Skolemization for Substructural Logics

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Abstract. The usual Skolemization procedure, which removes strong quantifiers by introducing new function symbols, is in general unsound for first-order substructural logics defined based on classes of complete residuated lattices. However, it is shown here (following similar ideas of Baaz and Iemhoff for first-order intermediate logics in [1]) that first-order substructural logics with a semantics satisfying certain witnessing conditions admit a "parallel" Skolemization procedure where a strong quantifier is removed by introducing a finite disjunction or conjunction (as appropriate) of formulas with multiple new function symbols. These logics typically lack equivalent prenex forms. Also, semantic consequence does not in general reduce to satisfiability. The Skolemization theorems presented here therefore take various forms, applying to the left or right of the consequence relation, and to all formulas or only prenex formulas.

1 Introduction

Skolemization is an important ingredient of automated reasoning methods in (fragments of) first-order classical logic. Crucially, a sentence $(\forall\bar{x})(\exists y)\varphi(\bar{x}, y)$ is classically satisfiable if and only if $(\forall\bar{x})\varphi(\bar{x}, f(\bar{x}))$ is satisfiable, where f is a function symbol not occurring in φ. The satisfiability of a sentence in prenex form therefore reduces to the satisfiability of a universal sentence; Herbrand's theorem then permits a further reduction to the satisfiability of a set of propositional formulas. For more details on the classical case, we refer the reader to [3].

For first-order non-classical logics, the situation is not so straightforward. Formulas are not always equivalent to prenex formulas and semantic consequence may not reduce to satisfiability, meaning that (non-prenex) sentences should be considered separately as premises and conclusions of consequences. A Skolemization procedure may in such cases be more carefully defined where strong occurrences of quantifiers in subformulas are replaced on the left, and weak occurrences on the right. However, satisfiability or, more generally, semantic consequence, may not be preserved. Notably, in first-order intuitionistic logic,

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formulas such as $\neg\neg(\forall x)P(x) \to (\forall x)\neg\neg P(x)$ do not skolemize (see, e.g., [15, 2], also for methods for addressing these problems).

The goal of this paper is to develop Skolemization theorems for first-order substructural logics based on residuated lattices, a family that spans first-order intermediate logics, exponential-free linear logic, relevance logics, fuzzy logics, and logics without contraction (see, e.g., [7, 9, 11, 14, 17]). Although these logics are in general undecidable, their (decidable) fragments provide foundations for knowledge representation and reasoning methods such as non-classical logic programming and description logics (see, e.g., [10, 12, 13, 18]). The work reported here aims to avoid duplicated research effort by providing a general approach to the development of automated reasoning methods in the substructural setting. A first step in this direction was taken in [6] which provides Herbrand theorems for these logics. Skolemization was also considered (briefly) in that paper, but unfortunately, the scope of the process was overstated in Theorem 1: the result applies only to first-order substructural logics based on classes of chains (totally ordered structures). An analysis of the failure of this theorem has, however, stimulated the new more general approach described in this paper. Future work will involve combining the various Herbrand and Skolem theorems obtained here and in [6] to develop resolution methods for a wide class of substructural logics.

The key idea of "parallel Skolemization" is to remove strong occurrences of quantifiers on the left of the consequence relation and weak occurrences of quantifiers on the right by introducing disjunctions and conjunctions, respectively, of formulas with multiple new function symbols. In particular, a sentence $(\forall \bar{x})(\exists y)\varphi(\bar{x}, y)$ occurring as the conclusion of a consequence is rewritten for some $n \in \mathbb{N}^+$ as $(\forall \bar{x}) \bigvee_{i=1}^{n} \varphi(\bar{x}, f_i(\bar{x}))$ where each function symbol f_i is new for $i = 1 \dots n$. Baaz and Iemhoff use this method in [1] to establish "full" Skolemization results for first-order intermediate logics whose Kripke models (with or without the constant domains condition) admit a finite model property. In this paper, we obtain full parallel Skolemization results for first-order substructural logics admitting certain new variants of the witnessed model property introduced by Hájek in [12]. We also obtain complete characterizations of full parallel Skolemization when these logics have a finitary consequence relation. We then turn our attention to first-order substructural logics that only partially satisfy a witnessing property and hence do not admit full parallel Skolemization. We show that under certain weaker conditions, these logics admit parallel Skolemization for prenex sentences occurring on the left or right of the consequence relation.

2 First-Order Substructural Logics

Predicates, interpreted classically as functions from the domain of a structure to the two element Boolean algebra **2**, are interpreted in first-order substructural logics as functions from the domain to algebras with multiple values that may represent, e.g., degrees of truth, belief, or confidence. For convenience, we consider here algebras for the full Lambek calculus with exchange – equivalently, intuitionistic linear logic without exponentials and additive constants – noting that a more general algebraic setting would lead to similar results, but complicate the presentation somewhat.

An $\mathrm{FL_e}$-*algebra* is an algebraic structure $\boldsymbol{A} = \langle A, \&, \to, \wedge, \vee, \overline{0}, \overline{1}\rangle$ such that:

1. $\langle A, \wedge, \vee\rangle$ is a lattice with an order defined by $x \leq y \Leftrightarrow x \wedge y = x$;
2. $\langle A, \&, \overline{1}\rangle$ is a commutative monoid;
3. $\to$ is the residuum of $\&$, i.e. $x \,\&\, y \leq z \;\Leftrightarrow\; x \leq y \to z$ for all $x, y, z \in A$.

$\boldsymbol{A}$ is called *complete* if $\bigvee X$ and $\bigwedge X$ exist in A for all $X \subseteq A$, and an $\mathrm{FL_e}$-*chain* if either $x \leq y$ or $y \leq x$ for all $x, y \in A$.

Example 1. Complete $\mathrm{FL_e}$-chains $\boldsymbol{A} = \langle [0,1], *, \to_*, \min, \max, d, e_*\rangle$ based on the real unit interval $[0,1]$ with the usual order have been studied intensively in mathematical fuzzy logic [7, 11, 14]. In this setting, $*$ is a residuated uninorm: an associative and commutative binary function on $[0,1]$ that is increasing in both arguments and has a unit e_* and residuum $\to_*$. (d is an arbitrary element in $[0,1]$.) Fundamental examples include the Łukasiewicz t-norm $\max(x+y-1, 0)$, the Gödel t-norm $\min(x,y)$, and the product t-norm $x \cdot y$.

The class $\mathbb{FL}_\mathrm{e}$ of $\mathrm{FL_e}$-algebras may be defined equationally and hence forms a *variety*: a class of algebras closed under taking homomorphic images, subalgebras, and products. Subvarieties of $\mathbb{FL}_\mathrm{e}$ provide algebraic semantics for a broad spectrum of substructural logics, including those defined via extensions of the sequent calculus for $\mathrm{FL_e}$. In particular, $\mathrm{FL_{ew}}$-algebras for $\mathrm{FL_e}$ with weakening are $\mathrm{FL_e}$-algebras satisfying $\overline{0} \leq x \leq \overline{1}$, and $\mathrm{FL_{ewc}}$-algebras for intuitionistic logic (term-equivalent to Heyting algebras) are $\mathrm{FL_{ew}}$-algebras satisfying $x \& x = x$. Further varieties consist of "involutive" $\mathrm{FL_e}$-algebras satisfying $(x \to \overline{0}) \to \overline{0} = x$ (corresponding to multiple-conclusion sequent calculi) and "semilinear" $\mathrm{FL_e}$-algebras satisfying $((x \to y) \wedge \overline{1}) \vee ((y \to x) \wedge \overline{1}) = \overline{1}$ (corresponding to hypersequent calculi). In particular, semilinear $\mathrm{FL_e}$-algebras, $\mathrm{FL_{ew}}$-algebras, and $\mathrm{FL_{ewc}}$-algebras provide algebraic semantics for, respectively, uninorm logic, monoidal t-norm logic, and Gödel logic (see [4, 5, 9, 14]).

