Can numerical analysis help in understanding piezoviscous hydrodynamic lubrication?

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Math Modelling Analysis Computing

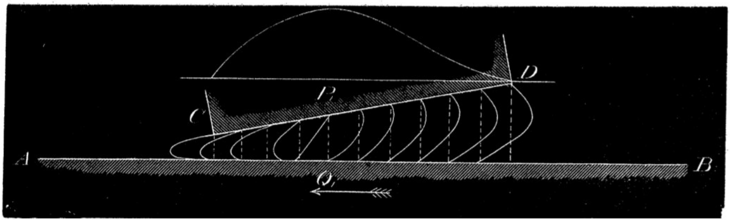
Hydrodynamic (thick-film) lubrication

IV. *On the Theory of Lubrication and its Application to Mr.* Beauchamp Tower's *Experiments, including an Experimental Determination of the Viscosity of Olive Oil.*

By Professor Osborne Reynolds, *LL.D., F.R.S.*

Received December 29, 1885,—Read February 11, 1886.

Fig. 9.



This is the explanation of continuous lubrication.

Standard approach

- focus (oversimplified): isothermal, slow, steady, full-film

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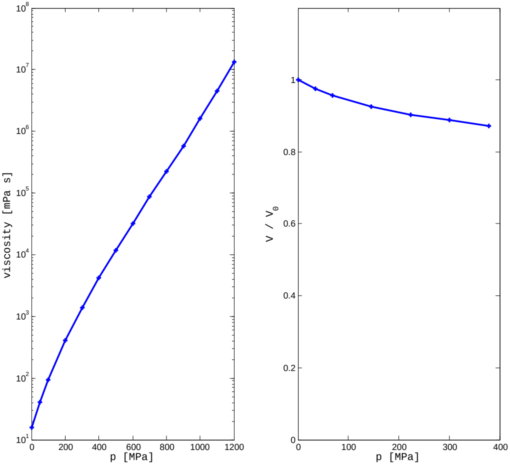
- dimensional reduction: Reynolds approximation
  - a single equation for the pressure
  - a number of assumptions on the flow characteristics
- focus: high pressures
  (EHL: pressure $\sim$ 2-3 GPa, shear rate $\sim 10^6$ ms^{-1})

Viscosity at large pressure and shear rate

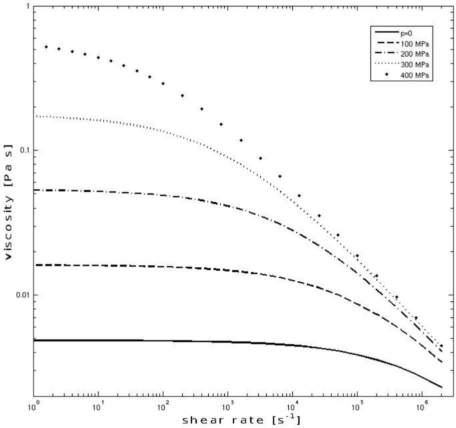
Viscosity and volume variation with pressure for *squalane*

("representing a low viscosity paraffinic mineral oil", S. Bair, *Tribology Letters*, 2006).



Viscosity at large pressure and shear rate

Viscosity for *SAE 10W/40 reference oil RL 88/1*, (partly) by Hutton, Jones, Bates, *SAE*, 1983.



Incompressible fluids with viscosity depending on pressure and shear rate

Mathematical formulation

inside $(0,T)\times\,\Omega$:

$$\begin{aligned} \operatorname{div}\boldsymbol{v} &= 0 \\ \partial_\tau\boldsymbol{v} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}\boldsymbol{S} &= -\nabla\pi + \boldsymbol{f}\,, \\ \boldsymbol{S} &= 2\,\nu(\pi, |\boldsymbol{D}(\boldsymbol{v})|^2)\,\boldsymbol{D}(\boldsymbol{v}) \end{aligned}$$

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Cauchy stress tensor

$$\boldsymbol{T} = -\pi \boldsymbol{I} + 2\nu(\pi, |\boldsymbol{D}|^2)\boldsymbol{D}, \quad \operatorname{tr} \boldsymbol{D} = 0$$

