Prescribing the behavior of the GMRES method and the Arnoldi method simultaneously

Jurjen Duintjer Tebbens
Institute of Computer Science
Academy of Sciences of the Czech Republic

joint work with

Gérard Meurant

2012 SIAM Conference on Applied Linear Algebra, Valencia, June 20, 2012.

Outline

1. Motivation

2. Prescribing convergence behavior for Arnoldi’s method

3. Prescribing convergence behavior for *both* Arnoldi’s method and GMRES

1. Motivation

Given a nonsingular matrix and nonzero vector

$$A \in \mathbb{C}^{n\times n}, \quad b \in \mathbb{C}^{n},$$

the kth iteration of the Arnoldi orthogonalization process [Arnoldi - 1951] (without breakdown) computes the decomposition

$$AV_k = V_{k+1}\tilde{H}_k,$$

where the columns of $V_k = [v_1, \ldots, v_k]$ (the Arnoldi vectors) contain an orthogonal basis for the kth Krylov subspace

$$\mathcal{K}_k(A, b) \equiv \text{span}\{b, Ab, \ldots, A^{k-1}b\}$$

and $\tilde{H}_k$ is rectangular upper Hessenberg; by deleting its last row we get the square matrix

$$H_k = V_k^* A V_k \in \mathbb{C}^{k\times k}.$$

1. Motivation

Essentially,

- for eigenpair approximations of A, the Arnoldi method [Arnoldi - 1951], [Saad - 1980] uses the eigenvalues and eigenvectors of H_k and the first k Arnoldi vectors,

- for approximate solutions to linear systems $Ax = b$, the GMRES method [Saad, Schultz - 1986] solves least squares problems with $\tilde{H}_k$ and $\|b\|e_1$ and the first k Arnoldi vectors.

- Both the GMRES and the Arnoldi method are very popular methods that are successful for a large variety of problem classes.

- Nevertheless, convergence behavior of the two methods is not fully understood, analysis is particularly challenging with highly non-normal input matrices.

1. Motivation

- Often one tries to use the tools that are successful for analysis of hermitian counterparts of GMRES and Arnoldi like the Conjugate Gradients and the Lanczos method.
- For example, the basic tool for explaining Krylov subspace methods for hermitian linear systems is the eigenvalue distribution.
- However, for GMRES it is known for some time that if GMRES generates a certain residual norm history, the same history can be generated with any nonzero spectrum [Greenbaum, Strakoš - 1994].
- Complemented with the fact that GMRES can generate arbitrary non-increasing residual norms, this gives the result that any non-increasing convergence curve is possible with any nonzero spectrum [Greenbaum , Pták, Strakoš - 1996].
- A complete description of the class of matrices and right hand sides with prescribed convergence and eigenvalues was given in [Arioli , Pták, Strakoš - 1998].

1. Motivation

Other objects have been successful in explaining GMRES for particular problems, including:

- The pseudo-spectrum, see e.g. [Trefethen, Embree - 2005],
- the field of values, see e.g. [Eiermann - 1993],
- the numerical polynomial hull, see e.g. [Greenbaum - 2002],
- the Ritz values, i.e. the eigenvalues of the Hessenberg matrices generated by the underlying Arnoldi process, see e.g. [van der Vorst, Vuik - 1993].

Although in practice eigenvalues do often influence convergence of GMRES, they cannot be used as a universal tool for explaining GMRES and such a tool is unlikely to exist.

1. Motivation

An important tool for hermitian eigenproblems solved with Krylov subspace methods is the following interlacing property:

Consider a tridiagonal Jacobi matrix T_m and its leading principal submatrix T_k for some $k < m$. If the ordered eigenvalues of T_k are

$$\rho_1^{(k)} < \rho_2^{(k)} < \quad \dots \quad < \rho_k^{(k)},$$

then in every open interval between two subsequent eigenvalues

$$(\rho_{i-1}^{(k)}, \rho_i^{(k)}), \quad i = 2, \dots, k,$$

there lies at least one eigenvalue of T_m.

This interlacing property enables, among others, to prove the persistence theorem (see [Paige - 1971, 1976, 1980] or [Meurant, Strakoš - 2006]) which is crucial for controlling the convergence of Ritz values in the Lanczos method.

1. Motivation

- There are generalizations of the interlacing property to the non-hermitian but normal case [Fan, Pall - 1957], [Thompson- 1966], [Ericsson - 1990], [Malamud - 2005], though a geometric interpretation is difficult.
- There is no interlacing property for the principal submatrices of general non-normal matrices [de Oliveira - 1969], [Shomron, Parlett - 2009].
- This makes convergence analysis of the Arnoldi method for non-normal input matrices rather delicate, just as it is for the GMRES method.
- The GMRES and Arnoldi methods being closely related through the Arnoldi process, can we show that arbitrary convergence behavior of Arnoldi is possible?
- By arbitrary behavior we mean arbitrary Ritz values for *all* iterations (we do not consider eigenvectors). Note that this involves many more conditions than prescribing one residual norm per GMRES iteration.