A *(countable) predicate language* $\mathcal{P}$ is a triple $\langle \mathbf{P}, \mathbf{F}, \mathbf{ar}\rangle$ where $\mathbf{P}$ and $\mathbf{F}$ are non-empty countable sets of predicate and function symbols, respectively, and $\mathbf{ar}$ is a function assigning to each predicate and function symbol $\star$ an *arity* $\mathbf{ar}(\star) = n \in \mathbb{N}$ ($\star$ is called n-ary); nullary function symbols are called *object constants* and nullary predicate symbols are called *propositional atoms*. $\mathcal{P}$-*terms* $s, t, \ldots$, and *(atomic)* $\mathcal{P}$-*formulas* $\varphi, \psi, \chi, \ldots$ are defined as in classical logic using a fixed countably infinite set OV of *object variables* $x, y, \ldots$, quantifiers $\forall$ and $\exists$, binary connectives $\&, \to, \wedge, \vee$, and logical constants $\overline{0}, \overline{1}$. Also $\neg\varphi$ is defined as $\varphi \to \overline{0}$ and $\varphi \leftrightarrow \psi$ as $(\varphi \to \psi) \wedge (\psi \to \varphi)$.

Bound and free variables, closed terms, sentences, and substitutability are defined in the standard way. Instead of $\xi_1, \ldots, \xi_n$ (where the ξ_i's are terms or formulas and n is arbitrary or fixed by the context) we sometimes write just $\bar{\xi}$. By the notation $\varphi(\bar{z})$ we indicate that *all* free variables of φ occur in the list of distinct object variables $\bar{z}$. If $\varphi(x_1, \ldots, x_n, \bar{z})$ is a formula and all free occurrences of x_i's are replaced in φ by terms t_i, the resulting formula is denoted simply by

$\varphi(t_1, \dots, t_n, \bar{z})$. We write $\chi[\varphi]$ for a formula χ with a distinguished subformula φ and understand $\chi[\psi]$ as the result of replacing φ in χ with the formula ψ. A set of $\mathcal{P}$-formulas is called a *$\mathcal{P}$-theory*.

Classical notions of structure, evaluation, and truth are generalized relative to a complete $\mathrm{FL_e}$-algebra $\boldsymbol{A}$ as follows: a *$\mathcal{P}$-structure* $\mathfrak{S} = \langle \boldsymbol{A}, \mathbf{S}\rangle$ consists of a complete $\mathrm{FL_e}$-algebra $\boldsymbol{A}$ and a triple $\mathbf{S} = \langle S, \langle P^{\mathbf{S}}\rangle_{P\in\mathbf{P}}, \langle f^{\mathbf{S}}\rangle_{f\in\mathbf{F}}\rangle$ where S is a non-empty set, $P^{\mathbf{S}}$ is a function $S^n \to A$ for each n-ary predicate symbol $P \in \mathbf{P}$, and $f^{\mathbf{S}} \colon S^n \to S$ is a function for each n-ary function symbol $f \in \mathbf{F}$. An *$\mathfrak{S}$-evaluation* is a mapping $\mathrm{v}\colon OV \to S$. By $\mathrm{v}[x{\to}a]$ we denote the $\mathfrak{S}$-evaluation where $\mathrm{v}[x{\to}a](x) = a$ and $\mathrm{v}[x{\to}a](y) = \mathrm{v}(y)$ for each object variable $y \neq x$. Terms and formulas are evaluated in $\mathfrak{S}$ as follows:

$$
\begin{aligned}
\|x\|^{\mathfrak{S}}_{\mathrm{v}} &= \mathrm{v}(x) && \\
\|f(t_1,\dots,t_n)\|^{\mathfrak{S}}_{\mathrm{v}} &= f^{\mathbf{S}}(\|t_1\|^{\mathfrak{S}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathfrak{S}}_{\mathrm{v}}) && \text{for } f \in \mathbf{F}\\
\|P(t_1,\dots,t_n)\|^{\mathfrak{S}}_{\mathrm{v}} &= P^{\mathbf{S}}(\|t_1\|^{\mathfrak{S}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathfrak{S}}_{\mathrm{v}}) && \text{for } P \in \mathbf{P}\\
\|\varphi \circ \psi\|^{\mathfrak{S}}_{\mathrm{v}} &= \|\varphi\|^{\mathfrak{S}}_{\mathrm{v}} \circ^{\boldsymbol{A}} \|\psi\|^{\mathfrak{S}}_{\mathrm{v}} && \text{for } \circ \in \{\&, \to, \wedge, \vee\}\\
\|\star\|^{\mathfrak{S}}_{\mathrm{v}} &= \star^{\boldsymbol{A}} && \text{for } \star \in \{\overline{0}, \overline{1}\}\\
\|(\forall x)\varphi\|^{\mathfrak{S}}_{\mathrm{v}} &= \inf\nolimits_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathfrak{S}}_{\mathrm{v}[x\to a]} \mid a \in S\} &&\\
\|(\exists x)\varphi\|^{\mathfrak{S}}_{\mathrm{v}} &= \sup\nolimits_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathfrak{S}}_{\mathrm{v}[x\to a]} \mid a \in S\}. &&
\end{aligned}
$$

A $\mathcal{P}$-structure $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ is a *$\mathcal{P}$-model* of a $\mathcal{P}$-theory T, written $\mathfrak{M} \models T$, if for each $\varphi \in T$ and $\mathfrak{M}$-evaluation v, $\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}} \geq \overline{1}^{\boldsymbol{A}}$.

Let us now fix an arbitrary class $\mathbb{K}$ of complete $\mathrm{FL_e}$-algebras. A $\mathcal{P}$-formula φ is a *semantic consequence* of a $\mathcal{P}$-theory T in $\mathbb{K}$, written $T \models^{\mathcal{P}}_{\mathbb{K}} \varphi$, if $\mathfrak{M} \models \varphi$ for each $\boldsymbol{A} \in \mathbb{K}$ and each $\mathcal{P}$-model $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ of T. We omit the prefixes for the class $\mathbb{K}$ or language $\mathcal{P}$ when known from the context.

To simplify notation, for a formula $\varphi(x_1, \dots, x_n)$ and an $\mathfrak{S}$-evaluation v with $\mathrm{v}(x_i) = a_i$, we write $\|\varphi(a_1, \dots, a_n)\|^{\mathfrak{S}}$ instead of $\|\varphi(x_1, \dots, x_n)\|^{\mathfrak{S}}_{\mathrm{v}}$. Observe that, as in classical logic, the truth value of a sentence does not depend on an evaluation. Also, $\mathfrak{M} \models \varphi \to \psi$ iff for each evaluation v, $\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}} \leq \|\psi\|^{\mathfrak{M}}_{\mathrm{v}}$, and $\mathfrak{M} \models \varphi \leftrightarrow \psi$ iff for each evaluation v, $\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}} = \|\psi\|^{\mathfrak{M}}_{\mathrm{v}}$.

The next lemma collects together some useful facts for $\mathrm{FL_e}$-algebras.