- π is not the thermodynamical pressure
- π is the mean normal stress, $\pi = -\frac{1}{3} \operatorname{tr} \boldsymbol{T}$,
- implicitely constituted model

$$\boldsymbol{T} - \tfrac{1}{3}(\operatorname{tr} \boldsymbol{T})\boldsymbol{I} - 2\nu(-\tfrac{1}{3} \operatorname{tr} \boldsymbol{T}, |\boldsymbol{D}|)\boldsymbol{D} = 0$$

see Rajagopal, *J. Fluid Mech.*, 2006 (and Málek, Rajagopal, 2006, 2007)

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Viscosity formulas used in applications

$$\nu = \nu(\pi, |\boldsymbol{D}(\boldsymbol{v})|^2) = \begin{cases} \sim \ \exp(\alpha\pi), \\ \sim \ (1 + |\boldsymbol{D}(\boldsymbol{v})|^2)^{\frac{p-2}{2}}, \quad 1 < p < 2 \end{cases}$$

So what. . .

. . . is the problem?

- the lubrication **works** and is used since before the invention of wheel
- the viscosity–pressure relation is **present in the very basis of the theory** of elastohydrodynamic lubrication
- **are there any fundamental questions left open?**

So what. . .

- **are there any fundamental questions left open?**

First, one may answer. . .

- lubrication is used everywhere (transportation, electricity production)
- any optimization can save energy consumption and prolongate the lifespan
- more and more precize **quantitative** predictions are needed

So what...

- **are there any fundamental questions left open?**

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...in fact, it is worse than that...

Bair, Gordon, 2006:
" ...there has been relatively little progress since the classic Newtonian solutions ...toward relating film thickness and traction to the properties of individual liquid lubricants

and it **not clear** at this time **that full numerical solutions can even be obtained** for heavily loaded contacts using accurate models.

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. . . in fact, even worse. . .
is the full system of governing equations well-posed?

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Problem well-posedness—first observations

$$\nu = \nu(\pi)$$

- M. Renardy, *Comm. Part. Diff. Eq.*, 1986.
- F. Gazzola, *Z. Angew. Math. Phys.*, 1997.
- F. Gazzola, P. Secchi, *Navier–Stokes eq.: th. and num. meth.* 1998.

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Problem well-posedness—a wave of succesful results

$$\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{D}} \sim (1 + |\boldsymbol{D}|^2)^{\frac{p-2}{2}} \qquad \left| \frac{\partial \boldsymbol{S}}{\partial \pi} \right| \le \gamma_0 \, (1 + |\boldsymbol{D}|^2)^{\frac{p-2}{4}} \qquad 1 < p < 2$$

- Málek, Nečas, Rajagopal, *Arch. Rational Mech. Anal.*, 2002.
- & Hron, Bulíček, Majdoub, . . .

Incompressible fluids with viscosity depending on pressure and shear rate

Mathematical formulation

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- Franta, Málek, Rajagopal, *Proc. Royal Soc. A*, 2005
- & Bulíček, Fišerová, Kaplický, Lanzendörfer, Stebel, Hirn, ...

Basic a priori estimates

Weak formulation

$$(q, \operatorname{div} \boldsymbol{w})_\Omega = 0$$
$$(\boldsymbol{S}(\pi, \boldsymbol{D}(\boldsymbol{v})), \boldsymbol{D}(\boldsymbol{w}))_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega = (\boldsymbol{f}, \boldsymbol{w})_\Omega - (\boldsymbol{b} \quad , \boldsymbol{w})_{\Gamma_P}$$

Inf–sup inequality and the boundedness of $\partial_\pi \boldsymbol{S}$

$$0 < \beta \leq \inf_{q \in L^{p'}_{b.c.}(\Omega)} \sup_{\boldsymbol{w} \in \mathbf{W}^{1,p}_{b.c.}(\Omega)} \frac{(q, \operatorname{div} \boldsymbol{w})_\Omega}{\|q\|_{p'} \|\boldsymbol{w}\|_{1,p}}$$

Pressure uniquely determined by velocity?

