ORTHOGONALIZATION WITH RESPECT TO THE INNER PRODUCTS AND BILINEAR FORMS IN RELATION TO CHOLESKY-LIKE FACTORIZATIONS

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Orthogonalization with respect to the standard inner product

$$A = (a_1, \ldots, a_n) \in \mathcal{R}^{m,n},\ m \geq n = rank(A)$$

orthogonal basis Q of $span(A)$:

$$Q = (q_1, \ldots, q_n) \in \mathcal{R}^{m,n},\ Q^T Q = I_n$$

$A = QR$, $R \in \mathcal{R}^{n,n}$ upper triangular,
factorization uniqueness: positive diagonal entries

$$\kappa(Q) = 1,\ \|R\| = \|A\|,\ \|R^{-1}\| = 1/\sigma_n(A),\ (\kappa(R) = \kappa(A))$$

$$C = A^T A = R^T R$$

Orthogonalization with respect to a non-standard inner product

$B \in \mathcal{R}^{m,m}$ symmetric positive definite, inner product $\langle \cdot, \cdot \rangle_B$

$$A = (a_1, \ldots, a_n) \in \mathcal{R}^{m,n},\ m \geq n = rank(A)$$

B-orthonormal basis of $span(A)$:

$$Q = (q_1, \ldots, q_n) \in \mathcal{R}^{m,n},\ Q^T BQ = I_n$$

$A = QR$, $R \in \mathcal{R}^{n,n}$ upper triangular with positive diagonal entries

$$B^{1/2}A = (B^{1/2}Q)R,\ \|B^{1/2}Q\| = \sigma_n(B^{1/2}Q) = 1\ \left(\kappa(Q) \leq \kappa^{1/2}(B)\right)$$

$$\|R\| = \|B^{1/2}A\|,\ \|R^{-1}\| = 1/\sigma_n(B^{1/2}A)\ \left(\kappa(R) = \kappa(B^{1/2}A)\right)$$

$$C = A^T BA = R^T R$$

Orthogonalization with respect to a symmetric bilinear form

$B \in \mathcal{R}^{m,m}$ symmetric indefinite and nonsingular, bilinear form

$$A = (a_1, \ldots, a_n) \in \mathcal{R}^{m,n},\ m \geq n = rank(A)$$

B-orthonormal basis of $span(A)$:

$$Q = (q_1, \ldots, q_n) \in \mathcal{R}^{m,n},\ Q^T B Q = \Omega \in \mathrm{diag}(\pm 1)$$

$A = QR$, $R \in \mathcal{R}^{n,n}$ upper triangular with positive diagonal

if no principal minor of C vanishes (if C is strongly nonsingular)

$$C = A^T B A = R^T \Omega R$$

Bunch 1971, Bunch-Parlett 1971
Della Dora 1975, Elsner 1979, Bunse-Gerstner 1981
Slapnicar 1999, Singer and Singer 2000, Singer 2006

Cholesky-like factorization of a symmetric indefinite matrix

$$C_j = A_j^T B A_j = \begin{pmatrix} C_{j-1} & c_{1:j-1,j} \\ c_{1:j-1,j}^T & c_{j,j} \end{pmatrix} =$$
$$\begin{pmatrix} R_{j-1}^T & 0 \\ r_{1:j-1,j}^T & r_{j,j} \end{pmatrix} \begin{pmatrix} \Omega_{j-1} & 0 \\ 0 & \omega_j \end{pmatrix} \begin{pmatrix} R_{j-1} & r_{1:j-1,j} \\ 0 & r_{j,j} \end{pmatrix}$$

$$r_{1:j-1,j} = \Omega_{j-1}^{-1} R_{j-1}^{-T} c_{1:j-1,j}$$
$$r_{j,j}^2 \omega_j = c_{j,j} - r_{1:j-1,j}^T \Omega_{j-1} r_{1:j-1,j} = c_{j,j} - c_{1:j-1,j}^T C_{j-1}^{-1} c_{1:j-1,j} = s_j$$

$$C_j^{-1} = \begin{pmatrix} C_{j-1}^{-1} + C_{j-1}^{-1} c_{1:j-1,j} s_j^{-1} c_{1:j-1,j}^T C_{j-1}^{-1} & -C_{j-1}^{-1} c_{1:j-1,j} s_j^{-1} \\ -s_j^{-1} c_{1:j-1,j}^T C_{j-1}^{-1} & s_j^{-1} \end{pmatrix}$$

$$\frac{1}{s_j} \leq \|C_j^{-1}\|,\ r_{j,j} = \sqrt{|s_j|} \geq \sqrt{\sigma_{min}(C_j)}$$

