On isothermal steady flows of incompressible, pressure-thickening and shear-thinning fluids and their Galerkin approximation.

M. Lanzendörfer [1,5]

presenting what he understood from
J. Málek[1,2] and M. Bulíček[1,2],

and collaborated on with
A. Hirn[3] and J. Stebel[2,4].

[1]Faculty of Mathematics and Physics, Charles University in Prague

[2]Jindřich Nečas Center for Mathematical Modeling

[3]University of Heidelberg,

[4]Mathematical Institute, AS CR

[5]Institute of Computer Science, AS CR

❶ Motivated by several practical applications, we consider ❷ incompressible fluids with viscosity depending on shear rate and on pressure. The latter dependence, in particular, leads to difficulties in both theory and numerical simulations. We will focus on isothermal steady flows of a subclass of such fluids; ❸ briefly discuss the known results on existence of weak solutions; ❹ show the connection of the viscosity–pressure relation with the inf–sup inequality and the stable Galerkin discretization; ❺ mention the relation of inf–sup inequality to the pressure boundary conditions. ❻ We will advert to open problems and disclose some troubles occurring in numerical experiments (motivated by the lubrication problems).

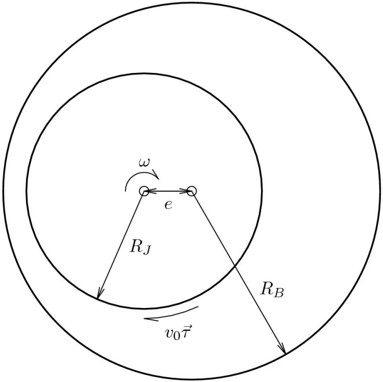
Incompressible fluids with viscosity depending on pressure and shear rate

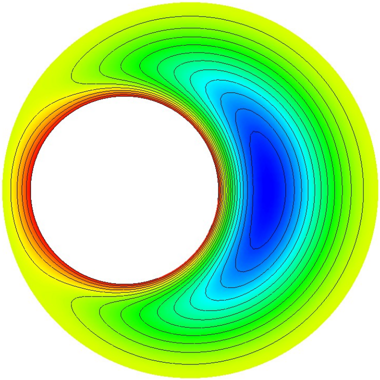
Mathematical formulation

inside $(0, T) \times \Omega$:

$$
\begin{aligned}
\operatorname{div} \boldsymbol{v} &= 0 \\
\partial_\tau \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \boldsymbol{S} &= -\nabla\pi + \boldsymbol{f}\,, \\
\boldsymbol{S} &= 2\,\nu(\pi, |\boldsymbol{D}(\boldsymbol{v})|^2)\,\boldsymbol{D}(\boldsymbol{v})
\end{aligned}
$$

Applications: lubrication problems, journal bearing





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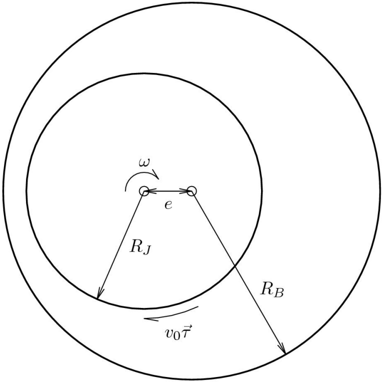
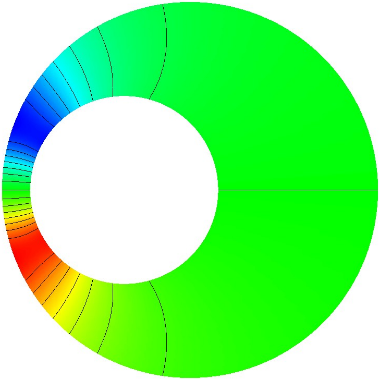
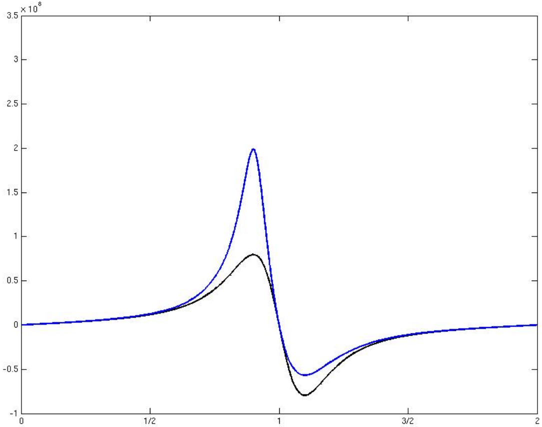


