1\}} = \mathsf{DMF}$.

Nevertheless, since $\mathcal{L}_{\{1\}}$ does not belong to any of the hierarchy typical of abstract algebraic logic (as it is neither protoalgebraic, nor truth-equational, neither selfextensional), it may be interesting to take a look to the structure of its Leibniz reduced models (whose algebraic components need not to coincide with DMF). Thank to a result of Font [8] about the structure of Leibniz reduced models (see [10]) of Belnap four-valued logic, we can prove the following.

Theorem 6. *Let $\boldsymbol{A}$ be non-trivial. $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^{*}\mathcal{L}_{\{1\}}$ if and only if the following conditions hold:*

1. *$\boldsymbol{A} \in \mathsf{DMF}$;*
2. *$\boldsymbol{A}$ has a maximum 1 and $F = \{1\}$;*
3. *if $a < b$, then there is $c \in \boldsymbol{A}$ such that $a \vee c < b \vee c = 1$ for every $a, b \geqslant n$.*

This result is indeed curious since it tells us that reduced models enjoy a local behaviour, in the sense that the fact that a model is reduced depends on the structure of the positive cone of the algebra. This can be stated in a nicer way making use of the characterisation of the Leibniz reduced models of the $\{\wedge, \vee\}$-fragment of classical logic given by Font, Guzmán and Verdú in [9].

Corollary 7. *Let $\boldsymbol{A} \in \mathsf{DMF}$. $\boldsymbol{A} \in \mathsf{Alg}^{*}\mathcal{L}_{\{1\}}$ if and only if $\uparrow(\boldsymbol{A}) \in \mathsf{Alg}^{*}\mathcal{CPC}_{\{\wedge,\vee,\bot\}}$.*

We conclude our semantical analysis of $\mathcal{L}_{\{1\}}$, by providing a fully adeguate, in the sense of [10], Gentzen system $\mathfrak{G}$ for it. Even if $\mathcal{L}_{\{1\}}$ is not algebrizable, we can prove that $\mathfrak{G}$ is. In order to explain this fact, let us denote by **Seq** the set of sequents whose premisses are non-empty finite sets of formulas and by **Eq** the set of equations. We let $\tau\colon \mathcal{P}(\mathbf{Seq}) \longrightarrow \mathcal{P}(\mathbf{Eq})\colon \rho$ be the residuated mappings defined as

$$\tau(\Gamma \rhd \alpha) = \bigwedge \Gamma \leqslant \alpha \vee n \qquad \rho(\alpha \approx \beta) = \{\alpha \lhd \rhd \beta, \neg\alpha \lhd \rhd \neg\beta\}$$

for every $\Gamma \rhd \alpha \in \mathbf{Seq}$ and $\alpha \approx \beta \in \mathbf{Eq}$.

Theorem 8. *$\mathfrak{G}$ is algebraizable with equivalent algebraic semantics DMF via τ and ρ.*

References

[1] R. Balbes and P. Dwinger. *Distributive lattices.* University of Missouri Press, Columbia, Mo., 1974.

[2] W. J. Blok and D. Pigozzi. Protoalgebraic logics. *Studia Logica*, 45:337–369, 1986.

[3] W. J. Blok and D. Pigozzi. *Algebraizable logics*, volume 396 of *Mem. Amer. Math. Soc.* A.M.S., Providence, January 1989.

[4] F. Bou and U. Rivieccio. The logic of distributive bilattices. *Logic Journal of the I.G.P.L.*, 19(1):183–216, 2011.

[5] S. Burris and H. P. Sankappanavar. *A course in Universal Algebra.* Springer-Verlag, New York, 1981.

[6] R. Cignoli. The class of Kleene algebras satisfying an interpolation property and Nelson algebras. *Algebra Universalis*, 23(3):262–292, 1986.

[7] J. P. Cleave. *A study of logics*, volume 18 of *Oxford Logic Guides.* The Clarendon Press Oxford University Press, New York, 1991. Oxford Science Publications.

[8] J. M. Font. Belnap's four-valued logic and De Morgan lattices. *Logic Journal of the I.G.P.L.*, 5(3):413–440, 1997.

[9] J. M. Font, F. Guzmán, and V. Verdú. Characterization of the reduced matrices for the $\{\wedge, \vee\}$-fragment of classical logic. *Bulletin of the Section of Logic*, 20:124–128, 1991.