2. Prescribed convergence for Arnoldi's method

Notation: Let the kth Hessenberg matrix H_k generated in Arnoldi's method have the eigenvalue ρ and eigenvector y,

$$H_k\, y = \rho\, y.$$

With the Arnoldi decomposition $AV_k = V_{k+1}\tilde{H}_k$, we obtain for the Ritz pair $\{\rho, V_k y\}$ the residual norm

$$\|A(V_k y) - \rho(V_k y)\| = \|A(V_k y) - V_k H_k y\| = \|V_{k+1}\tilde{H}_k y - V_k H_k y\| = h_{k+1,k}|e_k^T y|.$$

Often for small $h_{k+1,k}|e_k^T y|$, the Arnoldi method takes $\{\rho, V_k y\}$ as an approximate eigenvalue-eigenvector pair of A. Note that a small value $h_{k+1,k}|e_k^T y|$ needs not imply that ρ is close to a true eigenvalue of A, see e.g. [Chatelin - 1993], [Godet-Thobie - 1993]; convergence analysis cannot be based on this value but focusses instead on the quality of approximate invariant subspaces [Beattie, Embree, Sorensen - 2005].

2. Prescribed convergence for Arnoldi's method

Theorem 1 [DT, Meurant - 2012]. Let the set

$$\begin{aligned}\mathcal{R} = \{ \quad & \rho_1^{(1)}, \\ & (\rho_1^{(2)}, \rho_2^{(2)}), \\ & \vdots \\ & (\rho_1^{(n-1)}, \ldots, \rho_{n-1}^{(n-1)}), \\ & (\lambda_1, \ldots\ldots\ldots, \lambda_n) \},\end{aligned}$$

represent any choice of $n(n+1)/2$ complex Ritz values and denote by $C^{(k)}$ the companion matrix of the polynomial with roots $\rho_1^{(k)}, \ldots, \rho_k^{(k)}$, i.e.

$$C^{(k)} = \begin{pmatrix} 0 & \ldots & & 0 & -\alpha_0 \\ 1 & 0 & \ldots & 0 & -\alpha_1 \\ & \ddots & & \vdots & \vdots \\ & & & 1 & -\alpha_{k-1} \end{pmatrix}, \qquad \prod_{j=1}^{k}(z - \rho_j^{(k)}) = z^k + \sum_{j=0}^{k-1} \alpha_j z^j .$$

2. Prescribed convergence for Arnoldi's method

If we define the unit upper triangular matrix $U(\mathcal{S})$ through

$$U(\mathcal{S}) = I_n - \begin{bmatrix} 0 & C^{(1)}e_1 & \vdots & & \vdots \\ & 0 & C^{(2)}e_2 & & \vdots \\ & & 0 & & \\ & & & & C^{(n-1)}e_{n-1} \\ & & & & 0 \end{bmatrix},$$

then the upper Hessenberg matrix

$$H(\mathcal{R}) = U(\mathcal{S})^{-1} C^{(n)} U(\mathcal{S})$$

has the spectrum $\lambda_1, \ldots, \lambda_n$ and its kth leading principal submatrix has spectrum

$$\rho_1^{(k)}, \ldots, \rho_k^{(k)}, \qquad k = 1, \ldots, n-1.$$

It has unit subdiagonal.

2. Prescribed convergence for Arnoldi's method

Proof: The $k \times k$ leading principal submatrix of $H(\mathcal{R})$ is

$$[I_k, 0]\, H(\mathcal{R}) \begin{bmatrix} I_k \\ 0 \end{bmatrix} = [I_k, 0]\, U(\mathcal{S})^{-1} C^{(n)} U(\mathcal{S}) \begin{bmatrix} I_k \\ 0 \end{bmatrix}$$

$$= \quad [U_k^{-1}, \tilde{u}_{k+1}, \ldots, \tilde{u}_n] \begin{bmatrix} 0 \\ U_k \\ 0 \end{bmatrix} = [U_k^{-1}, \tilde{u}_{k+1}] \begin{bmatrix} 0 \\ U_k \end{bmatrix},$$

where U_k denotes the $k \times k$ leading principal submatrix of $U(\mathcal{S})$ and $\tilde{u}_j$ denotes the vector of the first k entries of the jth column of $U(\mathcal{S})^{-1}$ for $j > k$. Its spectrum is also the spectrum of the matrix

$$U_k [U_k^{-1}, \tilde{u}_{k+1}] \begin{bmatrix} 0 \\ U_k \end{bmatrix} U_k^{-1} = [I_k, U_k \tilde{u}_{k+1}] \begin{bmatrix} 0 \\ I_k \end{bmatrix},$$

which is a companion matrix with last column $U_k \tilde{u}_{k+1}$.