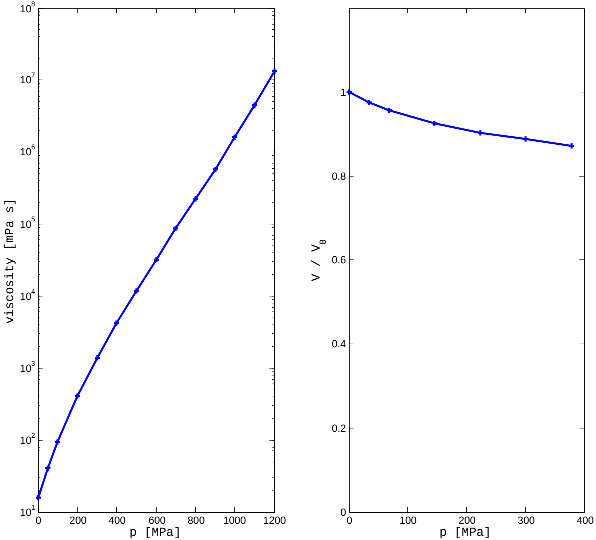


Applications: lubrication problems, journal bearing

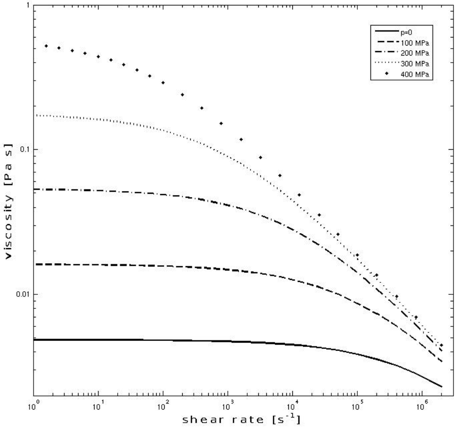


Viscosity and volume variation with pressure for *squalane*

(representing a low viscosity paraffinic mineral oil, see S. Bair, *Tribology Letters*, 2006).



Viscosity for *SAE 10W/40 reference oil RL 88/1*,
(partly) by Hutton, Jones, Bates, *SAE*, 1983



Incompressible fluids with viscosity depending on pressure and shear rate

Mathematical formulation

inside $(0, T) \times \Omega$:

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0 \\ \partial_\tau \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \boldsymbol{S} &= -\nabla \pi + \boldsymbol{f}\,, \\ \boldsymbol{S} &= 2\,\nu(\pi, |\boldsymbol{D}(\boldsymbol{v})|^2)\,\boldsymbol{D}(\boldsymbol{v}) \end{aligned}$$

Viscosity formulas used in applications

$$\nu = \nu(\pi, |\boldsymbol{D}(\boldsymbol{v})|^2) = \begin{cases} \sim \ \exp(\alpha\pi), \\ \sim \ (1 + |\boldsymbol{D}(\boldsymbol{v})|^2)^{\frac{p-2}{2}}, \quad 1 < p < 2 \end{cases}$$

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Problem well-posedness—first observations

$$\nu = \nu(\pi)$$

- M. Renardy, *Comm. Part. Diff. Eq.*, 1986.
- F. Gazzola, *Z. Angew. Math. Phys.*, 1997.
- F. Gazzola, P. Secchi, *Navier–Stokes eq.: th. and num. meth.* 1998.

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\boldsymbol{S} &= 2\,\nu(\pi, |\boldsymbol{D}(\boldsymbol{v})|^2)\,\boldsymbol{D}(\boldsymbol{v})
\end{aligned}
$$

Problem well-posedness—first positive results

$$
\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{D}} \sim (1 + |\boldsymbol{D}|^2)^{\frac{p-2}{2}} \qquad \left|\frac{\partial \boldsymbol{S}}{\partial \pi}\right| \le \gamma_0\,(1 + |\boldsymbol{D}|^2)^{\frac{p-2}{4}} \qquad 1 < p < 2
$$

- Málek, Nečas, Rajagopal, *Arch. Rational Mech. Anal.*, 2002.
- Hron, Málek, Nečas, Rajagopal, *Math. Comput. Simulation*, 2003.
- Málek, Rajagopal, *Handbook of mathematical fluid dynamics*, 2007.