[10] J. M. Font and R. Jansana. *A general algebraic semantics for sentential logics*, volume 7 of *Lecture Notes in Logic.* Springer-Verlag, second edition, 2009. `projecteuclid.org/euclid.lnl/1235416965`.

[11] J. M. Font and V. Verdú. Algebraic logic for classical conjunction and disjunction. *Studia Logica, Special Issue on Algebraic Logic*, 50:391–419, 1991.

[12] M. Negri. Dmf-algebras: representation and topological characterisation. *Boll. Un. Mat. Ital. 1-B (8)*, 1998.

[13] J. G. Raftery. The equational definability of truth predicates. *Reports on Mathematical Logic*, (41):95–149, 2006.

[14] J. Rebagliato and V. Verdú. Algebraizable Gentzen systems and the deduction theorem for Gentzen systems. Mathematics Preprint Series 175, University of Barcelona, June 1995.

[15] U. Rivieccio. *An Algebraic Study of Bilattice-based Logics.* Ph. D. Dissertation, University of Barcelona, 2010.

Dynamic many-valued logics for searching games with errors

Bruno Teheux[*]

Introduction

Providing concrete interpretations of many-valued logics has always been an intriguing problem. In [6], Mundici develops a model of the Rényi - Ulam searching games with lies in terms of Łukasiewicz logic and MV-algebras. In this game, a liar picks out a number in a given search space M. A detective has to guess this number by asking Yes/No questions to the liar who is allowed to lie a maximum given number of times.

In his model of the game, Mundici interprets the states of knowledge of the detective at a given step of the game as an element of an MV-algebra. Even though this model provides a way to interpret the effect of the liar's answers on the states of knowledge of the game, its language (the language of MV-algebras) is not rich enough to state specifications about a *whole round* of the game.

The starting point of this talk is the will to add a "dynamic" layer to this "static" interpretation of the game. We actually develop finitely-valued generalizations of Propositional Dynamic Logic, which is a multi-modal logic designed to reason about programs (see [2, 5]). Informally, these new logics are a mixture of many-valued modal logics (as introduced in [1, 3, 4]) and algebras of regular programs.

$n+1$-valued Kripke models

We fix $n \geq 1$ for the remainder of the paper and we denote by $Ł_n$ the sub-MV-algebra $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ of $[0,1]$.

We denote by Π a set of programs and by Form a set of formulas defined from a countable set Prop of propositional variables $p, q, \ldots$ and a countable set Π_0 of atomic programs $a, b, \ldots$ by the following Backus-Naur forms (where ϕ are formulas and α are programs) :

$$\phi ::= p \mid 0 \mid \neg\phi \mid \phi \to \phi \mid [\alpha]\phi$$
$$\alpha ::= a \mid \phi? \mid \alpha;\alpha \mid \alpha \cup \alpha \mid \alpha^*.$$

Definition 1. *An $n+1$-valued* Kripke *model $\mathcal{M} = \langle W, R, \mathrm{Val}\rangle$ is given by a non empty set W, a map $R : \Pi_0 \to 2^{W\times W}$ that assigns a binary relation R_a to any a of Π_0 and a map $\mathrm{Val} : W \times \mathsf{Prop} \to Ł_n$ that assigns a truth value to any propositional variable p of Prop in any world w of W.*

The maps R and Val are extended by mutual induction to formulas and programs by the following rules (where $\neg^{[0,1]}$ and $\to^{[0,1]}$ denote Łukasiewicz*'s interpretation of $\neg$ and $\to$ on $[0,1]$):*

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1. $R_{\alpha;\beta} = R_\alpha \circ R_\beta$;
2. $R_{\alpha\cup\beta} = R_\alpha \cap R_\beta$;
3. $R_{\psi?} = \{(u,u) \mid \mathrm{Val}(u,\psi) = 1\}$;
4. $R_{\alpha^*} = \bigcup_{n\in\omega}(R_\alpha)^n$;
5. $\mathrm{Val}(w, \phi \to \psi) = \mathrm{Val}(w,\phi) \to^{[0,1]} \mathrm{Val}(w,\psi)$;
6. $\mathrm{Val}(w, \neg\psi) = \neg^{[0,1]}\mathrm{Val}(w,\psi)$;
7. $\mathrm{Val}(w, [\alpha]\psi) = \bigwedge\{\mathrm{Val}(v,\psi) \mid (w,v) \in R_\alpha\}$

If w is a world of a Kripke *model $\mathcal{M}$ and if* $\mathrm{Val}(w,\phi) = 1$, *we write $\mathcal{M}, w \models \phi$ and say that ϕ* is true in w. *If ϕ is a formula that is true in each world of a model $\mathcal{M}$ then ϕ is* true in $\mathcal{M}$. *A formula that is true in every* Kripke *model is called a* tautology.