2. Prescribed convergence for Arnoldi's method

From

$$e_{k+1} = U_{k+1}U_{k+1}^{-1}e_{k+1} = \begin{bmatrix} U_k & -C^{(k)}e_k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_{k+1} \\ 1 \end{bmatrix} = \begin{bmatrix} U_k\tilde{u}_{k+1} - C^{(k)}e_k \\ 1 \end{bmatrix}$$

we obtain $U_k\tilde{u}_{k+1} = C^{(k)}e_k$. $\square$

Remark: The matrix

$$H(\mathcal{R}) = U(\mathcal{S})^{-1}C^{(n)}U(\mathcal{S}).$$

is the *unique* upper Hessenberg matrix $H(\mathcal{R})$ with the prescribed spectrum and Ritz values and the entry one along the subdiagonal (see also [Parlett, Strang - 2008] where $H(\mathcal{R})$ is constructed in a different way).

Note that $U(\mathcal{S})$ transforms the matrix $C^{(n)}$ with all Ritz values zero to the matrix $H(\mathcal{R})$ with prescribed Ritz values. It is composed of (columns of) companion matrices and we will call $U(\mathcal{S})$ the Ritz value companion transform.

2. Prescribed convergence for Arnoldi's method

Thus the Ritz values generated in the Arnoldi method can exhibit any convergence behavior: It suffices to apply the Arnoldi process with the initial Arnoldi vector e_1 and the matrix $H(\mathcal{R})$ with arbitrarily prescribed Ritz values. Then the method generates the Hessenberg matrix $H(\mathcal{R})$ itself.

Question: Can the same prescribed Ritz values be generated with positive entries other than one on the subdiagonal?

For $\sigma_1, \sigma_2, \ldots, \sigma_{n-1} > 0$ consider the diagonal similarity transformation

$$H \equiv \text{diag}\,(1, \sigma_1, \sigma_1\sigma_2, \ldots, \Pi_{j=1}^{n-1}\sigma_j)\, H(\mathcal{R})\, \big(\text{diag}\,(1, \sigma_1, \sigma_1\sigma_2, \ldots, \Pi_{j=1}^{n-1}\sigma_j)\big)^{-1}.$$

Then the subdiagonal of H has the entries $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and all leading principal submatrices of H are similar the corresponding leading principal submatrices of $H(\mathcal{R})$.

2. Prescribed convergence for Arnoldi's method

This immediately leads to a parametrization of the matrices and initial Arnoldi vectors that generate a given set of Ritz values $\mathcal{R}$:

Theorem 2 [DT, Meurant - 2012]. Assume we are given a set of tuples

$$
\begin{aligned}
\mathcal{R} = \{ \quad & \rho_1^{(1)}, \\
& (\rho_1^{(2)}, \rho_2^{(2)}), \\
& \vdots \\
& (\rho_1^{(n-1)}, \ldots, \rho_{n-1}^{(n-1)}), \\
& (\lambda_1, \ldots\ldots\ldots, \lambda_n)\},
\end{aligned}
$$

of complex numbers and $n-1$ positive real numbers

$$\sigma_1, \ldots, \sigma_{n-1}.$$

If A is a matrix of order n and b a nonzero n-dimensional vector, then the following assertions are equivalent:

2. Prescribed convergence for Arnoldi's method

1. The Hessenberg matrix generated by the Arnoldi method applied to A and initial Arnoldi vector b has eigenvalues $\lambda_1, \dots, \lambda_n$, subdiagonal entries $\sigma_1, \dots, \sigma_{n-1}$ and $\rho_1^{(k)}, \dots, \rho_k^{(k)}$ are the eigenvalues of its kth leading principal submatrix for all $k = 1, \dots, n-1$.
2. The matrix A and initial vector b are of the form

$$A = VD_\sigma U(\mathcal{S})^{-1} C^{(n)} U(\mathcal{S}) D_\sigma^{-1} V^*, \qquad b = \|b\| V e_1,$$

where V is unitary, $U(\mathcal{S})$ is the Ritz value companion transform,

$$D_\sigma = \text{diag}\,(1, \sigma_1, \sigma_1\sigma_2, \dots, \Pi_{j=1}^{n-1}\sigma_j),$$

and $C^{(n)}$ is the companion matrix of the polynomial with roots $\lambda_1, \dots, \lambda_n$.

This also shows how little on the quality of the Ritz value ρ needs be said by

$$\|A(V_k y) - \rho(V_k y)\| = h_{k+1,k}|e_k^T y|.$$

Any distance from ρ to the spectrum of A is possible with any value of $h_{k+1,k}$!