Incompressible fluids with viscosity depending on pressure and shear rate

Mathematical formulation

inside $(0,T)\times\,\Omega$:

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on the boundary $(0,T)\times\,\partial\Omega = \Gamma_D\cup\Gamma_N\cup\Gamma_P$:

$$\begin{aligned} &\boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ and } -\boldsymbol{T}\boldsymbol{n} = \sigma\,\boldsymbol{v} && \text{on } \Gamma_N && \\ &\boldsymbol{v} = \boldsymbol{v_D} && \text{on } \Gamma_D && \text{if } \Gamma_P = \emptyset, \\ &-\boldsymbol{T}\boldsymbol{n} = \boldsymbol{b}(\boldsymbol{v}) && \text{on } \Gamma_P && \text{then } \int_{\Omega_0}\pi\,\mathrm{d}\boldsymbol{x} = 0 \end{aligned}$$

- Bulíček, Málek, Rajagopal, *Indiana Univ. Math. J.*, 2007
- Bulíček, Málek, Rajagopal, *SIAM J. Math. Anal.*, 2009

Incompressible fluids with viscosity depending on pressure and shear rate

Mathematical formulation

inside Ω:

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\begin{aligned}
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\end{aligned}
$$

- Franta, Málek, Rajagopal, *Proc. Royal Soc. A*, 2005
- M. L., *Nonlin. Anal.: Real World App.*, 2009

Incompressible fluids with viscosity depending on pressure and shear rate

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inside Ω:

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on the boundary $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_P$:

$\boldsymbol{v} \cdot \boldsymbol{n} = 0$ and $-\boldsymbol{T}\boldsymbol{n} = \sigma\,\boldsymbol{v}$ on Γ_N

$\boldsymbol{v} = \boldsymbol{v_D}$ on Γ_D if $\Gamma_P = \emptyset$,

$-\boldsymbol{T}\boldsymbol{n} = \boldsymbol{b}(\boldsymbol{v})$ on Γ_P then $\int_{\Omega_0} \pi \, \mathrm{d}\boldsymbol{x} = 0$

Incompressible fluids with viscosity depending on pressure and shear rate

Mathematical formulation

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- Stebel & M. L., *Appl. Mat.–Czech.*, in print; 2009 *preprint NCMM*
- Stebel & M. L., *Math. Comput. Simulat.*, submitted 2009 *preprint NCMM*

Basic a priori estimates

Weak formulation

$$
\begin{aligned}
(q, \operatorname{div} \boldsymbol{w})_\Omega &= 0 \\
([\nabla \boldsymbol{v}]\boldsymbol{v}, \boldsymbol{w})_\Omega + (\boldsymbol{S}(\pi, \boldsymbol{D}(\boldsymbol{v})), \boldsymbol{D}(\boldsymbol{w}))_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega &= (\boldsymbol{f}, \boldsymbol{w})_\Omega - (\boldsymbol{b}(\boldsymbol{v}), \boldsymbol{w})_{\Gamma_P}
\end{aligned}
$$

Basic a priori estimates

Weak formulation

$$(q, \operatorname{div} \boldsymbol{w})_\Omega = 0$$
$$(\boldsymbol{S}(\pi, \boldsymbol{D}(\boldsymbol{v})), \boldsymbol{D}(\boldsymbol{w}))_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega = (\boldsymbol{f}, \boldsymbol{w})_\Omega - (\boldsymbol{b} \quad , \boldsymbol{w})_{\Gamma_P}$$

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Test eq. by solution

$$(\boldsymbol{S}(\pi, \boldsymbol{D}(\boldsymbol{v})), \boldsymbol{D}(\boldsymbol{v}))_\Omega \sim |\boldsymbol{D}(\boldsymbol{v})|^p \pm 1$$

$\Longrightarrow$

$$\|\boldsymbol{D}(\boldsymbol{v})\|_p \leq K \qquad \Longrightarrow \qquad \|\boldsymbol{v}\|_{1,p} + \|\boldsymbol{S}\|_{p'} \leq K$$

Basic a priori estimates

Weak formulation

$$(q, \operatorname{div} \boldsymbol{w})_\Omega = 0$$
$$(\boldsymbol{S}(\pi, \boldsymbol{D}(\boldsymbol{v})), \boldsymbol{D}(\boldsymbol{w}))_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega = (\boldsymbol{f}, \boldsymbol{w})_\Omega - (\boldsymbol{b} \quad , \boldsymbol{w})_{\Gamma_P}$$

