Inverse Ill–Posed Problems in Image Processing

☞ Image Deblurring ☜

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Schola Ludus — Nové Hrady — July 25, 2012

Goals of this lecture

- What is the inverse ill-posed problem?
  (Image deblurring as an example of such problem.)

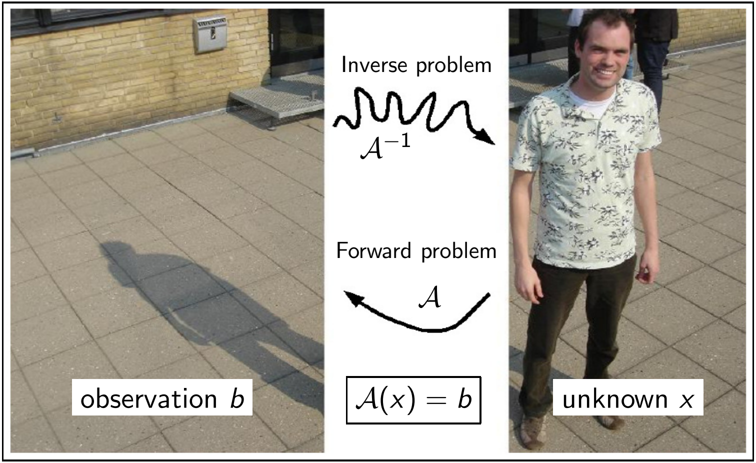
- What is the regularization?
  How / why it works?

Motivation. A gentle start ...

What is it an inverse problem?

Motivation. A gentle start ...

What is it an inverse problem?



[Kjøller: M.Sc. thesis, DTU Lyngby, 2007].

More realistic examples of inverse ill-posed problems

Computer tomography in medical sciences

Computer tomograph (CT) maps a 3D object of $M \times N \times K$ voxels by ℓ X-ray measurements on ℓ pictures with $m \times n$ pixels,

$$\mathcal{A}(\cdot) \equiv$$

$$: \mathbb{R}^{M \times N \times K} \longrightarrow \bigotimes_{j=1}^{\ell} \mathbb{R}^{m \times n}.$$

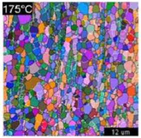
Simpler 2D tomography problem leads to the **Radon transform**. The inverse problem is ill-posed. (3D case is more complicated.)

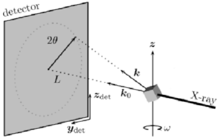
The mathematical problem is **extremely sensitive** to errors which are **always** present in the (measured) data: *discretization error* (finite ℓ, m, n); *rounding errors*; *physical sources of noise* (electronic noise in semiconductor PN-junctions in transistors, ...).

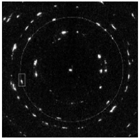
More realistic examples of inverse ill-posed problems

Transmission computer tomography in crystalographics

Reconstruction of an *unknown* orientation distribution function (ODF) of grains in a given sample of a polycrystalline matherial,

$\mathcal{A}\Big($ 

$\Big) \equiv$ 

$\longrightarrow \Big($  $, \dots \Big).$

$\underbrace{\qquad\qquad}_{\text{observation} = \text{data} + \text{noise}}$

The *right-hand side* is a set of measured difractograms.
[Hansen, Sørensen, Südkösd, Poulsen: SIIMS, 2009].

Further analogous applications also in geology, e.g.:

- Seismic tomography (cracks in tectonic plates),
- Gravimetry & magnetometry (ore mineralization).

More realistic examples of inverse ill-posed problems

General framework

In general we deal with a linear problem

$$Ax = b$$

which typically arose as a discretization of a

Fredholm integral equation of the 1st kind

$$b(\mathbf{s}) = \int K(\mathbf{s}, \mathbf{t}) x(\mathbf{t}) d\mathbf{t} \equiv \mathcal{A}\big(x(\mathbf{t})\big),$$