Lemma 1 ([7, 14, 16]). *Given formulas φ, ψ, χ, a variable x not free in χ, and a term t substitutable for x in φ:*

1. $\models_{\mathbb{K}} (\forall x)\varphi(x) \to \varphi(t)$
2. $\models_{\mathbb{K}} \varphi(t) \to (\exists x)\varphi(x)$
3. $\models_{\mathbb{K}} (\forall x)(\chi \to \varphi) \leftrightarrow (\chi \to (\forall x)\varphi)$
4. $\models_{\mathbb{K}} (\forall x)(\varphi \to \chi) \leftrightarrow ((\exists x)\varphi \to \chi)$
5. $\{\varphi, \varphi \to \psi\} \models_{\mathbb{K}} \psi$
6. $\{\varphi\} \models_{\mathbb{K}} \varphi \wedge \overline{1}$
7. $\{\varphi\} \models_{\mathbb{K}} (\forall x)\varphi$
8. $\models_{\mathbb{K}} (\exists x)(\chi \to \varphi) \to (\chi \to (\exists x)\varphi)$
9. $\models_{\mathbb{K}} (\exists x)(\varphi \to \chi) \to ((\forall x)\varphi \to \chi)$
10. $\models_{\mathbb{K}} (\chi \mathbin{\&} (\exists x)\varphi) \leftrightarrow (\exists x)(\chi \mathbin{\&} \varphi)$
11. $\models_{\mathbb{K}} (\exists x)(\varphi \vee \psi) \leftrightarrow ((\exists x)\varphi \vee (\exists x)\psi)$
12. $\models_{\mathbb{K}} (\chi \vee (\forall x)\varphi) \to (\forall x)(\chi \vee \varphi)$
13. $\models_{\mathbb{K}} ((\forall x)\varphi \wedge (\forall x)\psi) \leftrightarrow (\forall x)(\varphi \wedge \psi)$
14. $\models_{\mathbb{K}} (\exists x)(\chi \wedge \varphi) \to (\chi \wedge (\exists x)\varphi)$.

Moreover, if $\mathbb{K}$ is a class of complete $\mathrm{FL_e}$*-chains:*

15. $\models_{\mathbb{K}} (\forall x)(\chi \vee \varphi) \leftrightarrow \chi \vee (\forall x)\varphi$
16. $\models_{\mathbb{K}} (\exists x)(\chi \wedge \varphi) \leftrightarrow \chi \wedge (\exists x)\varphi$.

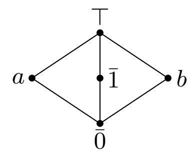


Fig. 1.

Let us emphasize that some quantifier shifts (8–14) are available for every choice of $\mathbb{K}$, and two more (15–16) if $\mathbb{K}$ consists of $\mathrm{FL_e}$-chains, but that, in general, the formulas $(\chi \to (\exists x)\varphi) \to (\exists x)(\chi \to \varphi)$, $((\forall x)\varphi \to \chi) \to (\exists x)(\varphi \to \chi)$, and $(\forall x)(\chi \,\&\, \varphi) \to (\chi \,\&\, (\forall x)\varphi)$ (where x is not free in χ) are not valid (see, e.g., [7]).

3 Parallel Skolemization

Skolemization fails in many first-order substructural logics. Consider, for example, a language with a binary predicate symbol P and object constants r and s, and a structure $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M} \rangle$ of this language where

- $\boldsymbol{A}$ is the $\mathrm{FL_e}$-algebra $\langle A, \&, \to, \wedge, \vee, \bar{0}, \bar{1} \rangle$ depicted in Figure 1 with

$$x \,\&\, y = \begin{cases} x \wedge y & \text{if } x, y \in \{0, a, b, \top\} \\ x & \text{if } y = \bar{1} \\ y & \text{if } x = \bar{1} \end{cases}$$

  and $\to$ is the residuum of $\&$;
- $M = \{r, s\}$ with $r^{\mathbf{M}} = r$, $s^{\mathbf{M}} = s$, $P^{\mathbf{M}}(s, s) = P^{\mathbf{M}}(r, s) = a$, $P^{\mathbf{M}}(r, r) = \bar{1}$, and $P^{\mathbf{M}}(s, r) = b$.

Then $\mathfrak{M}$ is a model of $(\forall x)(\forall z)(P(x, r) \vee P(z, s))$, but not of $(\exists y)(\forall x)P(x, y)$, since $\|(\exists y)(\forall x)P(x, y)\|^{\mathfrak{M}} = a \not\geq \bar{1}$, so

$$(\forall x)(\forall z)(P(x, r) \vee P(z, s)) \not\models_{\boldsymbol{A}} (\exists y)(\forall x)P(x, y).$$

On the other hand, for any unary function symbol f, we have

$$(\forall x)(\forall z)(P(x, r) \vee P(z, s)) \models_{\boldsymbol{A}} (\exists y)P(f(y), y).$$

Hence "ordinary" Skolemization in this case is not sound. Suppose, however, that we introduce *two* new unary function symbols f_1 and f_2. Then extending the same structure $\mathfrak{M}$ with interpretations $f_1^{\mathbf{M}}(r) = f_1^{\mathbf{M}}(s) = r$ and $f_2^{\mathbf{M}}(r) = f_2^{\mathbf{M}}(s) = s$, we obtain $\|(\exists y)(P(f_1(y), y) \wedge P(f_2(y), y))\|^{\mathfrak{M}} = a \not\geq \bar{1}$ and

$$(\forall x)(\forall z)(P(x, r) \vee P(z, s)) \not\models_{\boldsymbol{A}} (\exists y)(P(f_1(y), y) \wedge P(f_2(y), y)).$$

More generally (see Lemma 4) for any theory $T \cup \{(\exists \bar{y})(\forall x)\varphi(x, \bar{y})\}$ of this language and new function symbols f_1, f_2 of arity $|\bar{y}|$,

$$T \models_{\boldsymbol{A}} (\exists \bar{y})(\forall x)\varphi(x, \bar{y}) \quad \Leftrightarrow \quad T \models_{\boldsymbol{A}} (\exists \bar{y})(\varphi(f_1(\bar{y}), \bar{y}) \wedge \varphi(f_2(\bar{y}), \bar{y})).$$

We investigate here this "parallel Skolemization" procedure, introduced by Baaz and Iemhoff in [1] for intermediate logics, in the context of substructural logics.

Let us first recall some useful notions. An occurrence of a subformula ψ in a formula φ is *positive* (*negative*) if, inductively, one of the following holds:

1. φ is ψ;
2. φ is $\varphi_1 \wedge \varphi_2$, $\varphi_2 \wedge \varphi_1$, $\varphi_1 \vee \varphi_2$, $\varphi_2 \vee \varphi_1$, $\varphi_1 \mathbin{\&} \varphi_2$, $\varphi_2 \mathbin{\&} \varphi_1$, $(\forall x)\varphi_1$, $(\exists x)\varphi_1$, or $\varphi_2 \to \varphi_1$, and ψ is positive (negative) in $\varphi_1[\psi]$;
3. φ is $\varphi_1 \to \varphi_2$ and ψ is negative (positive) in $\varphi_1[\psi]$.

The following result is easily established by induction on formula complexity.

Lemma 2. *For $\mathcal{P}$-formulas φ, ψ, χ where ψ has the same free variables as χ:*

(i) *If ψ occurs positively in $\varphi[\psi]$, then $\{\chi \to \psi\} \models_{\mathbb{K}} \varphi[\chi] \to \varphi[\psi]$.*
(ii) *If ψ occurs negatively in $\varphi[\psi]$, then $\{\psi \to \chi\} \models_{\mathbb{K}} \varphi[\chi] \to \varphi[\psi]$.*

An occurrence of a quantified subformula $(Qx)\psi$ in a formula φ is called *strong* if either it is positive and $Q = \forall$, or it is negative and $Q = \exists$, *weak* otherwise.