Inf–sup inequality and boundedness of $\partial_\pi \boldsymbol{S}$

$$0 < \beta \le \inf_{q \in \mathrm{L}^{p'}_{b.c.}(\Omega)} \sup_{\boldsymbol{w} \in \mathbf{W}^{1,p}_{b.c.}(\Omega)} \frac{(q, \operatorname{div} \boldsymbol{w})_\Omega}{\|q\|_{p'} \|\boldsymbol{w}\|_{1,p}}$$

$\Longrightarrow$

$$\beta \|\pi\|_{p'} \le \|\boldsymbol{S}(\pi, \boldsymbol{D}(\boldsymbol{v}))\|_{p'} + \|\boldsymbol{f} + \boldsymbol{b}\| \le K$$

Basic a priori estimates

Weak formulation

$$(q, \operatorname{div} \boldsymbol{w})_\Omega = 0$$
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Pressure uniquelly determined by velocity?

$$\begin{aligned} \beta \|\pi^1 - \pi^2\|_{p'} &\leq \|\boldsymbol{S}(\pi^1, \boldsymbol{D}(\boldsymbol{v})) - \boldsymbol{S}(\pi^2, \boldsymbol{D}(\boldsymbol{v}))\|_{p'} \leq \left\| \int_{\pi^1}^{\pi^2} \partial_\pi \boldsymbol{S}(\pi, \boldsymbol{D}(\boldsymbol{v})) \mathrm{d}\pi \right\|_{p'} \\ &\leq \gamma_0 \|\pi^1 - \pi^2\|_{p'} \end{aligned}$$

Basic a priori estimates

Weak formulation

$$(q, \operatorname{div} \boldsymbol{w})_\Omega = 0$$
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Pressure and velocity uniquelly determined?

$$\beta \|\pi^1 - \pi^2\|_2 \quad \leq \quad \|\boldsymbol{S}(\pi^1, \boldsymbol{D}(\boldsymbol{v}^1)) - \boldsymbol{S}(\pi^2, \boldsymbol{D}(\boldsymbol{v}^2))\|_2$$

Basic a priori estimates

$$\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{D}} \sim (1 + |\boldsymbol{D}|^2)^{\frac{p-2}{2}} \qquad \left|\frac{\partial \boldsymbol{S}}{\partial \pi}\right| \le \gamma_0 \, (1 + |\boldsymbol{D}|^2)^{\frac{p-2}{4}} \qquad 1 < p < 2$$

write

$$\begin{aligned} \boldsymbol{S}^i &:= \boldsymbol{S}(\pi^i, \boldsymbol{D}(\boldsymbol{v}^i)), \ i = 1, 2 \\ d(\boldsymbol{v}^1, \boldsymbol{v}^2) &:= \int_\Omega \int_0^1 (1 + |\boldsymbol{D}(\boldsymbol{v}^1) + s\, \boldsymbol{D}(\boldsymbol{v}^2 - \boldsymbol{v}^1)|^2)^{\frac{p-2}{2}} |\boldsymbol{D}(\boldsymbol{v}^1 - \boldsymbol{v}^2)|^2 \, \mathrm{d}s \, \mathrm{d}\boldsymbol{x} \end{aligned}$$

then the assumptions imply

$$\|\boldsymbol{S}^1 - \boldsymbol{S}^2\|_2 \le \sigma_1 \, d(\boldsymbol{v}^1, \boldsymbol{v}^2) + \gamma_0 \|\pi^1 - \pi^2\|_2,$$

$$d(\boldsymbol{v}^1, \boldsymbol{v}^2)^2 \le \frac{2}{\sigma_0} \left(\boldsymbol{S}^1 - \boldsymbol{S}^2, \boldsymbol{D}^1 - \boldsymbol{D}^2\right)_\Omega + \frac{\gamma_0^2}{\sigma_0^2} \|\pi^1 - \pi^2\|_2^2$$

Basic a priori estimates

Weak formulation

$$(q, \operatorname{div} \boldsymbol{w})_\Omega = 0$$

$$(\boldsymbol{S}(\pi, \boldsymbol{D}(\boldsymbol{v})), \boldsymbol{D}(\boldsymbol{w}))_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega = (\boldsymbol{f}, \boldsymbol{w})_\Omega - (\boldsymbol{b} \quad , \boldsymbol{w})_{\Gamma_P}$$