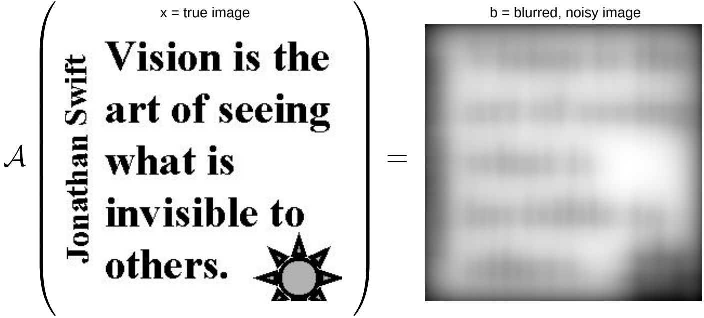
and the right-hand side b is typically contaminated by **noise**.

Our pilot application is the image deblurring problem.

Mathematical model of blurring

Image deblurring—Our pilot application

Our pilot application is the image deblurring problem [J. Nagy]:



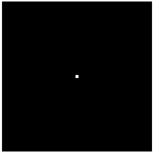
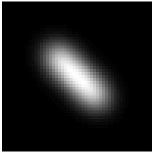
It leads to a linear system $Ax = b$ with square nonsingular matrix.

We consider **gray-scale** images, thus each pixel is represetned by one real number, e.g., from the interval $[0, 1] \equiv [\text{black}, \text{white}]$.

Mathematical model of blurring

Blurring as an operator of the vector space of images

Consider a **single-pixel-image (SPI)** and a blurring operator $\mathcal{A}$ as follows

$\mathcal{A}(X) = \mathcal{A}\Big($  $\Big) =$  $= B.$

and denote

$$X = [x_1, \dots, x_n], \qquad B = [b_1, \dots, b_n] \in \mathbb{R}^{m \times n}.$$

Consider a mapping $\mathrm{vec} : \mathbb{R}^{m \times n} \longrightarrow \mathbb{R}^{mn}$ such that

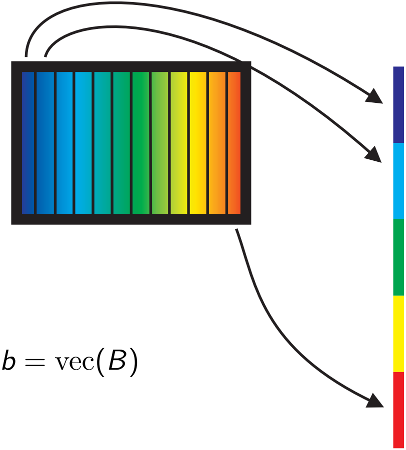
$$x = \mathrm{vec}(X) \equiv [x_1^T, \dots, x_n^T]^T.$$

The picutre B is called **point-spread-function (PSF)**.

Mathematical model of blurring

Reshaping

$$B = [b_1, \ldots, b_n]$$



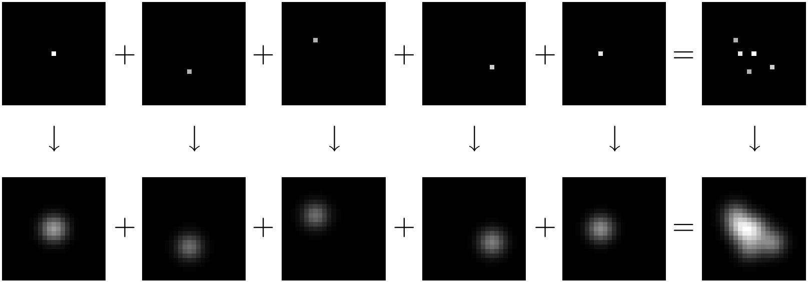
$$b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$b = \mathrm{vec}(B)$$

