Numerical behavior of inexact saddle point solvers

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Saddle point problems

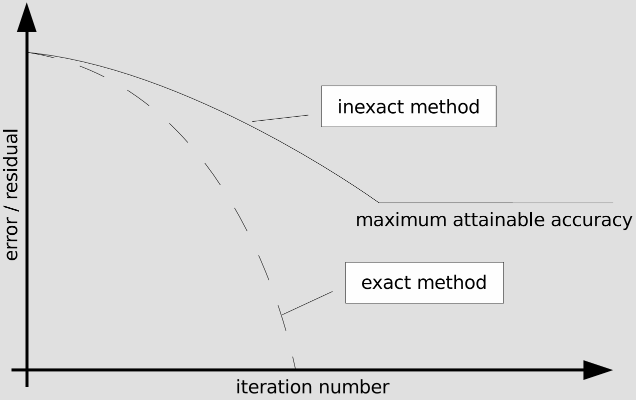
We consider a saddle point problem with the symmetric 2×2 block form

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

- A is a square $n \times n$ nonsingular (symmetric positive definite) matrix,
- B is a rectangular $n \times m$ matrix of (full column) rank m.

Applications: mixed finite element approximations, weighted least squares, constrained optimization etc. [Benzi, Golub, and Liesen, 2005].

inexact solutions of inner systems + rounding errors
$\rightarrow$ inexact saddle point solver



Schur complement reduction method

- Compute y as a solution of the Schur complement system

$$B^T A^{-1} B y = B^T A^{-1} f,$$

- compute x as a solution of

$$Ax = f - By.$$

Systems with A are solved inexactly, the computed solution $\bar{u}$ of $Au = b$ is interpreted an exact solution of a perturbed system

$$(A + \Delta A)\bar{u} = b + \Delta b,\ \|\Delta A\| \leq \tau \|A\|,\ \|\Delta b\| \leq \tau \|b\|,\ \tau\kappa(A) \ll 1.$$

choose y_0, solve $Ax_0 = f - By_0$

compute α_k and $p_k^{(y)}$

$y_{k+1} = y_k + \alpha_k p_k^{(y)}$

solve $Ap_k^{(x)} = -Bp_k^{(y)}$

back-substitution:

A: $x_{k+1} = x_k + \alpha_k p_k^{(x)}$,

B: solve $Ax_{k+1} = f - By_{k+1}$,

C: solve $Au_k = f - Ax_k - By_{k+1}$,

$x_{k+1} = x_k + u_k$.

inner iteration

$r_{k+1}^{(y)} = r_k^{(y)} - \alpha_k B^T p_k^{(x)}$

outer iteration

Measure of the limiting accuracy

The limiting (maximum attainable) accuracy is measured by the ultimate (asymptotic) values of:

1. **the Schur complement residual**: $B^T A^{-1} f - B^T A^{-1} B y_k$;
2. **the residuals in the saddle point system**: $f - Ax_k - By_k$ and $-B^T x_k$;
3. **the forward errors**: $x - x_k$ and $y - y_k$.

Numerical example:

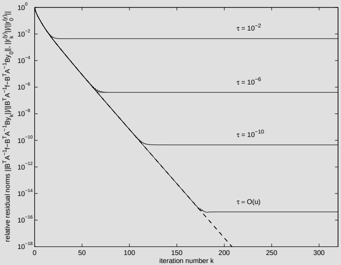
$$A = \mathrm{tridiag}(1, 4, 1) \in \mathbb{R}^{100\times 100},\ B = \mathrm{rand}(100, 20),\ f = \mathrm{rand}(100, 1),$$
$$\kappa(A) = \|A\| \cdot \|A^{-1}\| = 7.1695 \cdot 0.4603 \approx 3.3001,$$
$$\kappa(B) = \|B\| \cdot \|B^{\dagger}\| = 5.9990 \cdot 0.4998 \approx 2.9983.$$

Accuracy in the outer iteration process

$$B^T(A+\Delta A)^{-1}B\hat{y} = B^T(A+\Delta A)^{-1}f,$$

$$\|B^TA^{-1}f - B^TA^{-1}B\hat{y}\| \leq \frac{\tau\kappa(A)}{1-\tau\kappa(A)}\|A^{-1}\|\|B\|^2\|\hat{y}\|.$$

$$\|-B^TA^{-1}f + B^TA^{-1}By_k - r_k^{(y)}\| \leq \frac{O(\tau)\kappa(A)}{1-\tau\kappa(A)}\|A^{-1}\|\|B\|(\|f\| + \|B\|Y_k).$$