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$$0 < \beta \leq \inf_{q \in \mathrm{L}^{p'}_{b.c.}(\Omega)} \sup_{\boldsymbol{w} \in \mathbf{W}^{1,p}_{b.c.}(\Omega)} \frac{(q, \operatorname{div} \boldsymbol{w})_\Omega}{\|q\|_{p'} \|\boldsymbol{w}\|_{1,p}}$$

Pressure and velocity uniquelly determined?

$$\beta \, \|\pi^1 - \pi^2\|_2 \quad \leq \quad \|\boldsymbol{S}^1 - \boldsymbol{S}^2\|_2 \ \leq \ \sigma_1 \, d(\boldsymbol{v}^1, \boldsymbol{v}^2) + \gamma_0 \|\pi^1 - \pi^2\|_2$$

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Discrete problem—numerical approximation

Galerkin approximation

$$\boldsymbol{V}_h \subset \mathbf{W}^{1,p}_{b.c.}(\Omega), \qquad Q_h \subset \mathrm{L}^{p'}_{b.c.}(\Omega).$$

Find $\boldsymbol{v}_h \in \boldsymbol{V}_h$, $\pi_h \in Q_h$ such that:

$$(q_h, \operatorname{div} \boldsymbol{w_h})_\Omega = 0$$

$$(\boldsymbol{S}(\pi_h, \boldsymbol{D}(\boldsymbol{v}_h)), \boldsymbol{D}(\boldsymbol{w}_h))_\Omega - (\pi_h, \operatorname{div} \boldsymbol{w}_h)_\Omega = (\boldsymbol{f} - \boldsymbol{b}, \boldsymbol{w}_h)_\Omega$$

hold for all $\boldsymbol{w}_h \in \boldsymbol{V}_h$, $q_h \in Q_h$.

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hold for all $\boldsymbol{w}_h \in \boldsymbol{V}_h$, $q_h \in Q_h$.

Discrete inf–sup condition

$$0 < \tilde{\beta} \leq \inf_{q \in Q_h} \sup_{\boldsymbol{w} \in \boldsymbol{V}_h} \frac{(q, \operatorname{div} \boldsymbol{w})_\Omega}{\|q\|_{p'} \|\boldsymbol{w}\|_{1,p}}$$

Summary

- $\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{D}} \sim (1 + |\boldsymbol{D}|^2)^{\frac{p-2}{2}}$, $p < 2$ shear-thinning setting.
- inf–sup inequality ($\beta > 0$), discrete inf–sup condition ($\tilde{\beta} > 0$), boundary conditions
- $\left| \frac{\partial \boldsymbol{S}}{\partial \pi} \right| \leq \gamma_0 (1 + |\boldsymbol{D}|^2)^{\frac{p-2}{4}}$ to deal with pressure non-linearity

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- $\tilde{\beta} > 0 \implies \exists$ discrete solution $(\pi_h, \boldsymbol{v}_h)$ and

$$\|\boldsymbol{v_h}\|_{1,p} + \|\boldsymbol{S}(\pi_h, \boldsymbol{v}_h)\|_{p'} + \beta\|\pi_h\|_{p'} \leq K$$

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$$\|\boldsymbol{v_h}\|_{1,p} + \|\boldsymbol{S}(\pi_h, \boldsymbol{v}_h)\|_{p'} + \beta\|\pi_h\|_{p'} \leq K$$

- $\gamma_0 < \tilde{\beta}\frac{\sigma_0}{\sigma_0+\sigma_1} \implies (\pi_h, \boldsymbol{v}_h)$ unique, $\quad \gamma_0 < \beta\frac{\sigma_0}{\sigma_0+\sigma_1} \implies (\pi, \boldsymbol{v})$ at most one