$$\beta \, \|\pi^1 - \pi^2\|_{p'} \leq \|\boldsymbol{S}(\pi^1, \boldsymbol{D}(\boldsymbol{v})) - \boldsymbol{S}(\pi^2, \boldsymbol{D}(\boldsymbol{v}))\|_{p'} \leq \left\| \int_{\pi^1}^{\pi^2} \frac{\partial \boldsymbol{S}(\pi, \boldsymbol{D}(\boldsymbol{v}))}{\partial \pi} \mathrm{d}\pi \right\|_{p'}$$
$$\leq \gamma_0 \, \|\pi^1 - \pi^2\|_{p'} \qquad \text{where} \qquad \left| \frac{\partial \boldsymbol{S}}{\partial \pi} \right| \leq \gamma_0$$

Two major questions

…concerned with Reynolds equation

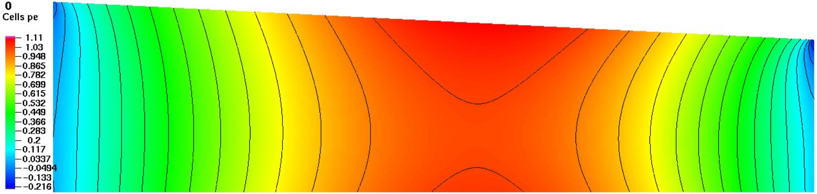
Is it based on well-posed system of equations?

- no success for $\nu(\pi)$
- successful analysis for a subclass of models within the **subcritical case:** $|\partial \boldsymbol{S}/\partial\pi| < \ldots < 1$
- **no success in the supercritical case** $|\partial \boldsymbol{S}/\partial\pi| > 1$.

Hydrodynamic (thick-film) lubrication

Flow in a converging channel

Newtonian model $\nu = \text{const}$



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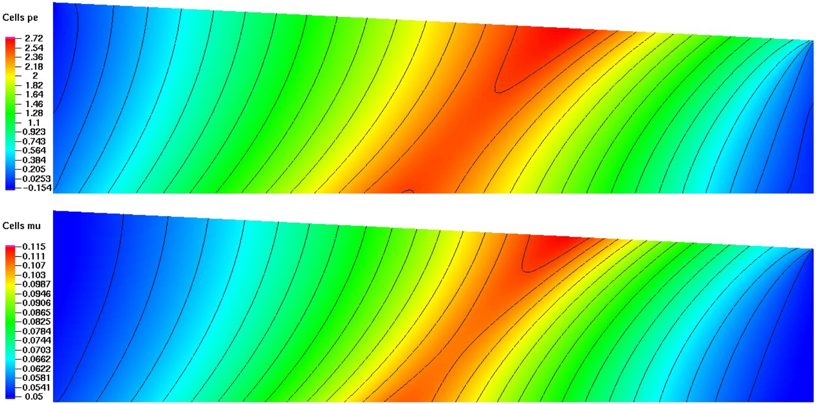
Flow in a converging channel

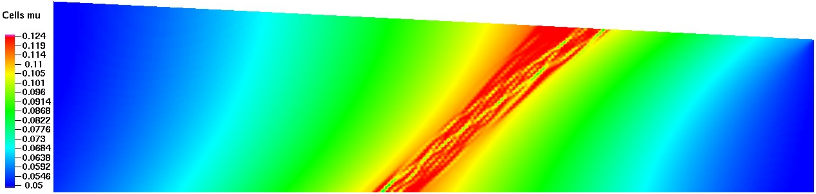
Barus model $\nu = \exp(\alpha\pi)$, $\alpha = 0.306$



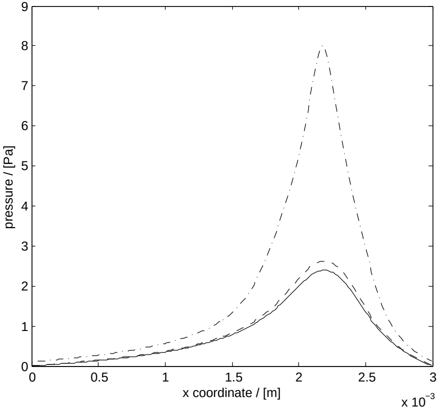
Hydrodynamic (thick-film) lubrication

Flow in a converging channel

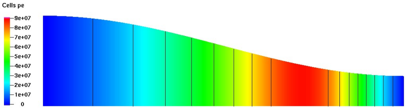
Barus model $\nu = \exp(\alpha\pi)$, $\alpha = 0.3061$