Fix $n \in \mathbb{N}^+$ and consider a $\mathcal{P}$-sentence φ with a subformula $(Qx)\psi(x, \bar{y})$ and function symbols $f_1, \ldots, f_n \notin \mathcal{P}$ of arity $|\bar{y}|$. Replace this subformula in φ by

$$\bigvee_{i=1}^{n} \psi(f_i(\bar{y}), \bar{y}) \;\text{ if } Q = \exists \quad \text{and} \quad \bigwedge_{i=1}^{n} \psi(f_i(\bar{y}), \bar{y}) \;\text{ if } Q = \forall.$$

The replacement strictly decreases the multiset of depths of occurrences of quantifiers according to the standard multiset well-ordering described in [8]. Hence applying this process repeatedly to leftmost *strong* occurrences of quantifiers in an arbitrary $\mathcal{P}$-sentence φ results in a unique (up to renaming of function symbols) $\mathcal{P}'$-sentence $sk^r_n(\varphi)$ for some extension $\mathcal{P}'$ of $\mathcal{P}$ that contains only *weak* occurrences of quantifiers. Similarly, let $sk^l_n(\varphi)$ be the result of applying this process repeatedly to leftmost weak occurrences of quantifiers in φ.

Example 2. Consider a sentence $\varphi = (\forall x)((\exists y)P(x, y) \to (\exists z)Q(x, z))$. Taking $n = 1$, the above process leads to

$$sk^l_1(\varphi) = (\forall x)((\exists y)P(x, y) \to Q(x, g(x))) \quad \text{and} \quad sk^r_1(\varphi) = P(c, d) \to (\exists z)Q(c, z).$$

On the other hand, considering $n = 2$ and applying the procedure to weak occurrences of quantifiers in φ, we produce the formula $sk^l_2(\varphi)$

$$(\forall x)((\exists y)P(x, y) \to (Q(x, g_1(x)) \vee Q(x, g_2(x)))),$$

while applying it to strong occurrences, we obtain first

$$((\exists y)P(c_1, y) \to (\exists z)Q(c_1, z)) \wedge ((\exists y)P(c_2, y) \to (\exists z)Q(c_2, z)),$$

and then a formula $sk^r_2(\varphi)$ of the form

$$((P(c_1, d^1_1) \vee P(c_1, d^1_2)) \to (\exists z)Q(c_1, z)) \wedge ((P(c_2, d^2_1) \vee P(c_2, d^2_2)) \to (\exists z)Q(c_2, z)).$$

Let us fix an arbitrary class of complete $\mathrm{FL_e}$-algebras $\mathbb{K}$. We say that the consequence relation $\models_{\mathbb{K}}$ admits *parallel Skolemization right of degree* n, for a $\mathcal{P}$-sentence φ if for any $\mathcal{P}$-theory T,

$$T \models_{\mathbb{K}} \varphi \quad \Leftrightarrow \quad T \models_{\mathbb{K}} sk_n^r(\varphi).$$

Similarly, we say that $\models_{\mathbb{K}}$ admits *parallel Skolemization left of degree* n for a $\mathcal{P}$-sentence φ if for any $\mathcal{P}$-theory $T \cup \{\psi\}$,

$$T \cup \{\varphi\} \models_{\mathbb{K}} \psi \quad \Leftrightarrow \quad T \cup \{sk_n^l(\varphi)\} \models_{\mathbb{K}} \psi.$$

Note that there exists the following relationship between the left and right forms of parallel Skolemization.

Lemma 3. *If $\models_{\mathbb{K}}$ admits parallel Skolemization left of degree n for all sentences, then $\models_{\mathbb{K}}$ admits parallel Skolemization right of degree n for all sentences.*

Proof. For any $\mathcal{P}$-theory T, $\mathcal{P}$-sentence φ, and propositional atom P not occurring in $T \cup \{\varphi\}$:

$$\begin{aligned} T \models_{\mathbb{K}} \varphi \quad &\Leftrightarrow \quad T \cup \{\varphi \to P\} \models_{\mathbb{K}} P && (1)\\ &\Leftrightarrow \quad T \cup \{sk_n^l(\varphi \to P)\} \models_{\mathbb{K}} P && (2)\\ &\Leftrightarrow \quad T \cup \{sk_n^r(\varphi) \to P\} \models_{\mathbb{K}} P && (3)\\ &\Leftrightarrow \quad T \models_{\mathbb{K}} sk_n^r(\varphi). && (4) \end{aligned}$$

Equivalences (1) and (4) follow from [6, Corollary 1], (2) follows from the assumption that $\models_{\mathbb{K}}$ admits parallel Skolemization left of degree n for all $\mathcal{P}$-sentences, and (3) follows inductively from the definitions of $sk_n^l(\cdot)$ and $sk_n^r(\cdot)$. $\square$

We are unable to prove the converse direction to this lemma. Suppose, however, that $\models_{\mathbb{K}}$ admits the weaker version of the classical deduction theorem stating that for any $\mathcal{P}$-theory $T \cup \{\psi\}$ and $\mathcal{P}$-sentence φ:

$$T \cup \{\varphi\} \models_{\mathbb{K}} \psi \quad \Leftrightarrow \quad T \models_{\mathbb{K}} (\varphi \wedge \bar{1}) \to \psi.$$

Then if $\models_{\mathbb{K}}$ admits parallel Skolemization right of degree n for all $\mathcal{P}$-sentences, also $\models_{\mathbb{K}}$ admits parallel Skolemization left of degree n for all $\mathcal{P}$-sentences. Just note that for any $\mathcal{P}$-theory $T \cup \{\psi\}$ and $\mathcal{P}$-sentence φ:

$$\begin{aligned} T \cup \{\varphi\} \models_{\mathbb{K}} \psi \quad &\Leftrightarrow \quad T \models_{\mathbb{K}} (\varphi \wedge \bar{1}) \to \psi && (1)\\ &\Leftrightarrow \quad T \models_{\mathbb{K}} sk_n^r((\varphi \wedge \bar{1}) \to \psi) && (2)\\ &\Leftrightarrow \quad T \models_{\mathbb{K}} (sk_n^l(\varphi) \wedge \bar{1}) \to sk_n^r(\psi) && (3)\\ &\Leftrightarrow \quad T \cup \{sk_n^l(\varphi)\} \models_{\mathbb{K}} sk_n^r(\psi) && (4)\\ &\Leftrightarrow \quad T \cup \{sk_n^l(\varphi)\} \models_{\mathbb{K}} \psi. && (5) \end{aligned}$$

Equivalences (1) and (4) follow from the deduction theorem, (2) and (5) follow from the fact that $\models_{\mathbb{K}}$ admits parallel Skolemization right of degree n for all $\mathcal{P}$-sentences, and (3) follows inductively from the definitions of $sk_n^l(\cdot)$ and $sk_n^r(\cdot)$.

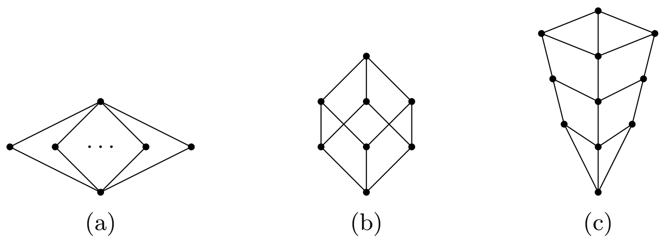


Fig. 2. Examples of 2-compact and 3-compact systems

4 Parallel Skolemization for All Formulas

In this section, we investigate consequence relations $\models_{\mathbb{K}}$ that admit parallel Skolemization of some fixed degree on the left and right for *any* sentence. This is a rather strong property for a consequence relation, but includes all cases where $\models_{\mathbb{K}}$ is equivalent to $\models_{\mathbb{K}'}$ for some finite class $\mathbb{K}'$ of finite algebras, as well as certain non-finite cases.