Mathematical model of blurring

Linear and spatial invariant operator

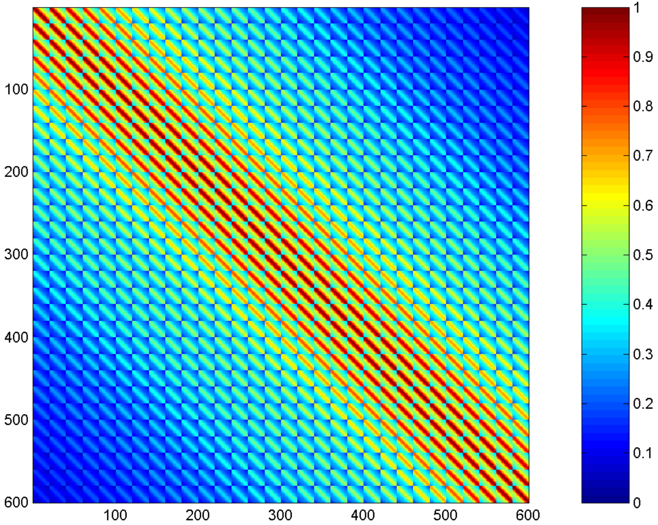
Linearity + spatial invariance:



First row: Original (SPI) images (matrices X).
Second row: Blurred (PSF) images (matrices $B = \mathcal{A}(X)$).

Mathematical model of blurring

Matrix A

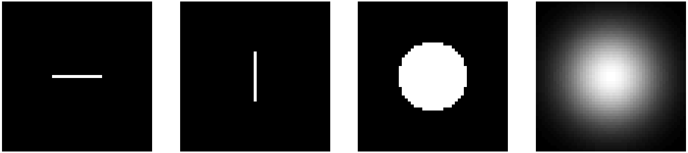


Mathematical model of blurring

Point—spread—function (PSF)

Examples of $PSF_{\mathcal{A}}$:

horizontal motion blur | vertical motion blur | out-of-focus blur | **Gaußian blur**



(Note: Action of the linear and spatial invariant blurring operator $\mathcal{A}(X)$ on the given image X is done by **2D convolution** of the image with the PSF corresponding to the operator.)

System of linear algebraic equations

Smoothing properties

If $\mathcal{A}$ is linear, then $\mathcal{A}(X) = B$, the problem $\mathcal{A}(X) = B$ can be rewritten as a system of linear algebraic equations

$$Ax = b, \qquad A \in \mathbb{R}^{mn \times mn}, \quad x = \mathrm{vec}(X), \; b = \mathrm{vec}(B) \in \mathbb{R}^{mn}.$$

The kernel $K(\mathbf{s}, \mathbf{t})$ in the underlying Fredholm equation

$$b(\mathbf{s}) = \int K(\mathbf{s}, \mathbf{t}) x(\mathbf{t}) d\mathbf{t},$$

- has **smoothing property**,
- thus the function $b(\mathbf{s})$ is smooth (recall the blurred image).

Because A and b are restrictions of $K(\mathbf{s}, \mathbf{t})$, and $y(\mathbf{s})$; the linear system $Ax = b$ in some sense inherits these properties.

System of linear algebraic equations

Singular valued decomposition

For *any* matrix $A \in \mathbb{R}^{M\times N}$, $r = \mathrm{rank}(A)$ there exist orthogonal matrices

$$U = [u_1, \dots, u_M] \in \mathbb{R}^{M\times M}, \quad U^{-1} = U^T$$
$$V = [v_1, \dots, v_N] \in \mathbb{R}^{N\times N}, \quad V^{-1} = V^T$$

and diagonal matrix

$$\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_N) \in \mathbb{R}^{M\times N}, \quad \sigma_1 \geq \dots \geq \sigma_r > 0$$

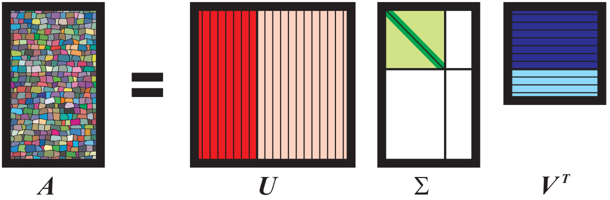
with r positive nonincreasing entries on the diagonal, such that

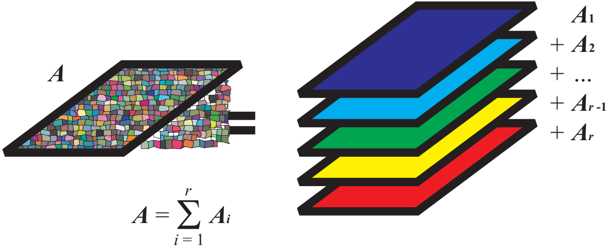
$$A = U\,\Sigma\,V^T, \qquad A = \underbrace{u_1\,\sigma_1\,v_1^T}_{A_1} + \dots + \underbrace{u_r\,\sigma_r\,v_r^T}_{A_r}.$$

It is called the **singular value decomposition (SVD)**.
(See also the principal component analysis (PCA).)