Accuracy in the saddle point system

$$-B^T A^{-1} f + B^T A^{-1} B y_k = -B^T x_k - B^T A^{-1}(f - Ax_k - By_k)$$

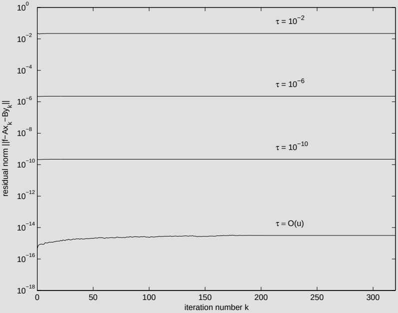
$$\|f - Ax_k - By_k\| \leq \frac{O(\alpha_1)\kappa(A)}{1 - \tau\kappa(A)}(\|f\| + \|B\| Y_k),$$

$$\|-B^T x_k - r_k^{(y)}\| \leq \frac{O(\alpha_2)\kappa(A)}{1 - \tau\kappa(A)}\|A^{-1}\|\|B\|(\|f\| + \|B\| Y_k),$$

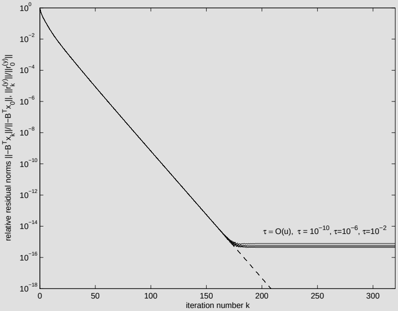
$$Y_k \equiv \max\{\|y_i\| \,|\, i = 0, 1, \ldots, k\}.$$

Back-substitution scheme	α_1	α_2
A: Generic update $x_{k+1} = x_k + \alpha_k p_k^{(x)}$	τ	u
B: Direct substitution $x_{k+1} = A^{-1}(f - By_{k+1})$	τ	τ
C: Corrected dir. subst. $x_{k+1} = x_k + A^{-1}(f - Ax_k - By_{k+1})$	u	τ

additional system with A

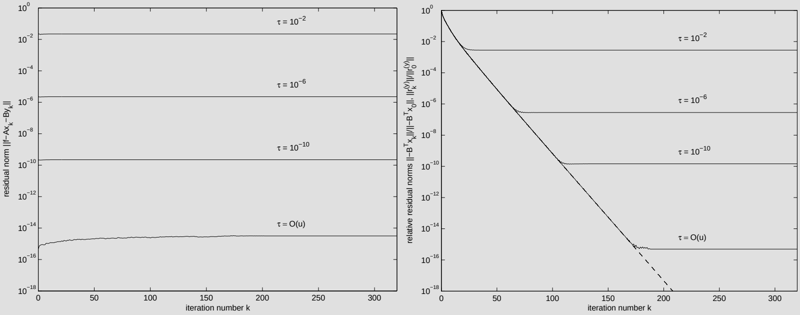


residual norm ||f−Ax_k−By_k||
iteration number k
τ = 10^−2
τ = 10^−6
τ = 10^−10
τ = O(u)

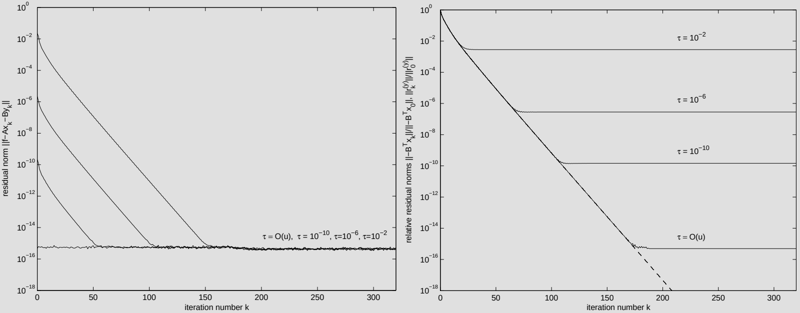


relative residual norms ||−B^T x_k||/||−B^T x_0||, ||r_k^(y)||/||r_0^(y)||
iteration number k
τ = O(u), τ = 10^−10, τ=10^−6, τ=10^−2