The inverse of the triangular factor in Cholesky-like factorization

$$R_j^{-1} = \begin{pmatrix} R_{j-1}^{-1} & -R_{j-1}^{-1} r_{1:j-1,j}/r_{j,j} \\ 0 & 1/r_{j,j} \end{pmatrix} = \begin{pmatrix} R_{j-1}^{-1} & -C_{j-1}^{-1} c_{1:j-1,j}/\sqrt{|s_j|} \\ 0 & 1/\sqrt{|s_j|} \end{pmatrix}$$

$$r_{j,j}^2 \omega_j = s_j$$

$$(R_j^T R_j)^{-1} = \begin{pmatrix} (R_{j-1}^T R_{j-1})^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \omega_j \left[C_j^{-1} - \begin{pmatrix} C_{j-1}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \right]$$

$$\|R_j^{-1}\|^2 \leq \|C_j^{-1}\| + 2 \sum_{i=1,\ldots,j-1;\ \omega_{i+1} \neq \omega_i} \|C_i^{-1}\|$$

The norm of the triangular factor in Cholesky-like factorization

$$R_j^T R_j =$$

$$\begin{pmatrix} I & 0 \\ c_{1:j-1,j}^T C_{j-1}^{-1} & 1 \end{pmatrix} \begin{pmatrix} R_{j-1}^T R_{j-1} & 0 \\ 0 & \omega_j s_j \end{pmatrix} \begin{pmatrix} I & C_{j-1}^{-1} c_{1:j-1,j} \\ 0 & 1 \end{pmatrix}$$

$$C_j = \begin{pmatrix} I & 0 \\ c_{1:j-1,j}^T C_{j-1}^{-1} & 1 \end{pmatrix} \begin{pmatrix} C_{j-1} & 0 \\ 0 & s_j \end{pmatrix} \begin{pmatrix} I & C_{j-1}^{-1} c_{1:j-1,j} \\ 0 & 1 \end{pmatrix}$$

$$R_j^T R_j = \omega_1 C_j + \sum_{i=1,\dots,j-1} (\omega_{i+1} - \omega_i) \begin{pmatrix} 0 & 0 \\ 0 & C_j \backslash C_i \end{pmatrix}$$

$$\|R_j\|^2 \leq \|C_j\| + 2 \sum_{i=1,\dots,j-1;\ \omega_{i+1} \neq \omega_i} \|C_j \backslash C_i\|,$$

Conditioning of the factors R and Q

$$\|R\| \le \|C\|\|R^{-1}\|$$

$$\kappa(R) \le \|C\| \left(\|C^{-1}\| + 2 \sum_{j;\ \omega_{j+1} \ne \omega_j} \|C_j^{-1}\| \right)$$

$$\|Q\| \le \|A\|\|R^{-1}\|, \quad \sigma_{min}(Q) \ge \tfrac{\sigma_{min}(A)}{\|R\|}$$

$$\kappa(Q) \le \kappa(A)\kappa(R)$$

$B = V\Lambda V^T$, $C = A^T B A = (A^T V|\Lambda|^{1/2})\Omega(|\Lambda|^{1/2}V^T A) = R^T \Omega R$

$|\Lambda|^{1/2}V^T A$ is a Ω-orthogonal matrix (in the case with $m = n$)

N. Higham, J-orthogonal matrices, SIAM Review 2003

I. Slapničar, K. Veselić, 1999

M. Fiedler, F.J. Hall, T. Markham, G-matrices, 2012-2013

Example with $\kappa(R) \approx \kappa^{1/2}(B)$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & \sqrt{\varepsilon} \\ \sqrt{\varepsilon} & -\varepsilon \end{pmatrix}$$

$$Q = R^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & \frac{1}{\sqrt{\varepsilon}} \end{pmatrix}, \quad R = Q^{-1} = \begin{pmatrix} 1 & \sqrt{\varepsilon} \\ 0 & \sqrt{\varepsilon} \end{pmatrix}, \quad \Omega = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\|B\| \approx 1 + \varepsilon \text{ and } \sigma_{min}(B) = 2\varepsilon$$

$$\|R\| \approx \sqrt{1+\varepsilon}, \;\; \sigma_{min}(R) \approx \sqrt{\varepsilon}, \;\; \kappa(R) = \kappa(Q) \approx \frac{1}{\sqrt{\varepsilon}}$$