System of linear algebraic equations

Singular valued decomposition





In our case the matrix A is square nonsingular (i.e. $M = N = r$), and, symmetric positive definite (i.e. the SVD is identical to the spectral decomposition of A).

Using the SVD the soution of

$$Ax = b, \qquad A \in \mathbb{R}^{N \times N}, \quad N = mn,$$

can be written as

$$x \;=\; A^{-1}b \;=\; V\Sigma^{-1}U^T b \;=\; \sum_{j=1}^{N} \frac{u_j^T b}{\sigma_j}\, v_j.$$

Recall that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_N > 0$.

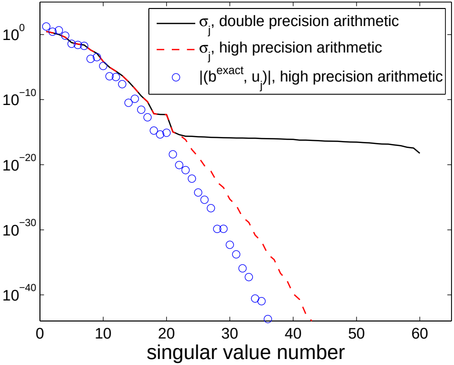
Properties of the problem

The Discrete Picard condition (DPC)

The singular values σ_j of A decay but the sum

$$x = \sum_{j=1}^{N} \frac{u_j^T b}{\sigma_j} v_j$$

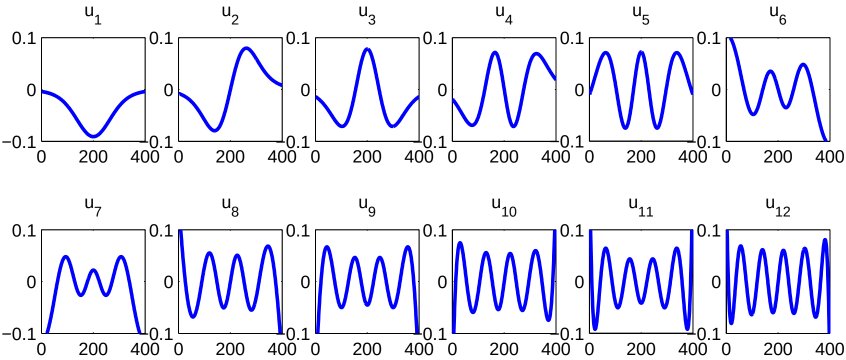
represents some "real data". By the **(discrete) Picard condition**: the projections $|u_i^T b|$ has to decay on average faster than σ_j.



Properties of the problem

Singular vectors of A

Left singular vectors of A represent bases with increasing frequencies:

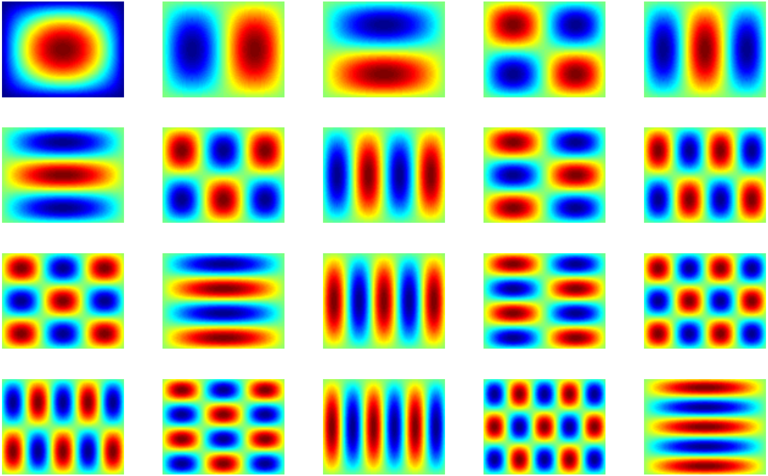


(1D ill-posed problem **SHAW(400)** [Regularization Toolbox]).