Direct substitution: $x_{k+1} = A^{-1}(f - By_{k+1})$



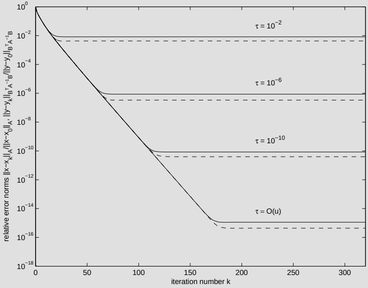
Corrected direct substitution: $x_{k+1} = x_k + A^{-1}(f - Ax_k - By_{k+1})$



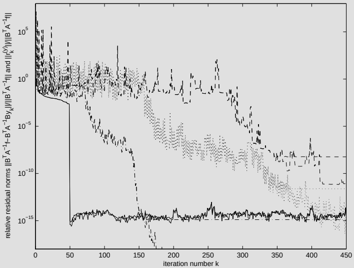
Forward error of computed approximate solution

$$\|x - x_k\| \leq \gamma_1 \|f - Ax_k - By_k\| + \gamma_2 \| - B^T x_k\|,$$
$$\|y - y_k\| \leq \gamma_2 \|f - Ax_k - By_k\| + \gamma_3 \| - B^T x_k\|,$$

$$\gamma_1 = \sigma_{min}^{-1}(A),\ \gamma_2 = \sigma_{min}^{-1}(B),\ \gamma_3 = \sigma_{min}^{-1}(B^T A^{-1} B).$$



- All bounds of the limiting accuracy depend on the maximum norm of computed iterates, cf. [Greenbaum, 1997].
- The accuracy measured by the residuals of the saddle point problem depends on the choice of the back-substitution scheme [J, R, 2008].
- Care must be taken when solving nonsymmetric systems [J, R, 2008b].



- The residuals in the outer iteration process and the forward errors of computed approximations are proportional to the backward error in solution of inner systems.

Thank you for your attention.

http://www.cs.cas.cz/∼miro

P. Jiránek and M. Rozložník. Maximum attainable accuracy of inexact saddle point solvers. *SIAM J. Matrix Anal. Appl.*, 29(4):1297–1321, 2008.

P. Jiránek and M. Rozložník. Limiting accuracy of segregated solution methods for nonsymmetric saddle point problems. *J. Comput. Appl. Math.*, 215:28–37, 2008.

Null-space projection method

- compute $x \in N(B^T)$ as a solution of the projected system

$$(I - \Pi)A(I - \Pi)x = (I - \Pi)f,$$

- compute y as a solution of the least squares problem

$$By \approx f - Ax,$$

  Π is the orthogonal projector onto $R(B)$.

The least squares with B are solved inexactly, i.e. the computed solution $\bar{v}$ of $Bv \approx c$ is an exact solution of a perturbed least squares problem

$$(B + \Delta B)\bar{v} \approx c + \Delta c, \ \|\Delta B\| \leq \tau\|B\|, \ \|\Delta c\| \leq \tau\|c\|, \ \tau\kappa(B) \ll 1.$$

Iterative solution of the null-space projected system

$$
\left.
\begin{array}{l}
\text{choose } x_0, \text{ solve } By_0 \approx f - Ax_0 \\
\text{compute } \alpha_k \text{ and } p_k^{(x)} \in N(B^T) \\
x_{k+1} = x_k + \alpha_k p_k^{(x)} \\
\quad \left.
\begin{array}{|l}
\text{solve } Bp_k^{(y)} \approx r_k^{(x)} - \alpha_k A p_k^{(x)} \\
\color{red}{\text{back-substitution:}} \\
\textbf{A: } y_{k+1} = y_k + p_k^{(y)}, \\
\textbf{B: } \text{solve } By_{k+1} \approx f - Ax_{k+1}, \\
\textbf{C: } \text{solve } Bv_k \approx f - Ax_{k+1} - By_k, \\
\qquad y_{k+1} = y_k + v_k.
\end{array}
\right\} \begin{array}{l}\text{inner}\\ \text{iteration}\end{array} \\
r_{k+1}^{(x)} = r_k^{(x)} - \alpha_k A p_k^{(x)} - Bp_k^{(y)}
\end{array}
\right\} \begin{array}{l}\text{outer}\\ \text{iteration}\end{array}
$$

$$\|f - Ax_k - By_k - r_k^{(x)}\| \leq \frac{O(\alpha_3)\kappa(B)}{1 - \tau\kappa(B)}(\|f\| + \|A\|X_k),$$