The crucial requirement for this form of Skolemization is the completeness of $\models_{\mathbb{K}}$ with respect to models based on algebras exhibiting some degree of "compactness". Let $\boldsymbol{L}$ be a lattice and $\mathcal{X} \subseteq \mathfrak{P}(L)$. We say that $\mathcal{X}$ is *n-compact* for some $n \in \mathbb{N}^+$ if for each $A \in \mathcal{X}$,

$$\bigvee A = a_1 \vee \ldots \vee a_n \quad \text{for some } a_1, \ldots, a_n \in A$$
$$\bigwedge A = a_1 \wedge \ldots \wedge a_n \quad \text{for some } a_1, \ldots, a_n \in A.$$

Example 3. It is easily seen that if the lattice $\boldsymbol{L}$ has *height* (the cardinality of a maximal chain in $\boldsymbol{L}$) smaller than $n+1$, then any $\mathcal{X} \subseteq \mathfrak{P}(L)$ is n-compact. If $\boldsymbol{L}$ contains no infinite chain and has *width* (the cardinality of a maximal anti-chain in $\boldsymbol{L}$) smaller than m, then any $\mathcal{X} \subseteq \mathfrak{P}(L)$ is m-compact. For example, the powerset of a lattice, as depicted in Figure 2(a), that consists of a (finite or infinite) set of incomparable elements together with a top element and a bottom element, is 2-compact (but not 1-compact). The powerset of the lattice in Figure 2(b), which may also be generalized by repeating many times the internal elements, is 3-compact (but not 2-compact). On the other hand, the powerset of the lattice in Figure 2(c) is 2-compact.

It is not necessary for parallel Skolemization that *all* sets of subsets of the algebras in $\mathbb{K}$ be n-compact, only that the set of definable sets of elements in a given $\mathcal{P}$-structure have this property. Let us call a $\mathcal{P}$-structure $\mathfrak{S} = \langle \boldsymbol{A}, \mathbf{S} \rangle$ *n-witnessed* if the following system is n-compact:

$$\{\{||\varphi(b, \bar{a})||^{\mathfrak{S}} \mid b \in S\} \mid \varphi(x, \bar{y}) \text{ a } \mathcal{P}\text{-formula and } \bar{a} \in S\}.$$

We say that the consequence relation $\models_{\mathbb{K}}$ has *the n-witnessed model property* if for any $\mathcal{P}$-theory $T \cup \{\varphi\}$,

$$T \models_{\mathbb{K}} \varphi \quad \Leftrightarrow \quad \text{each } n\text{-witnessed model } \mathfrak{M} \text{ of } T \text{ is a model of } \varphi.$$

Note that this new notion generalizes the (1-)witnessed model property introduced by Hájek in [12] (see also [7]).

Example 4. Suppose that $\mathbb{K}$ is a class of $\mathrm{FL_e}$-algebras whose underlying lattices *either* have height bounded by some fixed $n+1$, *or* contain no infinite chain and have width bounded by some fixed n (see Example 3). Then $\models_{\mathbb{K}}$ has the n-witnessed model property.

Example 5. Let us emphasize that it is not necessary for parallel Skolemization that all sets of subsets of the algebras in the class $\mathbb{K}$ are n-compact. Suppose, for example, that $\mathbb{K}$ consists of the standard Łukasiewicz algebra on $[0,1]$. The powerset of $[0,1]$ is clearly not n-compact for any $n \in \mathbb{N}^+$. However, $\models_{\mathbb{K}}$ has the 1-witnessed model property, as shown by Hájek in [12].

We turn our attention now to the relationship between the n-witnessed model property and parallel Skolemization left and right of degree n. We begin with a crucial lemma which can be seen as "one step" Skolemization on the left.

Lemma 4. *Suppose that $\models_{\mathbb{K}}$ has the n-witnessed model property.*

(a) *For any $\mathcal{P}$-theory $T \cup \{\chi, \psi[(\exists x)\varphi(x,\bar{y})]\}$ where $(\exists x)\varphi(x,\bar{y})$ occurs positively in ψ, for function symbols $f_1,\ldots,f_n \notin \mathcal{P}$ of arity $|\bar{y}|$,*

$$T \cup \{\psi[(\exists x)\varphi(x,\bar{y})]\} \models_{\mathbb{K}} \chi \quad \Leftrightarrow \quad T \cup \{\psi[\bigvee_{i=1}^{n} \varphi(f_i(\bar{y}),\bar{y})]\} \models_{\mathbb{K}} \chi.$$

(b) *For any $\mathcal{P}$-theory $T \cup \{\chi, \psi[(\forall x)\varphi(x,\bar{y})]\}$ where $(\forall x)\varphi(x,\bar{y})$ occurs negatively in ψ, for function symbols $f_1,\ldots,f_n \notin \mathcal{P}$ of arity $|\bar{y}|$,*

$$T \cup \{\psi[(\forall x)\varphi(x,\bar{y})]\} \models_{\mathbb{K}} \chi \quad \Leftrightarrow \quad T \cup \{\psi[\bigwedge_{i=1}^{n} \varphi(f_i(\bar{y}),\bar{y})]\} \models_{\mathbb{K}} \chi.$$

Proof. For the left-to-right directions for both (a) and (b), note that

$$\models_{\mathbb{K}} \bigvee_{i=1}^{n} \varphi(f_i(\bar{y}),\bar{y}) \to (\exists x)\varphi(x,\bar{y}) \quad \text{and} \quad \models_{\mathbb{K}} (\forall x)\varphi(x,\bar{y}) \to \bigwedge_{i=1}^{n} \varphi(f_i(\bar{y}),\bar{y}),$$

and hence, by Lemma 2, for (a) and (b), respectively,

$$\models_{\mathbb{K}} \psi[\textstyle\bigvee_{i=1}^{n} \varphi(f_i(\bar{y}),\bar{y})] \to \psi[(\exists x)\varphi(x,\bar{y})]$$
$$\text{and} \ \models_{\mathbb{K}} \psi[\textstyle\bigwedge_{i=1}^{n} \varphi(f_i(\bar{y}),\bar{y})] \to \psi[(\forall x)\varphi(x,\bar{y})].$$

We prove the right-to-left direction contrapositively just for (a), as (b) is very similar. Suppose that $T \cup \{\psi[(\exists x)\varphi(x,\bar{y})]\} \not\models_{\mathbb{K}} \chi$. So there is an n-witnessed model $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ of $T \cup \{\psi[(\exists x)\varphi(x,\bar{y})]\}$ such that $\mathfrak{M} \not\models_{\mathbb{K}} \chi$. Because $\mathfrak{M}$ is n-witnessed, for each $\bar{m} \in M$, there are $u_1^{\bar{m}},\ldots,u_n^{\bar{m}} \in M$ such that

$$\|(\exists x)\varphi(x,\bar{m})\|^{\mathfrak{M}} = \|\varphi(u_1^{\bar{m}},\bar{m})\|^{\mathfrak{M}} \vee \ldots \vee \|\varphi(u_n^{\bar{m}},\bar{m})\|^{\mathfrak{M}}.$$

Using the axiom of choice, we define $f_i(\bar{m}) = u_i^{\bar{m}}$ for each $i \in \{1,\ldots,n\}$. Then $\mathfrak{M}$, with these new interpretations, is a model of $T \cup \{\psi[\bigvee_{i=1}^{n} \varphi(f_i(\bar{y}),\bar{y})]\}$ and not χ. $\square$

Theorem 1. *If $\models_{\mathbb{K}}$ has the n-witnessed model property, then $\models_{\mathbb{K}}$ admits parallel Skolemization left and right of degree n for all sentences. Moreover, the converse implication also holds whenever $\models_{\mathbb{K}}$ is finitary, i.e., for any $\mathcal{P}$-theory $T \cup \{\varphi\}$,*

$$T \models_{\mathbb{K}} \varphi \quad \Leftrightarrow \quad T' \models_{\mathbb{K}} \varphi \ \text{ for some finite } T' \subseteq T.$$

Proof. Suppose that $\models_{\mathbb{K}}$ has the n-witnessed model property. Parallel Skolemization left of degree n for all $\mathcal{P}$-sentences follows from Lemma 4 and an induction on the multiset of depths of quantifier occurrences according to the standard multiset well-ordering from [8]. Parallel Skolemization right of degree n for all $\mathcal{P}$-sentences then follows from Lemma 3.