Hence, we intend to interpret the operator ';' as the concatenation program operator, the operator '∪' as the alternative program operator and the operator '∗' as the Kleene program operator.

$n+1$-valued propositional dynamic logics

The purpose of the talk is to characterize the theory of the $n+1$-valued Kripke models (Theorem 5).

Definition 2. *An $n+1$*-valued propositional dynamic logic *(or simply a* logic*) is a subset* L *of* Form *that is closed under the rules of* modus ponens, *uniform substitution and necessitation (generalization) and that contains the following axioms:*

1. *tautologies of the $n+1$-valued* Łukasiewicz *logic;*
2. *for any program α, axioms defining modality $[\alpha]$:*
   - (a) $[\alpha](p \to q) \to ([\alpha]p \to [\alpha]q)$,
   - (b) $[\alpha](p \oplus p) \leftrightarrow [\alpha]p \oplus [\alpha]p$,
   - (c) $[\alpha](p \odot p) \leftrightarrow [\alpha]p \odot [\alpha]p$,
3. *the axioms that define the program operators: for any programs α and β of Π:*
   - (a) $[\alpha \cup \beta]p \leftrightarrow [\alpha]p \wedge [\beta]p$,
   - (b) $[\alpha;\beta]p \leftrightarrow [\alpha][\beta]p$,
   - (c) $[q?]p \leftrightarrow (\neg q^n \vee p)$,
   - (d) $[\alpha^*]p \leftrightarrow (p \wedge [\alpha][\alpha^*]p)$,
   - (e) $[\alpha^*]p \to [\alpha^*][\alpha^*]p$,
4. *the induction axiom $(p \wedge [\alpha^*](p \to [\alpha]p)^n) \to [\alpha^*]p$ for any program α.*

We denote by PDL_n the smallest $n+1$-valued propositional dynamic logic.

As usual, a formula ϕ that belongs to a logic L *is called a* theorem *of* L.

Completeness result

The classical construction of the canonical model can be adapted for PDL_n. We denote by $\mathcal{F}_n$ the Lindenbaum - Tarski algebra of PDL_n. The reduct of $\mathcal{F}_n$ to the language of MV-algebras is an MV-algebra. We denote by $\mathcal{MV}(\mathcal{F}_n, \text{Ł}_n)$ the set of MV-homomorphisms from the MV-reduct of $\mathcal{F}_n$ to Ł_n.

Definition 3. *The* canonical model *of* PDL_n *is defined as the model* $\mathcal{M}^c = \langle W^c, R^c, \mathrm{Val}^c \rangle$ *where*

1. $W^c = \mathcal{MV}(\mathcal{F}_n, Ł_n)$;
2. *if* $\alpha \in \Pi$, *the relation* R^c_α *is defined by*

$$R^c_\alpha = \{(u, v) \mid \forall \phi \in \mathcal{F}_n \ (u([\alpha]\phi) = 1 \Rightarrow v(\phi) = 1)\};$$

3. *the map* Val^c *is defined by*

$$\mathrm{Val}^c : W^c \times \mathsf{Form} : (u, \phi) \mapsto u(\phi).$$

Even though the valuation in $\mathcal{M}^c$ is defined for any formula, it turns out that it is compatible with the inductive definition of a valuation in a KRIPKE model.

Proposition 4. 1. *If* $\phi \in \mathsf{Form}$, *if* $\alpha \in \Pi$ *and if* u *is a world of* W^c *then* $\mathrm{Val}^c(u, [\alpha]\phi) = \bigwedge\{\mathrm{Val}^c(v, \phi) \mid v \in R^c_\alpha u\}$.

2. *For any* $\alpha \in \Pi$, *the relation* R_{α^*} *is a reflexive and transitive extension of* R_α.