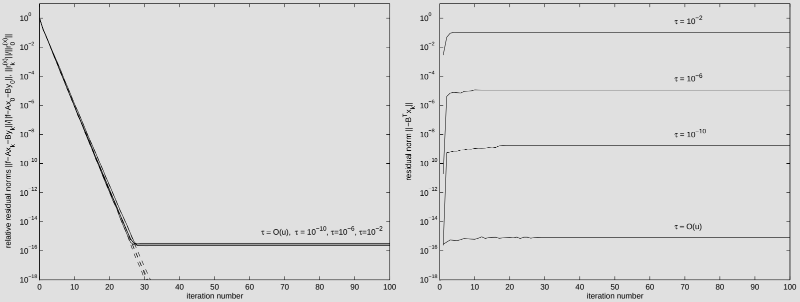
$$\| - B^T x_k\| \leq \frac{O(\tau)\kappa(B)}{1 - \tau\kappa(B)}\|B\|X_k,$$

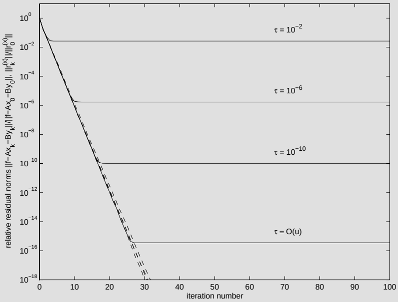
$$X_k \equiv \max\{\|x_i\| \mid i = 0, 1, \ldots, k\}.$$

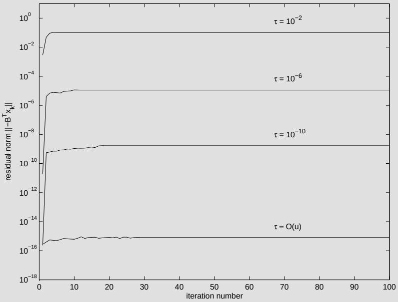
Back-substitution scheme		α_3
A:	Generic update $y_{k+1} = y_k + p_k^{(y)}$	u
B:	Direct substitution $y_{k+1} = B^\dagger(f - Ax_{k+1})$	τ
C:	Corrected dir. subst. $y_{k+1} = y_k + B^\dagger(f - Ax_{k+1} - By_k)$	u

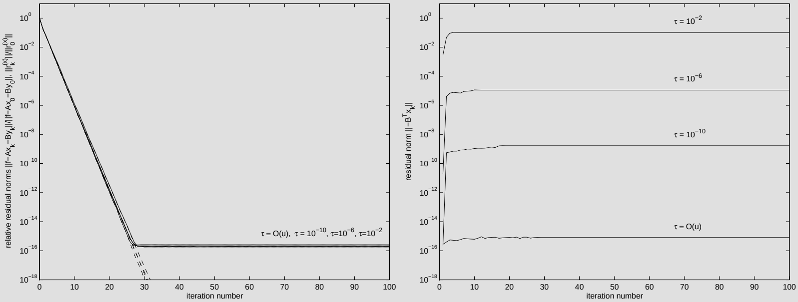
additional least square with B

Generic update: $y_{k+1} = y_k + p_k^{(y)}$




relative residual norms ||f−Ax_k−By_k||/||f−Ax_0−By_0||, ||r_k^(x)||/||r_0^(x)||
iteration number
τ = 10^−2
τ = 10^−6
τ = 10^−10
τ = O(u)


residual norm ||−B^T x_k||
iteration number
τ = 10^−2
τ = 10^−6
τ = 10^−10
τ = O(u)


relative residual norms $\|f-Ax_k-By_k\|/\|f-Ax_0-By_0\|$, $\|r_k^{(x)}\|/\|r_0^{(x)}\|$
iteration number
$\tau = O(u)$, $\tau = 10^{-10}$, $\tau=10^{-6}$, $\tau=10^{-2}$
residual norm $\|-B^Tx_k\|$
$\tau = 10^{-2}$
$\tau = 10^{-6}$
$\tau = 10^{-10}$
$\tau = O(u)$
iteration number

References

M. Benzi, G. H. Golub, and J. Liesen. Numerical solution of saddle point problems. *Acta Numer.*, 14:1–137, 2005.

A. Greenbaum. Estimating the attainable accuracy of recursively computed residual methods. *SIAM J. Matrix Anal. Appl.*, 18(3):535–551, 1997.

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