Properties of the problem

Right singular vectors ≡ Singular images

Reshaped right singular vectors of A (singular images)



(Image deblurring problem, Gaußian blur, zero BC).

Impact of noise

Noise, Sources of noise

Consider a problem of the form

$$Ax = b, \quad b = b^{\mathrm{exact}} + b^{\mathrm{noise}}, \quad \|b^{\mathrm{exact}}\| \gg \|b^{\mathrm{noise}}\|,$$

where b^{noise} is unknown and represents, e.g.,

- rounding errors,
- discretization error,
- noise with physical sources.

We want to compute (approximate)

$$x^{\mathrm{exact}} \equiv A^{-1} b^{\mathrm{exact}}.$$

The vector b^{noise} typically resebles **white noise**, i.e. it has flat frequency characteristics.

Impact of noise

Violation of the discrete Picard condition

Recall that the singular vectors of A represent frequencies.

Thus the white noise components in left singular subspaces are about the same order of magnitude.

The vector b^{noise}

violates the discrete Picard condition.

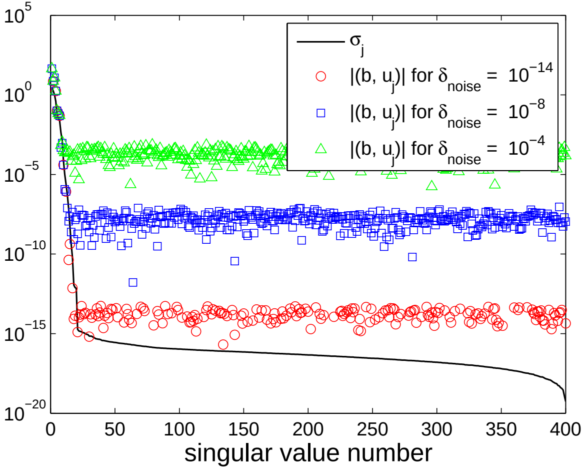
Summarizing:

- b^{exact} has some real pre-image x^{exact}, it satifies DPC
- b^{noise} does not have any real pre-image, it violates DPC.

Impact of noise & regularization

Violation of the discrete Picard condition

Violation of the discrete Picard condition by noise (SHAW(400)):



Impact of noise & regularization

Violation of the discrete Picard condition

Consider the **naive solution**

$$x \;=\; A^{-1}b \;=\; A^{-1}b^{\mathrm{exact}} + A^{-1}b^{\mathrm{noise}},$$

using the singular value expansion

$$x \;=\; A^{-1}b \;=\; \sum_{j=1}^{N} \frac{u_j^T b}{\sigma_j}\, v_j$$

$$= \underbrace{\sum_{j=1}^{N} \frac{u_j^T b^{\mathrm{exact}}}{\sigma_j}\, v_j}_{x^{\mathrm{exact}}} + \underbrace{\sum_{j=1}^{N} \frac{u_j^T b^{\mathrm{noise}}}{\sigma_j}\, v_j}_{\text{amplified noise}} .$$

Thus, even for $\|b^{\mathrm{exact}}\| \gg \|b^{\mathrm{noise}}\|$,

$$\|A^{-1}b^{\mathrm{exact}}\| \ll \|A^{-1}b^{\mathrm{noise}}\|,$$

the data are covered by the inverted noise.