2. Prescribed convergence for Arnoldi's method

Counterintuitive example 1: Convergence of interior Ritz values only:

$$
\begin{aligned}
\mathcal{R} = \{ \quad & 3, \\
& (3,3)\,, \\
& (2,3,4)\,, \\
& (3,3,3,3)\,, \\
& (1,2,3,4,5)\}\,.
\end{aligned}
$$

This gives the unit upper Hessenberg matrix

$$
H(\mathcal{R}) = U(\mathcal{S})^{-1} C^{(5)} U(\mathcal{S}) = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 \\ & 1 & 3 & -1 & 0 \\ & & 1 & 3 & 5 \\ & & & 1 & 3 \end{bmatrix}.
$$

2. Prescribed convergence for Arnoldi's method

Thus these Ritz values are generated by the Arnoldi method applied to

$$A = V\text{diag}\,(1, \sigma_1, \sigma_1\sigma_2, \ldots, \Pi_{j=1}^{n-1}\sigma_j) \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 \\ & 1 & 3 & -1 & 0 \\ & & 1 & 3 & 5 \\ & & & 1 & 3 \end{bmatrix} \text{diag}\,(1, \sigma_1, \sigma_1\sigma_2, \ldots, \Pi_{j=1}^{n-1}\sigma_j)^{-1} V^*,$$

with initial vector $b = Ve_1$ and for any unitary V and positive values $\sigma_1, \ldots, \sigma_{n-1}$.

This is not a highly non-normal example, for instance with $\sigma_i \equiv 1$:

$$\|A\|\|A^{-1}\| = 9.7137,$$

and the eigenvector basis W of A has condition number

$$\|W\|\|W^{-1}\| = 4.8003.$$

2. Prescribed convergence for Arnoldi's method

Counterintuitive example 2: We can prescribe the "diverging" Ritz values

$$\mathcal{R} = \{ \begin{array}{c} 1, \\ (0,2)\,, \\ (-1,1,3)\,, \\ (-2,0,2,4)\,, \\ (1,1,1,1,1)\}, \end{array} \quad \text{with}$$

$$H(\mathcal{R}) = U(\mathcal{S})^{-1}C^{(5)}U(\mathcal{S}) = \begin{bmatrix} 1 & 1 & 0 & -3 & 0 \\ 1 & 1 & 3 & 0 & -31 \\ & 1 & 1 & 6 & 0 \\ & & 1 & 1 & -10 \\ & & & 1 & 1 \end{bmatrix}.$$

They are generated by Arnoldi applied to $A = VH(\mathcal{R})V^*$, $b = Ve_1$ for unitary V.

2. Prescribed convergence for Arnoldi's method

The same "diverging" Ritz values are generated with the exponentially decreasing values 2^{-1}, 2^{-2}, 2^{-3} and 2^{-4} on the subdiagonal of the Hessenberg matrix:

$$A = V \begin{bmatrix} 1 & 2 & 0 & -192 & 0 \\ 0.5 & 1 & 12 & 0 & -15872 \\ & 0.25 & 1 & 48 & 0 \\ & & 0.125 & 1 & -160 \\ & & & 0.0625 & 1 \end{bmatrix} V^*, \qquad b = \|b\| V e_1.$$

Then the rounded residual norms $\|A(V_k y) - \rho(V_k y)\| = h_{k+1,k}|e_k^T y|$ seem to indicate convergence:

$$\{ \quad \tfrac{1}{2}, \\ (0.1118, 0.1118), \\ (0.011, 0.0052, 0.011), \\ (0.0006, 0.0001, 0.0001, 0.0006) \}.$$

3. Prescribed convergence for Arnoldi *and* GMRES

Starting with initial guess $x_0 = 0$, GMRES iterates x_k minimize the residual,

$$\|b - Ax_k\| = \min \|b - As\| \quad \text{over all} \quad s \in \mathcal{K}_k(A, b).$$

Writing x_k in the Arnoldi basis,

$$x_k = V_k y_k \in \mathcal{K}_k(A, r_0),$$

and using the Arnoldi decomposition $AV_k = V_{k+1}\tilde{H}_k$, we see that the residual norm is

$$\begin{aligned} \|b - Ax_k\| &= \|b - AV_k y_k\| = \|V_{k+1}\|b\|e_1 - AV_k y_k\| \\ &= \|V_{k+1}(\|b\|e_1 - \tilde{H}_k y_k)\| = \min_{y \in \mathbb{C}^k} \|\|b\|e_1 - \tilde{H}_k y\|. \end{aligned}$$

Thus the residual norms generated by the GMRES method are fully determined by the Hessenberg matrix $\tilde{H}_k$ and $\|b\|$.

3. Prescribed convergence for Arnoldi *and* GMRES

- We have seen that the subdiagonal entries of $\tilde{H}_k$ can be chosen arbitrarily, for any prescribed Ritz values in the kth iteration.
- Hence there is a chance we can modify the behavior of GMRES while maintaining the prescribed Ritz values.