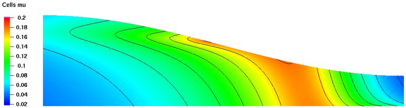
Sensitivity on boundary data



Pressure for $\pi(0) = \pi(L) = 0$ (full),
$1\ \mathrm{MPa}$ (dashed) a
$10\ \mathrm{MPa}$ (dash dotted).



pressure π

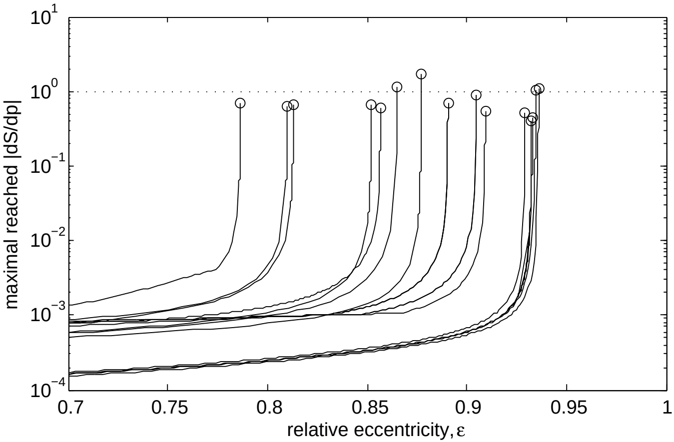


viskosity $\nu(\pi, |\boldsymbol{D}|)$

Hydrodynamic (thick-film) lubrication

Discrete problem stability.

Numerical experiments for journal bearing.



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When do the assumptions on the flow characteristics hold?

- **no success** with FEM simulations in the **supercritical case**
- interesting phenomena (e.g. the modified Reynolds equation) arise in the supercritical, $\implies$ validation of Reynolds approximation not possible
- in **subcritical case**, a priori error estimates for FEM derived (Hirn, Lanzendörfer, Stebel, 2012)
- a posteriori error analysis and adaptive FEM needed for quantitative studies (localized nonlinearity)

Thank you for your attention

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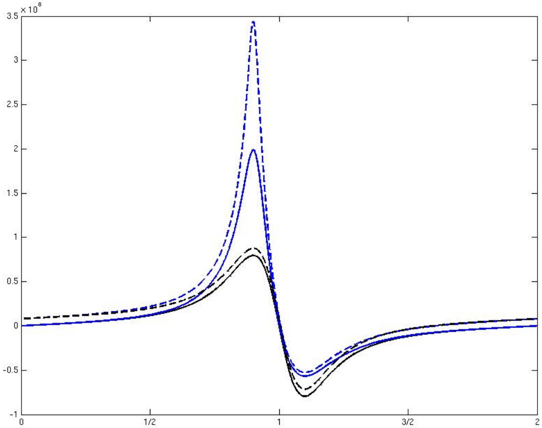
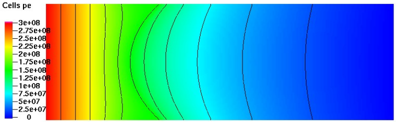
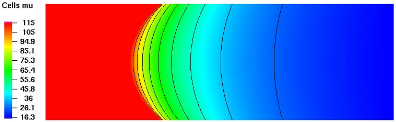


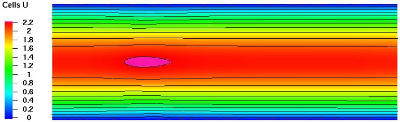
Poiseuille flow



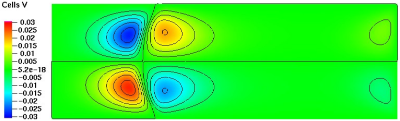
the pressure π



the viscosity $\nu(p)$



the velocity component $\boldsymbol{v} \cdot \boldsymbol{e}_x$



the velocity component $\boldsymbol{v} \cdot \boldsymbol{e}_y$