Regulaization

⟨MatLab demo⟩

Regulaization

Filtered solution, Truncated SVD filter

The impact of noise can be eliminated by filtering the solution

$$x_{\mathrm{Filt}} = \sum_{j=1}^{N} \phi_j \frac{u_j^T b}{\sigma_j} v_j \qquad \text{instead of} \qquad x = A^{-1} b = \sum_{j=1}^{N} \frac{u_j^T b}{\sigma_j} v_j.$$

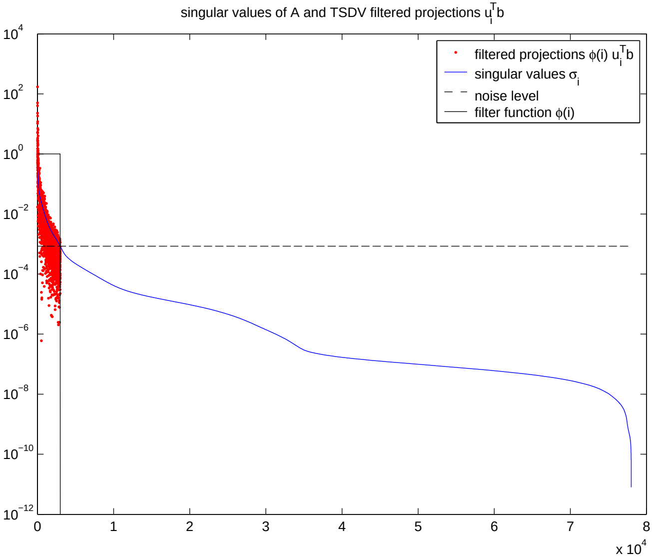
The simplest case is the **truncated SVD (TSVD)** filter

$$\phi_j = \begin{cases} 1, & \text{for} j \leq k \\ 0, & \text{for} j > k \end{cases} \qquad x_{\mathrm{TSVD}}(k) = \sum_{j=1}^{k} \frac{u_j^T b}{\sigma_j} v_j, \quad k \ll N.$$

Disadvantage: We have to know the SVD of A explicitely.

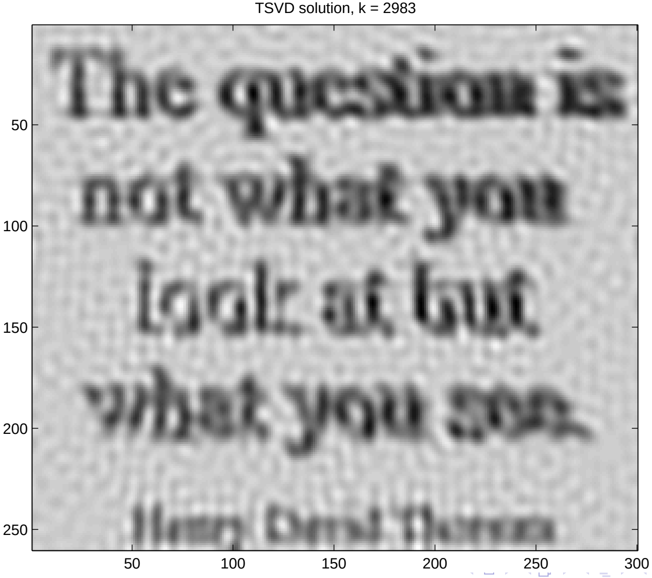
Regulaization

TSVD filter



Regulaization

TSVD solution



Regulaization

Tikhonov regularization

Recall that the norm $\|A^{-1}b\| \geq \|A^{-1}b^{\text{noise}}\|$ can be large.

The goal of **Tikhonov regularization** is to minimize both, the residual norm and the solution norm

$$x_{\text{Tikh}}(\lambda) = \arg\min_{x}\{\|b - Ax\|^2 + \lambda^2\|x\|^2\}, \qquad \text{for some} \quad \lambda.$$

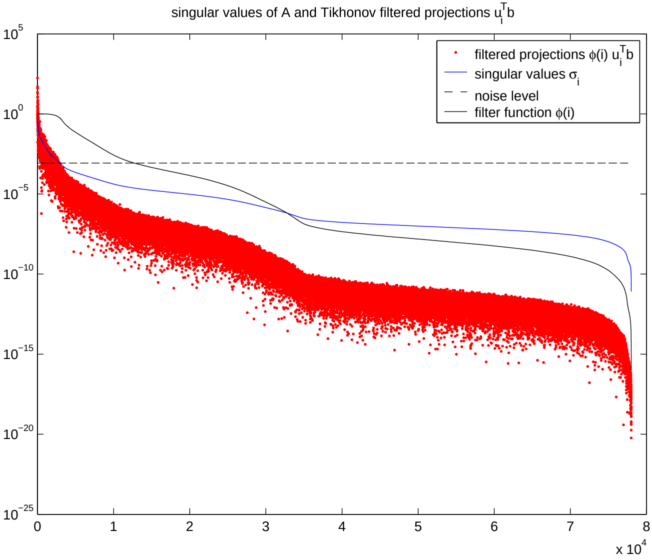
(It represents a least squares problem.)
Using the SVD of A the Tikhonov solution can be written as