Example with $\kappa(R) \gg \kappa^{1/2}(B)$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} \varepsilon & 1 \\ 1 & -\varepsilon \end{pmatrix}$$

$$Q = R^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\varepsilon}} & -\frac{1}{\sqrt{\varepsilon(1+\varepsilon^2)}} \\ 0 & \frac{\sqrt{\varepsilon}}{\sqrt{1+\varepsilon^2}} \end{pmatrix}, \quad R = Q^{-1} = \begin{pmatrix} \sqrt{\varepsilon} & \frac{1}{\sqrt{\varepsilon}} \\ 0 & \frac{\sqrt{1+\varepsilon^2}}{\sqrt{\varepsilon}} \end{pmatrix}, \quad \Omega = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\|B\| = \sigma_{min}(B) = \sqrt{1 + \varepsilon^2}$$

$$\|R\| \approx \frac{\sqrt{2}}{\sqrt{\varepsilon}}, \; \sigma_{min}(R) \approx \frac{\sqrt{\varepsilon}}{\sqrt{2}}, \quad \kappa(R) = \kappa(Q) \approx \frac{2}{\varepsilon}$$

Numerical experiments - model examples

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} R_{11}^T & 0 \\ R_{12}^T & R_{22}^T \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{pmatrix},$$

1. $\kappa(C_{11}) = 100 \ll \kappa(C) \approx 10^{2i}$, $\kappa(C_{12}) = 10^i$ for $i = 0, \dots, 8$; $C_{22} = 0$ ($\|C_{11}\| = \|C_{12}\| = 1$)

2. $\kappa(C_{11}) = 10^i \gg \kappa(C) = 1$ for $i = 0, \dots, 16$; $C_{11}^2 + C_{12}^2 = I$ $C_{22} = -C_{11}$ ($\|C_{11}\| = 1/2$)

The spectral properties of computed factors with respect to the conditioning of the submatrix C_{12} for Problem 1.

$\|C_{12}^{-1}\|$	$\|C^{-1}\|$	$\|S_{22}\|$	$\|\bar{R}\| = \|\bar{Q}^{-1}\|$	$\|\bar{R}^{-1}\| = \|\bar{Q}\|$
10^0	1.6180e+00	1.0000e+02	1.4142e+01	1.4142e+01
10^1	1.0099e+02	1.0000e+02	1.4142e+01	1.4142e+01
10^2	1.0001e+04	1.0000e+02	1.4142e+01	1.0001e+02
10^3	1.0000e+06	1.0000e+02	1.4142e+01	1.0000e+03
10^4	1.0000e+08	1.0000e+02	1.4142e+01	1.0000e+04
10^5	1.0000e+10	1.0000e+02	1.4142e+01	1.0000e+05
10^6	1.0000e+12	1.0000e+02	1.4142e+01	1.0000e+06
10^7	9.9808e+13	1.0000e+02	1.4142e+01	1.0000e+07
10^8	1.8925e+16	1.0000e+02	1.4142e+01	1.0000e+08

The spectral properties of computed factors with respect to the conditioning of the submatrix C_{11} for Problem 2.

$\|C_{11}^{-1}\|$	$\|C^{-1}\|$	$\|S_{22}\|$	$\|\bar{R}\| = \|\bar{Q}^{-1}\|$	$\|\bar{R}^{-1}\| = \|\bar{Q}\|$
10^0	1.0000e+00	2.0000e+00	1.9319e+00	1.9319e+00
10^1	1.0000e+00	2.0000e+01	6.3226e+00	6.3226e+00
10^2	1.0000e+00	2.0000e+02	2.0000e+01	2.0000e+01
10^3	1.0000e+00	2.0000e+03	6.3246e+01	6.3246e+01
10^4	1.0000e+00	2.0000e+04	2.0000e+02	2.0000e+02
10^5	1.0000e+00	2.0000e+05	6.3246e+02	6.3246e+02
10^6	1.0000e+00	2.0000e+06	2.0000e+03	2.0000e+03
10^7	1.0000e+00	2.0000e+07	6.3246e+03	6.3246e+03
10^8	1.0000e+00	2.0000e+08	2.0000e+04	2.0000e+04
10^9	1.0000e+00	2.0000e+09	6.3246e+04	6.3246e+04
10^{10}	1.0000e+00	2.0000e+10	2.0000e+05	2.0000e+05
10^{11}	1.0000e+00	2.0000e+11	6.3246e+05	6.3246e+05
10^{12}	1.0000e+00	2.0000e+12	2.0000e+06	2.0000e+06
10^{13}	1.0000e+00	1.9999e+13	6.3245e+06	6.3245e+06
10^{14}	1.0000e+00	2.0004e+14	2.0188e+07	2.0520e+07
10^{15}	1.0000e+00	2.0011e+15	6.6349e+07	5.2040e+07

