BIF: Balanced Incomplete Factorization

Miroslav Tůma

Institute of Computer Science,Academy of Sciences of the Czech Republic
Pod Vodárenskou věží 2, 182 07 Praha 8 - Libeň, Czech Republic

1 Introduction

Here we consider a new incomplete factorization which is intended for preconditioning of the conjugate gradient method for the solution of the system

$$Ax = b,$$

where $b \in \mathbb{R}^n$ is the right-hand side vector, $x \in \mathbb{R}^n$ is the vector of unknowns, and $A \in \mathbb{R}^{n\times n}$ is the large sparse system matrix. Our goal is to get a robust preconditioner rather by *new formulation* of the algorithmic scheme of the factorization than by the means of matrix preprocessings or ad hoc modifications of matrix factors.

Incomplete factorizations represent a class of algebraic preconditioners important from both theoretical and practical points of view. While state-of-the-art preconditioners of this kind can be very useful, very large problems, arising, e.g., from discretizations of complicated operators on unstructured grids typically, still may require more robust preconditioners. High-quality incomplete factorizations are also needed inside some multilevel schemes and in non-PDE applications.

Some more recent incomplete factorizations find links to factorized approximate inverses. It may be the case that in order to increase their robustness we need some information from the matrix inverse. For example, information from the inverse factors forms a crucial part of condition estimators. It is well-known that the condition estimators may be successfully used for dropping control in the incomplete factorization [1], [2]. Another step in this direction is represented by the computation of breakdown-free incomplete factorization RIF [3] via an intermediate factorized approximate inverse. In this case, the inherently breakdown-free computation of the approximate inverse provides the RIF factors as a side-effect.

In this talk we will describe the new algorithm which computes both direct and inverse incomplete factors of A. The order of computation of intermediate quantities in this algorithm enables to monitor conditioning of these factors, and, in addition, enables to use them for mutual corrections. Consider here, the simplest case of this factorization of the symmetric and positive definite A. Let $A = LDL^T$ be the factorization, where L is unit lower triangular and D is diagonal. The algorithm then computes the factors L, D, as well as the inverse of L. The basic overview of the result can be found in [4]. Its derivation was based on the theory for Sherman-Morrison-based approximate inverses [5].

2 The algorithm

Consider the left-looking algorithm which computes the columns v_k of V for $k = 1, \ldots, n$ as

$$v_k = a_k - e_k - \sum_{i=1}^{k-1} \frac{(a_k - e_k)^T u_i}{v_i^T e_i} v_i, \tag{1}$$

where a_k denotes k-th column of A and u_i denotes the i-th column of $triu(L - VD^{-1})$. It can be proved that $D = diag(V + I)$ and $L = tril(V + I)D^{-1}$. Clearly, the columns of U needed in the computation can be readily obtained from the previously computed part of V. In addition, it can be shown that $U = L^{-T}D^{-1}$, i.e., U is the scaled inverse factor of A.

During the course of the algorithm, the partially computed direct factor L is used in computation of L^{-1}, and partially computed L^{-1} is used to compute L. Moreover, both L and L^{-1} are involved in decisions related to the incompleteness of the evaluation. In the following we will show our dropping rules. They can be cheaply implemented due to the favorable order of evaluation of the factors.

Let the incomplete algorithm provides approximate quantities as $\hat{L}$, $\hat{D}$, $\widehat{L^{-1}}$, possibly in a scaled form. Denote $\hat{l}_{jk} = (\hat{L})_{jk}$, $\hat{\ell}_{jk} = (\widehat{L^{-1}})_{jk}$ for appropriate indices j, k. Motivated by [1], we drop the entries $\hat{l}_{jk}$ of the column k of $\hat{L}$ if they satisfy

$$|\hat{l}_{jk}| \parallel e_k^T \widehat{L^{-1}} \parallel \leq \tau \tag{2}$$

for a given drop tolerance τ. Similarly, we drop entries $\hat{\ell}_{jk}$ of the column of k of $\widehat{L^{-1}}$ if they satisfy

$$|\hat{\ell}_{jk}| \parallel e_k^T \hat{L} \parallel \leq \tau \tag{3}$$

The new algorithm allows straightforward embedding of both these rules.

We call the basic algorithm in the special case of SPD A the *balanced* incomplete factorization since the algorithm may monitor conditioning of both factors (approximations to L and L^{-1}) and balance their sizes and conditioning based on this information. The talk will present some merits of the new approach.

3 Collaboration

The content of the talk will be based on the joint work with Rafael Bru, José Marín and José Mas from Universitat Politècnica de València.

Acknowledgement: This work has been supported in part by the National Program of Research "Information Society" under project 1ET400300415 and by the project No. IAA100300802 of the Grant Agency of the Academy of Sciences of the Czech Republic.

References

[1] M. Bollhöfer, and Y. Saad. On the relations between ILUs and factored approximate inverses. *SIAM J. Matrix Anal. Appl.*, 24(1):219–237, 2002.

[2] M. Bollhöfer and Y. Saad. Multilevel preconditioners constructed from inverse-based ILUs. *SIAM J. Sci. Comput.*, 27(5):1627–1650, 2006.

[3] M. Benzi and M. Tůma. A robust incomplete factorization preconditioner for positive definite matrices. *Numer. Linear Algebra Appl.*, 10(5-6):385–400, 2003.

[4] R. Bru, J. Cerdán, J. Marín, and J. Mas. Preconditioning sparse nonsymmetric linear systems with the Sherman-Morrison formula. *SIAM J. Sci. Comput.*, 25(2):701–715, 2003.

[5] R. Bru, J. Marín, J. Mas, and M. Tůma. Balanced incomplete factorization. *submitted to SIAM J. Sci. Comput.*, see `http://www.cs.cas.cz/~tuma/ps/bif.pdf`, 2007.