$$x_{\text{Tikh}} = \sum_{j=1}^{N} \frac{\sigma_j^2}{\sigma_j^2 + \lambda^2} \frac{u_j^T b}{\sigma_j} v_j.$$

Which is the filtered solution with filter factors $\phi_j = \frac{\sigma_j^2}{\sigma_j^2+\lambda^2}$.

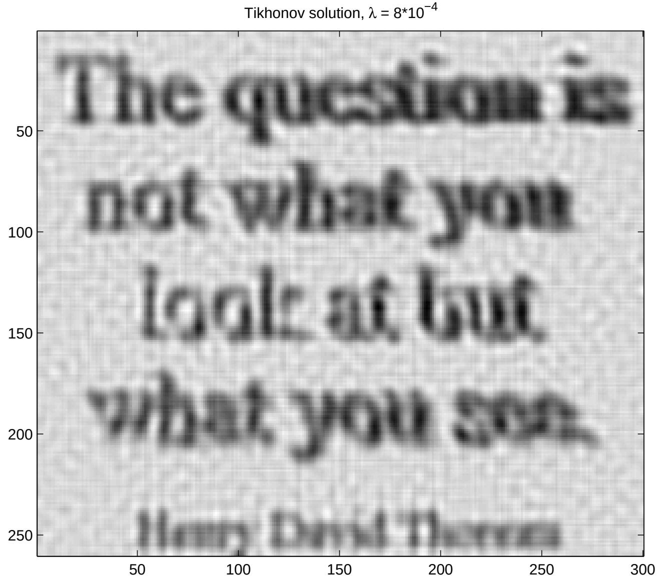
Regulaization

Tikhonov filter



Regulaization

Tikhonov solution



Regulaization

Choosing the parameter

Choosing of the regularization parameter (k or λ):

- Discrepancy principle [Morozov '66, '84]
- Generalized cross validation (GCV) [Chung, Nagy, O'Leary '04], [Golub, Von Matt '97], [Nguyen, Milanfar, Golub '01]
- L-curve [Calvetti, Golub, Reichel '99]
- Normalized cumulative periodograms (NCP) [Rust '98, '00], [Rust, O'Leary '08], [Hansen, Kilmer, Kjeldsen '06]
- ...

Regulaization

Iterative and hybrid approaches

Stationary iterative methods:

- Simultaneous iterative reconstruction techniques (SIRT) (Landweber, Cimminio iteration)
- Kaczmar's method / algebraic reconstuction techniques (ART)

Projection methods (Krylov subspace metods):

- CGLS, LSQR, CGNE

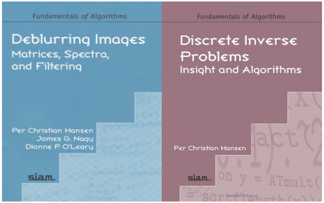
Hybrid methods [O'Leary, Simmons '81], [Hansen '98], [Fiero, Golub, Hansen, O'Leary '97], ... *and many others*

References

Textbooks + software

Textbooks:

- Hansen, Nagy, O'Leary: *Deblurring Images, Spectra, Matrices, and Filtering*, SIAM, FA03, 2006.
- Hansen: *Discrete Inverse Problems, Insight and Algorithms*, SIAM, FA07, 2010.



Sofwtare (MatLab toolboxes):

- HNO package,
- Regularization tools,
- AIRtools,
- ...

(software available on the homepage of P. C. Hansen).

Thank You for Your Attention!