According to the second item of the previous proposition, the canonical model may not be KRIPKE model. Nevertheless, it is possible to use a filtration lemma in order to use the canonical model to obtain a completeness result for PDL_n.

Theorem 5. *The logic* PDL_n *is complete with respect to the* $n+1$*-valued* KRIPKE *models,* i.e., *a formula* ϕ *is a theorem of* PDL_n *if and only if* ϕ *is a tautology.*

References

[1] Félix Bou, Francesc Esteva, Lluís Godo, and Ricardo Oscar Rodríguez. On the minimum many-valued modal logic over a finite residuated lattice. *J. Logic Comput.*, 21(5):739–790, 2011.

[2] Michael J. Fischer and Richard E. Ladner. Propositional dynamic logic of regular programs. *J. Comput. System Sci.*, 18(2):194–211, 1979.

[3] Georges Hansoul and Bruno Teheux. Completeness results for many-valued łukasiewicz modal systems and relational semantics. 2006. arXiv:math/0612542.

[4] Georges Hansoul and Bruno Teheux. Extending łukasiewicz logics with a modality: Algebraic approach to relational semantics. *Studia Logica*, 2012. in press, http://dx.doi.org/10.1007/s11225-012-9396-9.

[5] David Harel, Dexter Kozen, and Jerzy Tiuryn. *Dynamic logic.* Foundations of Computing Series. MIT Press, Cambridge, MA, 2000.

[6] Daniele Mundici. The logic of Ulam's game with lies. In *Knowledge, belief, and strategic interaction (Castiglioncello, 1989)*, Cambridge Stud. Probab. Induc. Decis. Theory, pages 275–284. Cambridge Univ. Press, Cambridge, 1992.

Lattice BCK logics with Modus Ponens as the only rule

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Lattice BCK logic is the expansion of the well known Meredith implicational logic BCK [4] expanded with lattice conjunction and disjunction obtained by the following calculus.

Axioms:

B $(\varphi \to \psi) \to ((\psi \to \xi) \to (\varphi \to \xi)$

C $(\varphi \to (\psi \to \xi)) \to (\psi \to (\varphi \to \xi))$

K $\varphi \to (\psi \to \varphi)$

$\vee$**1** $\varphi \to \varphi \vee \psi$

$\vee$**2** $\psi \to \varphi \vee \psi$

$\wedge$**1** $\varphi \wedge \psi \to \varphi$

$\wedge$**2** $\varphi \wedge \psi \to \psi$

Rules:

M.P. $\{\varphi, \varphi \to \psi\} \vdash \psi$

$\vee$ **rule** $\{\varphi \to \xi, \psi \to \xi\} \vdash \varphi \vee \psi \to \xi$

$\wedge$ **rule** $\{\xi \to \varphi, \xi \to \psi\} \vdash \xi \to \varphi \wedge \psi$

We recall that BCK-logic is an algebraizable logic and its equivalent semantics is the quasivariety of BCK-algebras [1, 5]. Similarly, lattice BCK logic is algebraizable and its equivalent semantics is the class of all BCK-lattices.

An algebra $\langle A; \to, \wedge, \vee, \top\rangle$ of type $(2, 2, 2, 0)$ is a **BCK-lattice** provided that its reduct $\langle A; \to \top\rangle$ is a BCK-algebra whose natural order gives a lattice, with $\wedge$ as meet and $\vee$ as join [2].

In this case the class of BCK-lattices is equational and hence it is a variety. Therefore axiomatic extensions of Lattice BCK-logic are in one to one correspondence with varieties of BCK-lattices.

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It is easy to see that in the previous axiomatic presentation of lattice BCK-logic the $\vee$-rule can be omitted and exchanged by the following axiom

$$(\varphi \to \xi) \wedge (\psi \to \xi) \to (\varphi \vee \psi \to \xi).$$

However this can not be done for the $\wedge$-rule. In fact, using algebraizability and a matrix model obtained from a BCK-lattice given by Idziak in [3], we show that there is no axiomatic presentation of lattice BCK-logic with Modus Ponens as the only rule.