Example from earlier: Consider the prescribed 'diverging' Ritz values

$$\begin{aligned} \mathcal{R} = \{ \quad & 1, \\ & (0,2)\,, \\ & (-1,1,3)\,, \\ & (-2,0,2,4)\,, \\ & (1,1,1,1,1)\}\,, \end{aligned}$$

and the prescribed subdiagonal entries of the generated Hessenberg matrix

$$\sigma_1 = 2^{-1}, \quad \sigma_2 = 2^{-2}, \quad \sigma_3 = 2^{-3}, \quad \sigma_4 = 2^{-4}.$$

3. Prescribed convergence for Arnoldi *and* GMRES

The corresponding GMRES convergence curve is

$$\|r^{(0)}\| = 1, \quad \|r^{(1)}\| = \sqrt{\frac{1}{5}}, \quad \|r^{(2)}\| = \sqrt{\frac{1}{5}}, \quad \|r^{(3)}\| = 0.0052, \quad \|r^{(4)}\| = 0.0052.$$

Question: Can we force any GMRES convergence speed with arbitrary Ritz values by modifying the subdiagonal entries?

Not any, because there is a relation between GMRES stagnation and zero Ritz values: A singular Hessenberg matrix corresponds to stagnation in the parallel GMRES process, see [Brown - 1991]. In our example we have

$$\begin{aligned}
\rho_1^{(1)} = 1, \quad \|r^{(1)}\| &= \frac{1}{\sqrt{5}} \\
(\rho_1^{(2)}, \rho_2^{(2)}) = (0, 2), \quad \|r^{(2)}\| &= \frac{1}{\sqrt{5}} \\
(\rho_1^{(3)}, \rho_2^{(3)}, \rho_3^{(3)}) = (-1, 1, 3), \quad \|r^{(3)}\| &= 0.0052 \\
(\rho_1^{(4)}, \rho_2^{(4)}, \rho_3^{(4)}, \rho_4^{(4)}) = (-2, 0, 2, 4), \quad \|r^{(4)}\| &= 0.0052.
\end{aligned}$$

3. Prescribed convergence for Arnoldi *and* GMRES

However, this is the *only* restriction Ritz values put on GMRES residual norms:

Theorem 3 [DT, Meurant - 2012]. Consider a set of tuples of complex numbers

$$\begin{aligned}\mathcal{R} = \{ \quad & \rho_1^{(1)}, \\ & (\rho_1^{(2)}, \rho_2^{(2)}), \\ & \vdots \\ & (\rho_1^{(n-1)}, \dots, \rho_{n-1}^{(n-1)}), \\ & (\lambda_1, \dots\dots\dots, \lambda_n)\},\end{aligned}$$

such that $(\lambda_1, \dots, \lambda_n)$ contains no zero number and n positive numbers

$$f(0) \geq f(1) \geq \cdots \geq f(n-1) > 0,$$

such that the k-tuple $(\rho_1^{(k)}, \dots, \rho_k^{(k)})$ contains a zero number if and only if

$$f(k-1) = f(k).$$

3. Prescribed convergence for Arnoldi *and* GMRES

Let A be a square matrix of size n and let b be a nonzero n-dimensional vector.
The following assertions are equivalent:

1. The GMRES method applied to A and right-hand side b with zero initial guess yields residuals $r^{(k)},\ k = 0, \dots, n-1$ such that

$$\|r^{(k)}\| = f(k), \quad k = 0, \dots, n-1,$$

A has eigenvalues

$$\lambda_1, \dots, \lambda_n,$$

and

$$\rho_1^{(k)}, \dots, \rho_k^{(k)}$$

are the Ritz values generated at the kth iteration for $k = 1, \dots, n-1$.

3. Prescribed convergence for Arnoldi *and* GMRES

2. The matrix A and right hand side b are of the form

$$A = V \text{diag}(f(0), D_c^{-*})\, U(\mathcal{S})^{-1} C^{(n)} U(\mathcal{S})\, \text{diag}(f(0)^{-1}, D_c^*) V^*, \qquad b = \|b\| V e_1,$$

where V is a unitary matrix, $U(\mathcal{S})$ is the the Ritz value companion transform for $\mathcal{R}$ and $C^{(n)}$ is the companion matrix of the polynomial with roots $\lambda_1, \ldots, \lambda_n$. D_c is a nonsingular diagonal matrix such that

$$R_h^{-T} \hat{h} = -f(0)^2 D_c c,$$

$$\hat{h} = [\eta_1, \ldots, \eta_{n-1}]^T, \qquad \eta_k = (f(k-1)^2 - f(k)^2)^{1/2},$$

R_h being the upper triangular factor of the Cholesky decomposition

$$R_h^T R_h = I - \frac{\hat{h}\hat{h}^T}{f(0)^2},$$

and c is the first row of $U(\mathcal{S})$ without its diagonal entry.