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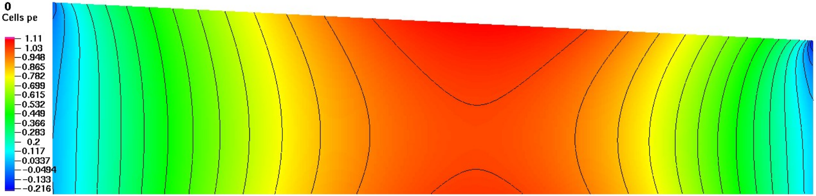
- $\gamma_0 < \tilde{\beta}\frac{\sigma_0}{\sigma_0+\sigma_1} \implies$ a priori error estimates

$$\|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,p} \lessapprox \inf_{\boldsymbol{w}_h \in \boldsymbol{V}_h} \|\boldsymbol{v} - \boldsymbol{w}_h\|_{1,p} + \inf_{q_h \in Q_h} \|\pi - q_h\|_{p'}$$

$$\|\pi - \pi_h\|_{p'} \lessapprox \inf_{q_h \in Q_h} \|\pi - q_h\|_{p'} + \|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,p}^{2/p'}$$

Applications: lubrication problems, journal bearing

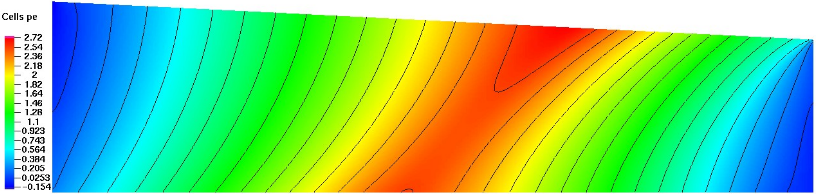
Flow in a converging channel

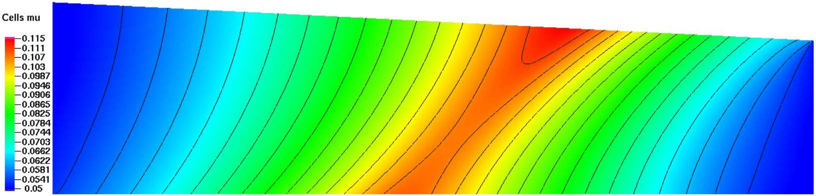


Applications: lubrication problems, journal bearing

Flow in a converging channel

Barus model $\nu = \exp(\alpha\pi)$, $\alpha = 0.306$

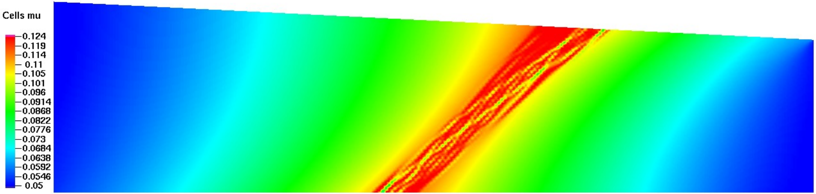




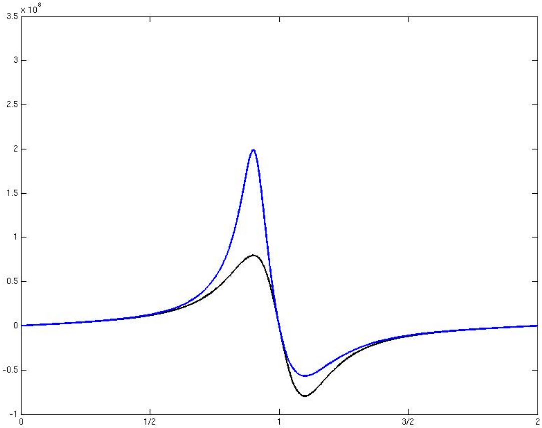
Applications: lubrication problems, journal bearing

Flow in a converging channel

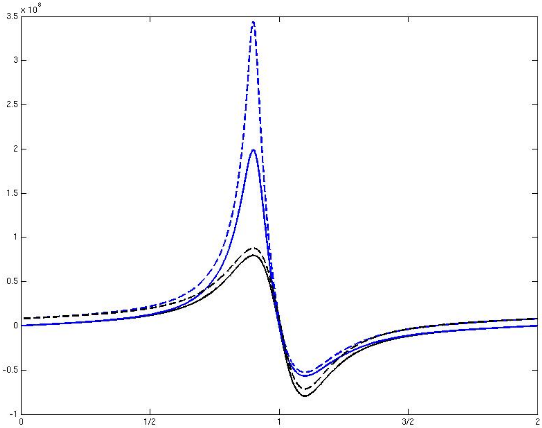
Barus model $\nu = \exp(\alpha\pi)$, $\alpha = 0.3061$



Applications: lubrication problems, journal bearing



Applications: lubrication problems, journal bearing



Thank you for your attention!