Meanwhile there are (wide enough) known axiomatic extensions of lattice BCK-logic admitting modus ponens as the only rule. For instance all axiomatic extensions of the implication, conjunction, disjunction fragment of the FL_{ew} logic [3] which can be obtained from Lattice BCK logic axioms plus

$$(\xi \to \varphi) \wedge (\xi \to \psi) \to (\xi \to \varphi \wedge \psi).$$

For every n, m natural numbers we denote by $\mathbf{LatBCK_{n,m}}$ the axiomatic extension of lattice BCK-logic by adding the axiom

$$(\xi \to \varphi)^n \to ((\xi \to \psi)^m \to (\xi \to \varphi \wedge \psi)).$$

Theorem 1.

1. *An axiomatic extension of Lattice BCK logic admits a presentation with modus ponens as the only rule if and only if it is an axiomatic extension of $LatBCK_{n,m}$ for some n, m.*

2. *If $n < m$, $LatBCK_{n,n}$ is a proper axiomatic extension of $LatBCK_{m,m}$.*

3. *There is no weakest axiomatic extension of Lattice BCK-logic admitting modus ponens as the only rule.*

References

[1] W. Blok and J.G. Raftery. On the quasivariety of BCK-algebras and its subvarieties. *Algebra Universalis*, **33**(1995), 68–90.

[2] P.M. Idziak. Lattice operations in BCK-algebras. *Math. Japonica*, **29** (1984), 839–846.

[3] P.M. Idziak. Filters and congruences in BCK-semilattices. *Math. Japonica*, **29** (1984), 975–980.

[4] C.A. Meredith and A.N. Prior. Notes on the axiomatics of propositional calculus. *Notre Dame Journal of formal logic*, **4** (1963), 171–187.

[5] A. Wroński. BCK algebras do not form a variety, *Math. Japonica*, **28** (1983), 211–213.

Image-finite first-order structures

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One of the well known difficulties (see [4, p. 236] or [1]) when dealing with first-order many-valued logics is the requirement to consider *safe* structures, i.e., those ones which have the necessary infima and suprema for computing the values of all first-order formulas.

Besides trivial cases like witnessed structures (which include finite ones) [2], in general it is quite difficult to show that a particular structure is safe. Another quite simple case, and apparently not previously considered in the literature, of safe structures is the one provided by what we call here image-finite. By definition, an structure is *image-finite* when, for each one of the predicate symbols in the vocabulary, its interpretation only takes a finite number of values in the many-valued chain considered. As previously pointed, it is not difficult to prove that all image-finite structures are safe (indeed witnessed), and another straightforward result is the following one.

Lemma. Let us assume that K_1 and K_2 are two classes of MTL-chains generating the same variety. Then, image-finite structures over chains in K_1 and image-finite structures over chains in K_2 share the same 1-valid sentences.

The previous result can be used to prove the statements below.

Theorem (Łukasiewicz Standard Semantics). The following families of first-order structures have the same set of 1-valid sentences.

- image-finite structures over the class of all MV-chains.
- (image-finite) first-order structures over the standard MV-chain.
- (image-finite) first-order structures over the rational standard MV-chain.
- (image-finite) first-order structures over some subalgebra of the standard MV-chain generated by one irrational.

Theorem (Monadic vocabulary with just one variable). Let us assume that the vocabulary only consists on unary predicate symbols, and let us just consider formulas using only one variable. Then, the family of structures given in the previous theorem and sharing the very set of 1-valid sentences can be enlarged with:

- (finite) first-order structures over the class of all MV-chains.
- finite first-order structures over the standard MV-chain.

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The statement in this last result was already obtained by a different (and very messy) method by Rutledge in his PhD dissertation [5, Chapter IV] (see also [3]), but up to now there were no alternative proofs available in the literature.

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References

[1] P. Hájek. *Metamathematics of fuzzy logic*, volume 4 of *Trends in Logic—Studia Logica Library.* Kluwer Academic Publishers, Dordrecht, 1998.

[2] P. Hájek. On witnessed models in fuzzy logic. *Mathematical Logic Quarterly*, 53(1):66–77, 2007.

[3] P. Hájek. On fuzzy modal logics $S5(\mathcal{C})$. *Fuzzy Sets and Systems*, 161(18):2389–2396, 2010.

[4] H. Rasiowa and R. Sikorski. *The mathematics of metamathematics.* PWN—Polish Scientific Publishers, Warsaw, third edition, 1970. Monografie Matematyczne, Tom 41.

[5] J. D. Rutledge. *A preliminary investigation of the infinitely-many-valued predicate calculus.* Ph. D. Dissertation, Cornell University, 1959.