Next we prove the converse: suppose that $\models_{\mathbb{K}}$ is finitary and admits parallel Skolemization left of degree n for all $\mathcal{P}$-sentences. (Note that only Skolemization for certain formulas is needed for the proof). First we establish the following:

Claim. For each $\mathcal{P}$-theory $T \cup \{\varphi\}$ such that $T \not\models_{\mathbb{K}} \varphi$, there exist a language $\mathcal{P}' \supseteq \mathcal{P}$ and a $\mathcal{P}'$-theory $T' \supseteq T$ such that $T' \not\models_{\mathbb{K}} \varphi$ and, for each $\mathcal{P}$-formula $(Qx)\chi(x, \bar{y})$:

$$T' \models_{\mathbb{K}} (\forall \bar{y})((Qx)\chi(x, \bar{y}) \leftrightarrow \bigcirc_{i=1}^{n} \chi(f_i^{\chi}(\bar{y}), \bar{y})),$$

where $\bigcirc = \begin{cases} \bigvee & \text{if } Q = \exists \\ \bigwedge & \text{if } Q = \forall \end{cases}$, and $f_1^{\chi}, \dots, f_n^{\chi}$ are function symbols from $\mathcal{P}' \setminus \mathcal{P}$.

Proof of the claim. Let $\varphi_0, \varphi_1, \dots$ be an enumeration of all $\mathcal{P}$-formulas of the form $(\forall x)\chi(x, \bar{y})$ or $(\exists x)\chi(x, \bar{y})$ (recalling that $\mathcal{P}$ is always a countable language). We construct an increasing series of languages $\mathcal{P}_i$ and $\mathcal{P}_i$-theories T_i such that $T_i \not\models_{\mathbb{K}} \varphi$. Let $T_0 = T$ and $\mathcal{P}_0 = \mathcal{P}$. If φ_j has the form $(\forall x)\chi(x, \bar{y})$, then as $\models_{\mathbb{K}}$ admits parallel Skolemization left of degree n for all $\mathcal{P}$-sentences,

$$\begin{aligned} T_j \models_{\mathbb{K}} \varphi \quad &\Leftrightarrow \quad T_j \cup \{(\forall \bar{y})((\forall x)\chi(x, \bar{y}) \to (\forall x)\chi(x, \bar{y}))\} \models_{\mathbb{K}} \varphi \\ &\Leftrightarrow \quad T_j \cup \{(\forall \bar{y})(\bigwedge_{i=1}^{n} \chi(f_i^{\chi}(\bar{y}), \bar{y}) \to (\forall x)\chi(x, \bar{y}))\} \models_{\mathbb{K}} \varphi. \end{aligned}$$

We define $\mathcal{P}_{j+1}$ as the extension of $\mathcal{P}_j$ with the function symbols $f_1^{\chi}, \dots, f_n^{\chi}$ and

$$T_{j+1} = T_j \cup \{(\forall \bar{y})(\bigwedge_{i=1}^{n} \chi(f_i^{\chi}(\bar{y}), \bar{y}) \to (\forall x)\chi(x, \bar{y}))\}.$$

The case where φ_j has the form $(\exists x)\chi(x, \bar{y})$ is dealt with similarly. We then let $\mathcal{P}' = \bigcup_{j<\omega} \mathcal{P}_j$ and $T' = \bigcup_{j<\omega} T_j$. Because $\models_{\mathbb{K}}$ is finitary, $T' \not\models_{\mathbb{K}} \varphi$. Moreover, for a formula $(Qx)\chi(x, \bar{y}) = \varphi_j$ for some j and assuming that $Q = \exists$, we have $(\forall \bar{y})((\exists x)\chi(x, \bar{y}) \to \bigvee_{i=1}^{n} \chi(f_i^{\chi}(\bar{y}), \bar{y})) \in T'$ and as the converse implication is always provable the claim follows.

To complete the proof of the theorem, we just iterate the above claim over ω. We obtain a theory $\hat{T}$ whose models are clearly n-witnessed and $\hat{T} \not\models_{\mathbb{K}} \varphi$. ☐

A natural question to ask at this point is whether the requirement that $\models_{\mathbb{K}}$ be finitary is really necessary to obtain an equivalence in the previous theorem. We do not have an answer. Observe, however, that this requirement could be avoided if we allow Skolemization of infinitely many formulas on the left *simultaneously*.

Theorem 1 and Example 4 establish parallel Skolemization of some finite degree for $\models_{\mathbb{K}}$ for a broad family of classes $\mathbb{K}$ of $\mathrm{FL_e}$-algebras. Also, using Example 5, first-order Łukasiewicz logic based on the standard Łukasiewicz algebra on $[0,1]$ admits parallel Skolemization of degree 1. However, the consequence relation of this logic is not finitary, so we cannot obtain the 1-witnessed model property directly from the fact that it admits Skolemization left of degree 1.

5 Parallel Skolemization for Prenex Formulas

In the previous section, we proved that consequence relations satisfying a rather strong witnessed model property admit parallel Skolemization to some degree for all formulas. In this section, we investigate the (broader) scope of parallel Skolemization restricted to prenex formulas.

First we show that parallel Skolemization for prenex formulas on the right holds in the presence of a weaker witnessed model property. Let $\boldsymbol{L}$ be a lattice and consider $\mathcal{X} \subseteq \mathfrak{P}(L)$. We say that $\mathcal{X}$ is *n-$\wedge$-precompact* for some $n \in \mathbb{N}^+$ if for all $A \in \mathcal{X}$ and $b \in L$,

$$\bigwedge A < b \quad \Longrightarrow \quad a_1 \wedge \ldots \wedge a_n < b \;\text{ for some } a_1, \ldots, a_n \in A.$$

Example 6. The powerset of the (infinite) lattice depicted in Figure 3(a) is 1-$\wedge$-precompact (but not n-compact for any n), while the powerset of the (infinite) lattice in Figure 3(b) is 2-$\wedge$-precompact (but neither n-compact for any n, nor 1-$\wedge$-precompact).

We call a $\mathcal{P}$-structure $\mathfrak{S} = \langle \boldsymbol{A}, \mathbf{S} \rangle$ *n-$\wedge$-prewitnessed* if the following system is n-$\wedge$-precompact:

$$\{\{||\varphi(b,\bar{a})||^{\mathfrak{S}} \mid b \in S\} \mid \varphi(x,\bar{y}) \text{ a } \mathcal{P}\text{-formula and } \bar{a} \in S\}.$$

Then $\models_{\mathbb{K}}$ has the *n-$\wedge$-prewitnessed model property* if for any $\mathcal{P}$-theory $T \cup \{\varphi\}$,

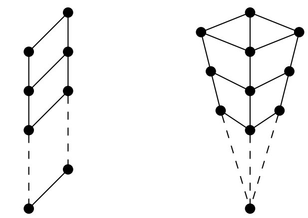
$$T \models_{\mathbb{K}} \varphi \quad \Leftrightarrow \quad \text{every } n\text{-}\wedge\text{-prewitnessed model } \mathfrak{M} \text{ of } T \text{ is a model of } \varphi.$$

Example 7. If $\boldsymbol{L}$ is a chain, then $\mathfrak{P}(L)$ is 1-$\wedge$-precompact and hence any logic based on chains enjoys the 1-$\wedge$-prewitnessed model property.