Note we exhausted all freedom modulo unitary transformation.

Example: Standardly converging Ritz values and 'nearly stagnating' GMRES:

$$\mathcal{R} = \{ \quad 5, \quad (1,5), \quad (1,4,5), \quad (1,3,4,5), \quad (1,2,3,4,5)\},$$

$\|r^{(0)}\| = 1, \quad \|r^{(1)}\| = 0.9, \quad \|r^{(2)}\| = 0.8, \quad \|r^{(3)}\| = 0.7, \quad \|r^{(4)}\| = 0.6, \quad \|r^{(5)}\| = 0$ gives

$$A = V \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 10.3237 & 1 & 0 & 0 & 0 \\ & 0.8458 & 4 & 0 & 0 \\ & & 3.312 & 3 & 0 \\ & & & 2.4169 & 2 \end{bmatrix} V^*, \qquad b = \|b\| V e_1.$$

3. Prescribed convergence for Arnoldi *and* GMRES

Again, this is not a highly non-normal example:

$$\|A\|\|A^{-1}\| = 28.9498,$$

and the eigenvector basis W of A has condition number

$$\|W\|\|W^{-1}\| = 57.735.$$

The residual norms $\|A(V_k y) - \rho(V_k y)\| = h_{k+1,k}|e_k^T y|$ for the Ritz pairs are

$$10.3237,$$
$$(0.8458, 0.7886)\,,$$
$$(0.8987, 3.312, 2.0509)\,,$$
$$(0.9906, 2.4169, 2.3137, 1.7303)\ .$$

respectively, i.e. they give misleading information.

Conclusions and future work:

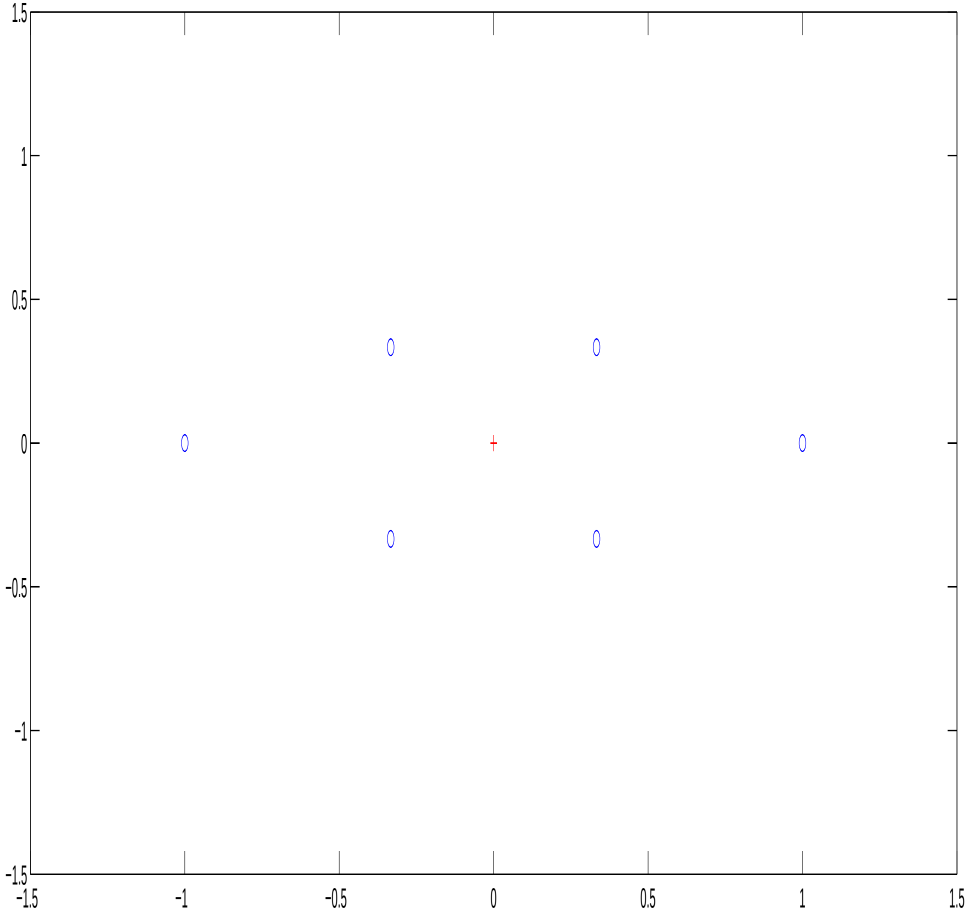
- There is no interlacing property for the Hessenberg matrices in the Arnoldi method.
- The Ritz values generated in the Arnoldi method can behave arbitrarily badly.
- Convergence of Ritz values need not say anything about the behavior of GMRES residual norms (zero Ritz values excepted). For close to normal matrices, the opposite has been suggested [van der Vorst, Vuik - 1993].
- Extension to harmonic Ritz values which determine the GMRES polynomials?
- It is desirable to have similar results for popular restarted versions of GMRES and the Arnoldi method (see e.g. the failure of restarted Arnoldi with exact shifts explained in [Embree - 2009]).

