On the Way Towards Robust Algebraic Preconditioners

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Abstract

Rapidly increasing sizes of discrete problems arising from solving partial differential equations in three dimensions imply strong need for fast and efficient iterative methods with moderate memory demands. It is well-known that iterative methods must be preconditioned in order to be really useful. Algebraic preconditioners represent a large group of approaches which are important both theoretically and practically, despite their generality and possible lack of specificity for solving particular problems. In addition, such preconditioners are often combined with the problem-based solving strategies, possibly enhanced by the multilevel framework or used to solve non-PDE applications.

In our talk we deal with the problem of robustness of algebraic preconditioners, in particular we consider direct and inverse incomplete decompositions. The development of these decompositions, e.g., standard direct incomplete Cholesky and LU decompositions is traditionally well connected with our community. Their subsequent improvements from early stencil-based relaxed factorizations towards fully general techniques with additional enhancements enabled to successfully solve important applications. The role of both factorized and non-factorized matrix inverses in this context may be less visible, but it starts to shape the field of contemporary preconditioning techniques as well. One example may be the recent development of inverse-based dropping and related pivoting strategies [1].

Contemporary implementations of incomplete decompositions are typically improved by various enhancements like by permutations which enforce stronger diagonal dominance or by sophisticated modifications of matrix entries. Nevertheless, many of such approaches may improve the preconditioners just quantitatively and not qualitatively. The quantitative improvement may be very important from the practical point of view but it does not answer the important question whether we are able to improve incomplete decompositions adaptively by understanding the complex interplay between the direct decomposition and its inverse counterpart. Note that these two types of decomposition do not differ locally so much as we could suppose. For example, it is well-known that the corresponding direct and inverse elementary elimination matrices differ just by the sign of the off-diagonal entry and the results discussed below are heavily based on this fact. We believe that it is possible to understand the complementary role of both of these decompositions and use this knowledge to develop more robust algebraic preconditioners.

Our presentation will be mainly based on the two papers [2], [3] devoted to the orthogonalization techniques with a non-standard inner product. In our case we consider the inner product with the matrix $(D - A^{-1})^{-1}$, where A is the system matrix of the considered system of linear algebraic equations and D is a chosen auxiliary diagonal matrix. Such inner product puts together both direct and inverse elementary elimination matrices in each step of the decomposition. It was shown that this orthogonalization leads to the simultaneous computation of the direct and inverse decomposition. All the computed factors are approximate if the decomposition is incomplete. Practical computational procedure based on this technique connects the computation of the direct and inverse incomplete factors by dropping of some evaluated entries such that the actual dropping in the direct factor is directed by the growth of the inverse factor and vice versa. The paper [3] shows that it is possible to influence mutually the computation of both factors even without the inverse-based dropping. Nevertheless, purely theoretical considerations of this kind should be followed by practical preconditioning procedures. That is, they should take into account the possibility to

apply the theoretical rules inside practical implementations. The implementation should not have excessive memory demands and, at the same time, memory restrictions should not pose difficulties for fast access to the partially computed quantities which are used in the orthogonalization.

Our implementation clearly shows that it is possible to enhance the robustness of the incomplete decompositions in this way. The resulting procedure is cost-efficient and we believe that it can be used not only independently, but it can be a useful building block, for example, in more complex solvers. Nevertheless, a significant number of open questions remains and we would like to discuss at least some of them. First, there is a problem whether this approach can be efficiently combined with some kind of pivoting in order to provide even more powerful preconditioners. If this would be feasible then we could think, for example, about coupling of the new approach with the multilevel framework. Another problem connected to our search for efficient algebraic solvers can be posed as follows: we ask whether is it possible to compute a direct incomplete Cholesky decomposition similarly to the computation of the stabilized factorized inverse SAINV, namely via orthogonal projections. The question is motivated by the existence of the breakdown-free inverse incomplete decomposition based on the generalized modified Gram-Schmidt algorithm which performs reasonably well within the class of inverse decompositions. The answer is positive, but the algorithmic scheme is not straightforward since it includes some quantities which may be available only approximately. Our talk will include some of these developments including also applications of the discussed direct-inverse decomposition for condition estimation based on the joint work with Jurjen Duintjer Tebbens.

[1] M. Bollhöfer, Y. Saad. A factored approximate inverse preconditioner with pivoting, *SIAM J. Matrix Anal. Appl., 23 (2002), 692-705.*

[2] R. Bru, J. Marín, J. Mas and M. Tůma. Balanced incomplete factorization, *SIAM J. on Scientific Computing, 30 (2008), 2302-2318.*

[3] R. Bru, J. Marín, J. Mas and M. Tůma. Improved balanced incomplete factorization, *SIAM J. Matrix Anal. Appl., 2010, to appear.*