Orthogonalization with respect to a skew-symmetric bilinear form

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathcal{R}^{2m,2m}$$ skew-symmetric and orthogonal,

$$A = (a_1, \ldots, a_n) \in \mathcal{R}^{m,n},\ m \geq n = rank(A)$$

J-orthonormal basis of $span(A)$: $Q = (q_1, \ldots, q_n) \in \mathcal{R}^{m,n}$

$$Q^T J Q = \mathrm{diag}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \in \mathcal{R}^{2n,2n}$$

$A = QR$, $R \in \mathcal{R}^{n,n}$ upper triangular with positive diagonal

if no minor of C with even dimension vanishes

$$C = A^T J A = R^T \mathrm{diag}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) R$$

Della Dora 1975, Elsner 1979, Bunse-Gerstner 1981
Mehrmann 1979, Bunse-Gerstner and Mehrmann 1986
Benner, Byers, Fassbender, Mehrmann, Watkins 2000

Orthogonalization with respect to a skew-symmetric bilinear form

B skew-symmetric and nonsingular, bilinear form
Schur-like factorization of skew-symmetric and nonsingular B

$$B = V \begin{pmatrix} 0 & \Sigma^2 \\ -\Sigma^2 & 0 \end{pmatrix} V^T$$

$V \in \mathcal{R}^{2m,2m}$ orthogonal with $V^T V = V V^T = I$
$\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_m) \in \mathcal{R}^{m,m}$ with positive entries

$$B = V \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix} V^T$$

$$C = A^T B A = R^T \mathrm{diag}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) R$$

$\begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix} V^T A$ is a J-orthogonal matrix

Uniqueness of the Cholesky-like factorization

$$C = R^T J R = \begin{pmatrix} r_{11} & 0 \\ r_{12} & r_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & r_{11}r_{22} \\ -r_{11}r_{22} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \pm\|C\| \\ \mp\|C\| & 0 \end{pmatrix} = \pm\|C\| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\kappa^2(R) = \frac{\|R\|_F^2 + \sqrt{\|R\|_F^4 - 4r_{11}^2 r_{22}^2}}{\|R\|_F^2 - \sqrt{\|R\|_F^4 - 4r_{11}^2 r_{22}^2}}$$

As $r_{11}r_{22}$ is a fixed and $\kappa(R)$ is an increasing function of $\|R\|_F$, it is minimized if $r_{12} = 0$ and $r_{11} = \pm\sqrt{\|C\|}$, $r_{22} = \pm\sqrt{\|C\|}$. Then $R^T R = \|C\| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Bunse-Gerstner and Mehrmann 1986, Fassbender 2000
Bhatia 1994, Chang, 1998

Example with a different choice of block normalization

$$A = \begin{pmatrix} \sqrt{\varepsilon} & 1 \\ 1 & 0 \\ 0 & \sqrt{\varepsilon} \\ 0 & 0 \end{pmatrix}, \; A^T J A = \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix}, \; R = \begin{pmatrix} \sqrt{\varepsilon} & 0 \\ 0 & \sqrt{\varepsilon} \end{pmatrix}$$

$$r_{11} = \|a_1\| = \sqrt{1+\varepsilon}, \; q_1 = \frac{1}{\sqrt{1+\varepsilon}} \begin{pmatrix} \sqrt{\varepsilon} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; r_{12} = q_1^T a_2 = \frac{\sqrt{\varepsilon}}{\sqrt{1+\varepsilon}},$$