For more details see: DUINTJER TEBBENS J, MEURANT G: *Any Ritz value behavior is possible for Arnoldi and for GMRES*, to appear in SIMAX, available at www.cs.cas.cz/duintjertebbens/duintjertebbens_pub.html

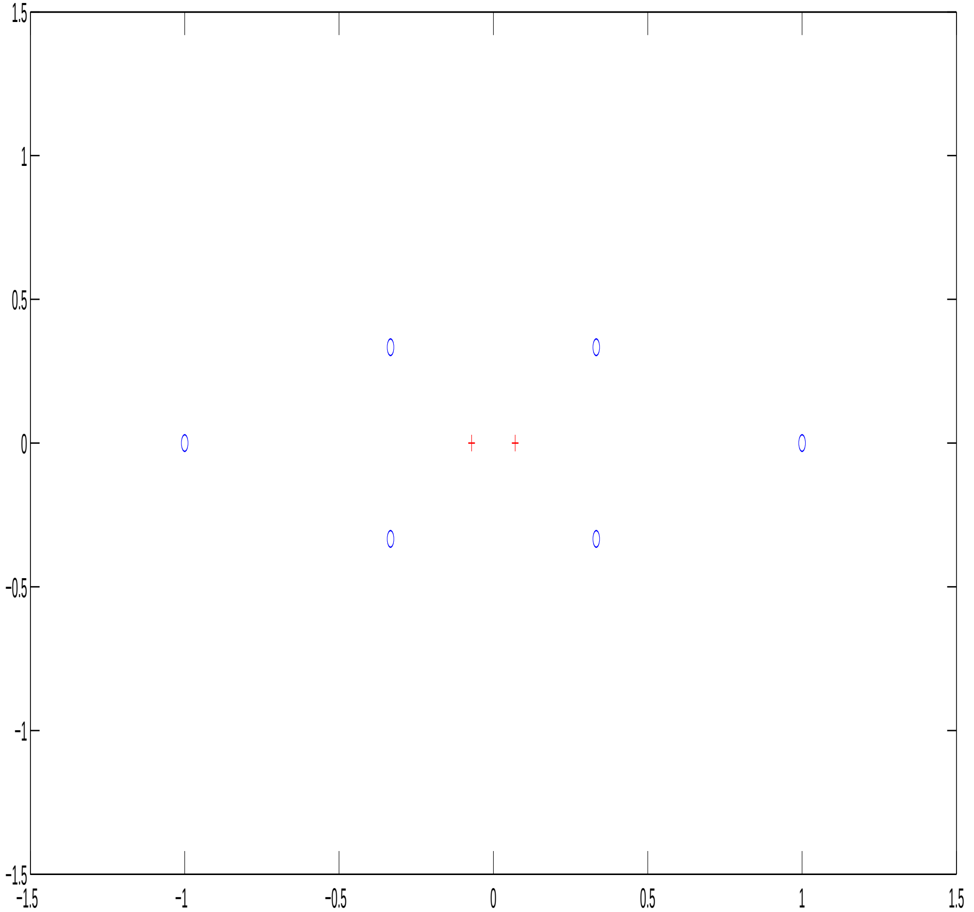
Thank you for your attention!

Supported by projects numbers M100300901 and IAA100300802 of the grant agency of the Academy of Sciences of the Czech Republic.