We show first that the n-$\wedge$-prewitnessed model property suffices to guarantee "one step" parallel Skolemization of degree n for formulas of a certain form occurring on the right of the consequence relation.

Theorem 2. *If $\models_{\mathbb{K}}$ has the n-$\wedge$-prewitnessed model property, then for any $\mathcal{P}$-theory $T \cup \{\varphi(x,\bar{y}), \psi\}$ and function symbols $f_1, \ldots, f_n \notin \mathcal{P}$ of arity $|\bar{y}|$:*

$$T \models_{\mathbb{K}} (\exists \bar{y})(\forall x)\varphi(x,\bar{y}) \quad \Leftrightarrow \quad T \models_{\mathbb{K}} (\exists \bar{y})(\bigwedge_{i=1}^{n} \varphi(f_i(\bar{y}),\bar{y})).$$



(a) (b)

Fig. 3. Examples of 1-$\wedge$-precompact and 2-$\wedge$-precompact systems

Proof. The left-to-right direction follows directly using Lemma 2. We prove the right-to-left direction contrapositively, assuming without loss of generality that T consists of $\mathcal{P}$-sentences. Suppose that $T \not\models_{\mathbb{K}} (\exists\bar{y})(\forall x)\varphi(x,\bar{y})$. Then there is an n-$\wedge$-prewitnessed model $\mathfrak{M} = \langle \boldsymbol{A}, \mathbf{M}\rangle$ of T such that $V = \|(\exists\bar{y})(\forall x)\varphi(x,\bar{y})\|^{\mathfrak{M}} \not\geq \bar{1}$, i.e., $V < V \vee \bar{1}$.

Suppose first that $V < V' < V \vee \bar{1}$ for some $V' \in A$. Clearly, for each $\bar{m} \in M$, $\|(\forall x)\varphi(x,\bar{m})\|^{\mathfrak{M}} \leq V < V'$. Since $\mathfrak{M}$ is n-$\wedge$-prewitnessed, for each $\bar{m} \in M$, there are $u_1^{\bar{m}}, \ldots, u_n^{\bar{m}} \in M$ such that $\|\varphi(u_1^{\bar{m}},\bar{m})\|^{\mathfrak{M}} \wedge \ldots \wedge \|\varphi(u_n^{\bar{m}},\bar{m})\|^{\mathfrak{M}} < V'$. Now for $i \in \{1,\ldots,n\}$, define, using the axiom of choice, $f_i(\bar{m}) = u_i^{\bar{m}}$. But then

$$\|(\exists\bar{y})(\textstyle\bigwedge_{i=1}^{n} \varphi(f_i(\bar{y}),\bar{y}))\|^{\mathfrak{M}} = \bigvee_{\bar{m}\in M} \bigwedge_{i=1}^{n} \|\varphi(f_i(\bar{m}),\bar{m})\|^{\mathfrak{M}} \leq V' < V \vee \bar{1}.$$

So $\|(\exists\bar{y})(\bigwedge\limits_{i=1}^{n} \varphi(f_i(\bar{y}),\bar{y}))\|^{\mathfrak{M}} \not\geq \bar{1}$.

Now suppose that no $V' \in A$ satisfies $V < V' < V \vee \bar{1}$. Clearly, for each $\bar{m} \in M$, $\|(\forall x)\varphi(x,\bar{m})\|^{\mathfrak{M}} \leq V < V \vee \bar{1}$. If $\|(\forall x)\varphi(x,\bar{m})\|^{\mathfrak{M}} < V$, then, as $\mathfrak{M}$ is n-$\wedge$-prewitnessed, we have $u_1^{\bar{m}}, \ldots, u_n^{\bar{m}} \in M$ such that $\|\varphi(u_1^{\bar{m}},\bar{m})\|^{\mathfrak{M}} \wedge \ldots \wedge \|\varphi(u_n^{\bar{m}},\bar{m})\|^{\mathfrak{M}} < V$. If $\|(\forall x)\varphi(x,\bar{m})\|^{\mathfrak{M}} = V$, then for some $u_1^{\bar{m}}, \ldots, u_n^{\bar{m}} \in M$,

$$\|(\forall x)\varphi(x,\bar{m})\|^{\mathfrak{M}} = V \leq \|\varphi(u_1^{\bar{m}},\bar{m})\|^{\mathfrak{M}} \wedge \ldots \wedge \|\varphi(u_n^{\bar{m}},\bar{m})\|^{\mathfrak{M}} < V \vee \bar{1}.$$

Hence, by assumption, $V = \|\varphi(u_1^{\bar{m}},\bar{m})\|^{\mathfrak{M}} \wedge \ldots \wedge \|\varphi(u_n^{\bar{m}},\bar{m})\|^{\mathfrak{M}}$. In both cases, for each $i \in \{1,\ldots,n\}$, define, using the axiom of choice, $f_i(\bar{m}) = u_i^{\bar{m}}$. But then

$$\|(\exists\bar{y})(\textstyle\bigwedge_{i=1}^{n} \varphi(f_i(\bar{y}),\bar{y}))\|^{\mathfrak{M}} = \bigvee_{\bar{m}} \bigwedge_{i=1}^{n} \|\varphi(f_i(\bar{m}),\bar{m})\|^{\mathfrak{M}} \leq V < V \vee \bar{1}.$$

So $\|(\exists\bar{y})(\bigwedge\limits_{i=1}^{n} \varphi(f_i(\bar{y}),\bar{y}))\|^{\mathfrak{M}} \not\geq \bar{1}$. □

In order to repeat this one step Skolemization process and obtain skolemized formulas for any prenex formula, we require an additional assumption, satisfied in particular whenever all algebras in $\mathbb{K}$ are frames (e.g., chains).

Theorem 3. *Suppose that $\models_{\mathbb{K}}$ has the n-$\wedge$-prewitnessed model property and for all $\mathcal{P}$-formulas φ and χ such that x is not free in χ:*

$$\models_{\mathbb{K}} (\chi \wedge (\exists x)\varphi) \to (\exists x)(\chi \wedge \varphi).$$

Then $\models_{\mathbb{K}}$ admits parallel Skolemization right of degree n for prenex sentences.

Proof. First we define $\wedge$-prenex $\mathcal{P}$-formulas as follows: every quantifier-free $\mathcal{P}$-formula is $\wedge$-prenex, and if φ, ψ are $\wedge$-prenex, then so are $\varphi \wedge \psi$, $(\exists x)\varphi$, and $(\forall x)\varphi$ for any variable x.