$$q_2 = \frac{1}{r_{22}} \begin{pmatrix} \frac{1}{1+\varepsilon} \\ -\frac{\sqrt{\varepsilon}}{1+\varepsilon} \\ \sqrt{\varepsilon} \\ 0 \end{pmatrix}, \; r_{22} = \frac{a_1^T J a_2}{r_{11}} = \frac{\varepsilon}{\sqrt{1+\varepsilon}}, \; q_2 = \frac{1}{\varepsilon} \begin{pmatrix} \frac{1}{\sqrt{1+\varepsilon}} \\ -\frac{\sqrt{\varepsilon}}{\sqrt{1+\varepsilon}} \\ \sqrt{\varepsilon}\sqrt{1+\varepsilon} \\ 0 \end{pmatrix}$$

$$R = \begin{pmatrix} \sqrt{1+\varepsilon} & \frac{\sqrt{\varepsilon}}{\sqrt{1+\varepsilon}} \\ 0 & \frac{\varepsilon}{\sqrt{1+\varepsilon}} \end{pmatrix}, \; \lambda(R^T R) \approx 1 + 2\varepsilon, \varepsilon^2/16, \; \kappa(R) \approx \frac{4}{\varepsilon}$$

Cholesky-like factorization of a skew-symmetric matrix

$$C_{2k} = A_{2k}^T J A_{2k} = \begin{pmatrix} & & & C_{1,k} \\ & C_{2(k-1)} & & \vdots \\ & & & C_{k-1,k} \\ -C_{1,k}^T & \dots & -C_{k-1,k}^T & \hat{C}_{k,k} \end{pmatrix} =$$

$$R_{2k}^T \begin{pmatrix} \hat{J}_{2(k-1)} & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} R_{2k}, R_{2k} = \begin{pmatrix} & R_{1,k} \\ R_{2(k-1)} & \vdots \\ & R_{k-1,k} \\ 0 & R_{k,k} \end{pmatrix}$$

$$R_{i,k} = \hat{J}_{2(k-1)}^{-1} R_{2(k-1)}^{-T} C_{i,k}$$

$$R_{k,k}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} R_{k,k} = C_{k,k} + \sum_{i=1}^{k-1} C_{i,k}^T C_{2(k-1)}^{-1} C_{i,k} = C_{2k} \backslash C_{2(k-1)}$$

$$= \pm \| C_{2k} \backslash C_{2(k-1)} \| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The inverse of the triangular factor in Cholesky-like factorization

$$R_{2k}^{-1} = \begin{pmatrix} R_{2(k-1)}^{-1} & -R_{2(k-1)}^{-1} \begin{pmatrix} R_{1,k} \\ \vdots \\ R_{k-1,k} \end{pmatrix} R_{k,k}^{-1} \\ 0 & R_{k,k}^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} R_{2(k-1)}^{-1} & -C_{2(k-1)}^{-1} \begin{pmatrix} C_{1,k} \\ \vdots \\ C_{k-1,k} \end{pmatrix} R_{k,k}^{-1} \\ 0 & R_{k,k}^{-1} \end{pmatrix}$$

$$\|R_{2n}^{-1}\|^2 \leq \|C_{2n}^{-1}\| + \sqrt{2} \sum_{k=1}^{n-1} \frac{\left(\left\|C_{2(k-1)}^{-1} \begin{pmatrix} C_{1,k} \\ \vdots \\ C_{k-1,k} \end{pmatrix}\right\| + 1\right)^2}{\|C_{2k} \backslash C_{2(k-1)}\|}$$

$$\|R_{2n}\| \leq \|C_{2n}\| \|R_{2n}^{-1}\|$$

The norm of the triangular factor in Cholesky-like factorization

$$R_{2n}^T R_{2n} - C_{2n} = R_{2n}^T \mathrm{diag}(\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}) R_{2n}$$

$$\|R_{2n}\|^2 \leq \|C_{2n}\| + \sqrt{2} \sum_{k=1}^{n} \frac{(\|C_{2k} \backslash C_{2(k-1)}\| + \|C_{2n} \backslash C_{2(k-1)}\|)^2}{\|C_{2k} \backslash C_{2(k-1)}\|},$$

$$\|Q\| \leq \|A\| \|R^{-1}\|, \quad \sigma_{min}(Q) \geq \tfrac{\sigma_{min}(A)}{\|R\|}$$