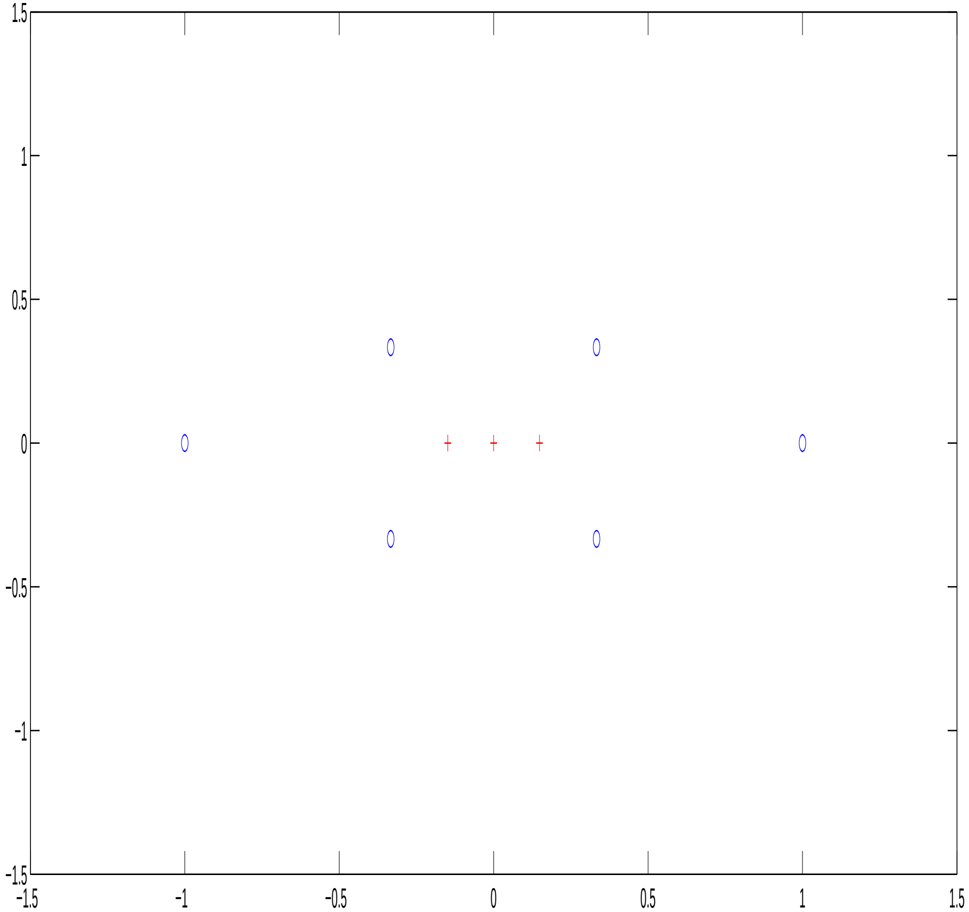
Counterintuitive normal example



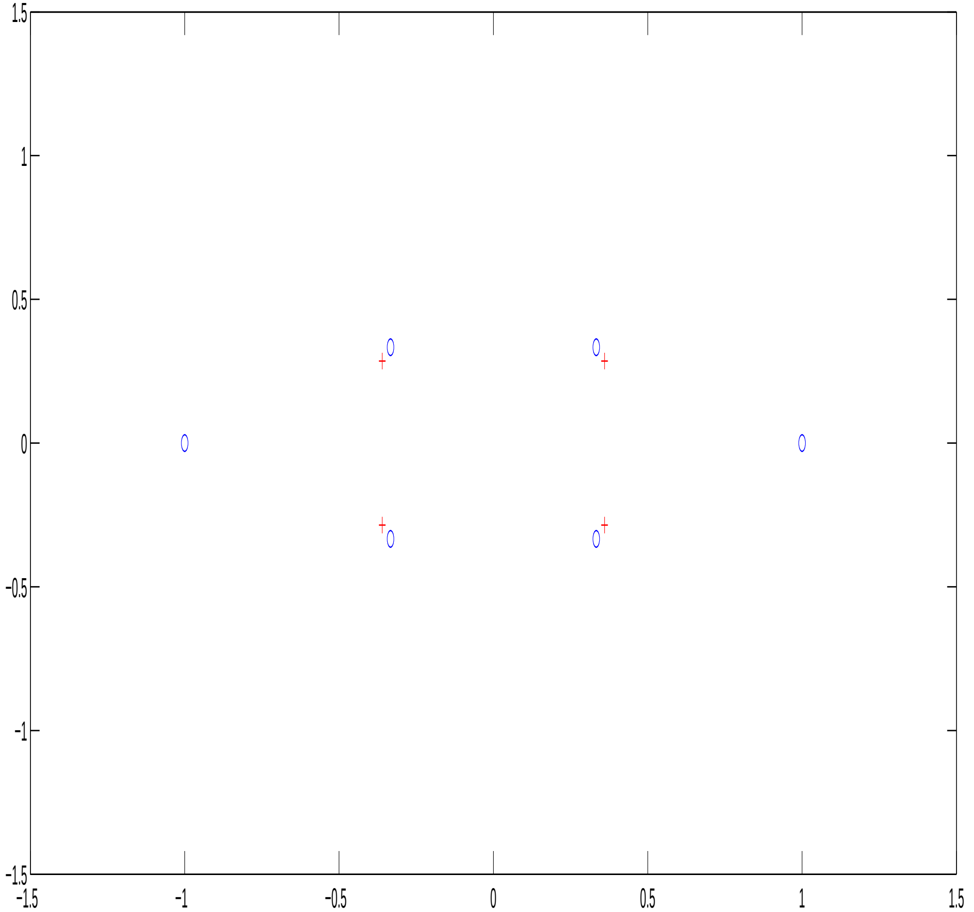
Counterintuitive normal example



Counterintuitive normal example



Counterintuitive normal example



Counterintuitive normal example