Now consider a $\mathcal{P}$-theory T and a $\wedge$-prenex $\mathcal{P}$-sentence χ with a leftmost strong quantifier occurrence $(\forall x)\varphi(x, \bar{y})$. Rewriting variables if necessary and using quantifier shifts, χ is equivalent to a sentence of the form

$$(\exists \bar{y})(\forall x)(\varphi(x, \bar{y}) \wedge \varphi'(\bar{y}))$$

and by Theorem 2,

$$T \models_{\mathbb{K}} (\exists \bar{y})(\forall x)(\varphi(x, \bar{y}) \wedge \varphi'(\bar{y})) \quad \Leftrightarrow \quad T \models_{\mathbb{K}} (\exists \bar{y})(\bigwedge_{i=1}^{n} \varphi(f_i(\bar{y}), \bar{y}) \wedge \varphi'(\bar{y})).$$

But then, shifting the existential quantifiers back to their original positions,

$$T \models_{\mathbb{K}} \chi[(\forall x)\varphi(x, \bar{y})] \quad \Leftrightarrow \quad T \models_{\mathbb{K}} \chi[\bigwedge_{i=1}^{n} \varphi(f_i(\bar{y}), \bar{y})].$$

Note that $\chi[\wedge_{i=1}^{n}\varphi(f_i(\bar{y}), \bar{y})]$ is also a $\wedge$-prenex formula. Hence, the claim follows by an induction on the multiset of depths of quantifier occurrences according to the standard multiset well-ordering from [8]. $\square$

Now we turn our attention to parallel Skolemization for prenex formulas on the *left*, using again a further weaker witnessed model property. Let $\boldsymbol{L}$ be a lattice and consider $\mathcal{X} \subseteq \mathfrak{P}(L)$. We say that an element b in L is n-$\vee$-*compact* for some $n \in \mathbb{N}^+$ if for all $A \in \mathcal{X}$,

$$\bigvee A \geq b \quad \Longrightarrow \quad a_1 \vee \ldots \vee a_n \geq b \ \text{ for some } a_1, \ldots, a_n \in A.$$

We will call a $\mathcal{P}$-structure $\mathfrak{S} = \langle \boldsymbol{A}, \mathbf{S} \rangle$ n-$(\exists)$-*witnessed* if the element $\bar{1}^{\boldsymbol{A}}$ is n-$\vee$-compact in the following system:

$$\{\{||\varphi(b, \bar{a})||^{\mathfrak{S}} \mid b \in S\} \mid \varphi(x, \bar{y}) \text{ a } \mathcal{P}\text{-formula and } \bar{a} \in S\}.$$

Then $\models_{\mathbb{K}}$ has *the n-$(\exists)$-witnessed model property* if for any $\mathcal{P}$-theory $T \cup \{\varphi\}$,

$$T \models_{\mathbb{K}} \varphi \quad \Leftrightarrow \quad \text{every } n\text{-}(\exists)\text{-witnessed model } \mathfrak{M} \text{ of } T \text{ is a model of } \varphi.$$

Example 8. It is easy to generate examples of $\mathrm{FL_e}$-algebras $\boldsymbol{A}$ whose powerset is not n-compact for any n but where $\bar{1}$ is n-$\vee$-compact: e.g., it would be sufficient to assume that $\bar{1}^{\boldsymbol{A}}$ is the top element in $\boldsymbol{A}$, that the set $\{a \in A \mid a < \bar{1}^{\boldsymbol{A}}\}$ has a maximal element, and that there is an infinite chain in $\boldsymbol{A}$. These examples would then naturally yield logics with the n-$(\exists)$-witnessed model property which in general do not have the n-witnessed model property.

The next proposition (which follows directly from [7, Corollary 4.3.10 and Theorem 4.5.5]) presents an important class of logics with the 1-$(\exists)$-witnessed model property given by algebras where, in general, $\bar{1}$ is not 1-$\vee$-compact.

Proposition 1. *Let $\mathbb{K}$ be a class of complete chains that generates a variety in which the class of all chains admits regular completions, i.e., each such chain can be embedded into a complete one by an embedding preserving all (even infinite) existing joins and meets. Then $\models_{\mathbb{K}}$ has the 1-($\exists$)-witnessed model property.*

Theorem 4. *If $\models_{\mathbb{K}}$ has the n-($\exists$)-witnessed model property, then for each $\mathcal{P}$-theory $T \cup \{\varphi(x, \bar{y}), \psi\}$ and function symbols $f_1, \ldots, f_n \notin \mathcal{P}$ of arity $|\bar{y}|$,*

$$T \cup \{(\forall \bar{y})(\exists x)\varphi(x, \bar{y})\} \models_{\mathbb{K}} \psi \quad \Leftrightarrow \quad T \cup \{(\forall \bar{y}) \bigvee_{i=1}^{n} \varphi(f_i(\bar{y}), \bar{y})\} \models_{\mathbb{K}} \psi.$$

Proof. The left-to-right direction is easy. For the right-to-left direction, suppose that $T \cup \{(\forall \bar{y})(\exists x)\varphi(x, \bar{y})\} \not\models_{\mathbb{K}} \psi$. By assumption, there is an n-($\exists$)-witnessed model $\mathfrak{M}$ of $T \cup \{(\forall \bar{y})(\exists x)\varphi(x, \bar{y})\}$ such that $\mathfrak{M} \not\models \psi$. Since for each $\bar{m} \in M$, $\|(\exists x)\varphi(x, \bar{m})\|^{\mathfrak{M}} \geq \bar{1}$, there are $u_1^{\bar{m}}, \ldots, u_n^{\bar{m}} \in M$ such that

$$\|\varphi(u_1^{\bar{m}}, \bar{m})\|^{\mathfrak{M}} \vee \cdots \vee \|\varphi(u_n^{\bar{m}}, \bar{m})\|^{\mathfrak{M}} \geq \bar{1}.$$

But then, using the axiom of choice, we can define functions f_i and expand the model $\mathfrak{M}$ into a model $\mathfrak{M}'$ such that for each $\mathcal{P}$-formula χ and $\bar{m}, \bar{s} \in M$,

$$\| \bigvee_{i=1}^{n} \varphi(f_i(\bar{m}), \bar{m})\|^{\mathfrak{M}'} \geq \bar{1} \quad \text{and} \quad \|\chi(\bar{s})\|^{\mathfrak{M}'} = \|\chi(\bar{s})\|^{\mathfrak{M}}.$$

So $\mathfrak{M}'$ is a model of $T \cup \{(\forall \bar{y}) \bigvee_{i=1}^{n} \varphi(f_i(\bar{y}), \bar{y})\}$ and $\mathfrak{M}' \not\models \psi$. □

As in the case of Skolemization on the right, this "one step" theorem extends to all prenex formulas, assuming the additional quantifier shift condition, satisfied in particular whenever all algebras in in $\mathbb{K}$ are co-frames (e.g., chains).

Theorem 5. *Suppose that $\models_{\mathbb{K}}$ has the n-($\exists$)-witnessed model property and for all $\mathcal{P}$-formulas φ and χ such that x is not free in χ:*

$$\models_{\mathbb{K}} (\forall x)(\chi \vee \varphi) \to (\chi \vee (\forall x)\varphi)$$

Then $\models_{\mathbb{K}}$ admits parallel Skolemization left of degree n for prenex sentences.

Finally, putting together the results of this section for the special case of first-order substructural logics based on classes of chains, we obtain:

Corollary 1. *Suppose that $\mathbb{K}$ is a class of complete* $\mathrm{FL_e}$*-chains. Then $\models_{\mathbb{K}}$ admits parallel Skolemization* right *of degree* 1 *for all prenex sentences. Moreover, if $\mathbb{K}$ is a class of complete chains that generates a variety in which the class of all chains admits regular completions, then $\models_{\mathbb{K}}$ admits parallel Skolemization* left *of degree* 1 *for all prenex sentences.*

It follows in particular from this corollary that any logic axiomatized relative to the first-order version of the logic MTL (the logic of all FL_{ew}-chains, see [5]) by adding axioms from the class P_3 introduced in [4] admits parallel Skolemization left and right of degree 1 for all prenex sentences.

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