$$\kappa(Q) \leq \kappa(A)\kappa(R)$$

Xu 2003

Example with $\kappa(R) \approx \kappa(A)$

$$A = \begin{pmatrix} \sqrt{\varepsilon} & 0 & 0 & -\frac{1}{\sqrt{\varepsilon}} \\ 0 & 0 & 0 & \frac{\sqrt{1-\varepsilon^2}}{\sqrt{\varepsilon}} \\ 0 & \sqrt{\varepsilon} & \frac{1}{\sqrt{\varepsilon}} & 0 \\ 0 & 0 & \frac{\sqrt{1-\varepsilon^2}}{\sqrt{\varepsilon}} & 0 \end{pmatrix},\ A^T J A = \begin{pmatrix} 0 & \varepsilon & 1 & 0 \\ -\varepsilon & 0 & 0 & 1 \\ -1 & 0 & 0 & \varepsilon \\ 0 & -1 & -\varepsilon & 0 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} \sqrt{\varepsilon} & 0 & 0 & -\frac{1}{\sqrt{\varepsilon}} \\ 0 & \sqrt{\varepsilon} & \frac{1}{\sqrt{\varepsilon}} & 0 \\ 0 & 0 & \frac{\sqrt{1-\varepsilon^2}}{\sqrt{\varepsilon}} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{1-\varepsilon^2}}{\sqrt{\varepsilon}} \end{pmatrix}$$

$$\sigma(A) \approx \frac{\sqrt{2}}{\sqrt{\varepsilon}}, \frac{\sqrt{\varepsilon}}{\sqrt{2}},\ \kappa(A) \approx \frac{2}{\varepsilon},$$
$$\sigma(R) \approx \frac{\sqrt{2}}{\sqrt{\varepsilon}}, \frac{\sqrt{\varepsilon}}{\sqrt{2}},\ \kappa(R) \approx \frac{2}{\varepsilon},\ \kappa(Q) = 1$$

Example with $\kappa(R) \gg \kappa(A)$

$$A = \begin{pmatrix} \sqrt{\varepsilon} & 1 & 0 & 0 \\ 1 & 0 & 0 & -\varepsilon \\ 0 & \sqrt{\varepsilon} & 0 & 1 \\ 0 & 0 & 1 & -\sqrt{\varepsilon} \end{pmatrix}, \; A^T J A = \begin{pmatrix} 0 & \varepsilon & 1 & 0 \\ -\varepsilon & 0 & 0 & 1 \\ -1 & 0 & 0 & \varepsilon \\ 0 & -1 & -\varepsilon & 0 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & 0 & \frac{\sqrt{\varepsilon}}{\sqrt{1-\varepsilon^2}} & -\frac{1}{\sqrt{1-\varepsilon^2}} \\ \frac{1}{\sqrt{\varepsilon}} & 0 & 0 & -\frac{\sqrt{1-\varepsilon^2}}{\sqrt{\varepsilon}} \\ 0 & 0 & \frac{\sqrt{\varepsilon}}{\sqrt{1-\varepsilon^2}} & -\frac{\sqrt{\varepsilon}}{\sqrt{1-\varepsilon^2}} \\ 0 & \frac{1}{\sqrt{\varepsilon}} & -\frac{\sqrt{\varepsilon}}{\sqrt{1-\varepsilon^2}} & \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \end{pmatrix},$$

$$R = \begin{pmatrix} \sqrt{\varepsilon} & 0 & 0 & -\frac{1}{\sqrt{\varepsilon}} \\ 0 & \sqrt{\varepsilon} & \frac{1}{\sqrt{\varepsilon}} & 0 \\ 0 & 0 & \frac{\sqrt{1-\varepsilon^2}}{\sqrt{\varepsilon}} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{1-\varepsilon^2}}{\sqrt{\varepsilon}} \end{pmatrix}$$

$$\sigma(A) \approx 1, \; \kappa(A) \approx 1,$$
$$\sigma(R) \approx \frac{\sqrt{2}}{\sqrt{\varepsilon}}, \frac{\sqrt{\varepsilon}}{\sqrt{2}}, \; \kappa(R) \approx \frac{2}{\varepsilon}, \; \sigma(Q) \approx \frac{\sqrt{2}}{\sqrt{\varepsilon}}, \frac{\sqrt{\varepsilon}}{\sqrt{2}}$$

Thank you Martin!!!

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