On the convergence curves that can be generated by restarted GMRES

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Abstract

The GMRES method [7], in particular when it is restarted after a small number of iterations, is a very frequently used iterative solver for linear systems with nonsingular, non-Hermitian input matrices. Although its mathematical definition is based on a simple minimization property, convergence analysis is difficult in the sense that there do not seem to exist, in general, clearly defined characteristics of a given linear system that govern the convergence behavior of the method. With normal system matrices it is in the first place the eigenvalue distribution, and secondarily also the projection of the right-hand side vector in the eigenvector basis, that characterize convergence behavior. As we will explain below, this is not the case for general non-normal input matrices. Among a large variety of proposed approaches for analysis of convergence one can mention approaches based on the pseudo-spectrum [9], the field of values [3] or the numerical polynomial hull [4]. Convergence analysis of *restarted* GMRES also considers non-stagnation conditions, see, e.g., [8].

One approach to enhance insight in GMRES convergence behavior is to study matrices with initial vectors that generate the same residual norm history. This was first done in a 1994 paper by Greenbaum and Strakoš [6], where one can find several ways to construct the set of all linear systems with the same convergence curve. Surprisingly, the matrices of the set can have any nonzero spectrum. Together with the fact that any nonincreasing convergence curve is possible [5], this resulted in parametrizations of the set of linear systems generating prescribed residual norms and having prescribed eigenvalues [1]. In [2] it was shown that these parametrizations allow the additional prescription of Ritz values, that is of the spectra of the Hessenberg matrices generated in all subsequent iterations. This implies that the eigenvalue approximations in Arnoldi's method for eigenproblems can be arbitrarily far from the spectrum of the input matrix.

This type of analysis has been applied to the restarted GMRES method, which is more relevant for practice, in a 2011 paper by Vecharinsky and Langou [10]. Let GMRES(m) denote GMRES restarted after every m iterations with the current residual vector and assume that in every restart cycle, all residual norms are equal except for the very last residual norm of each cycle, which is strictly less than the previous residual norm. If we denote the last residual norm of the kth cycle with $\|r_m^{(k)}\|$, then Vecharinsky and Langou showed that any decreasing convergence curve $\|r_m^{(1)}\| > \|r_m^{(2)}\| > \cdots > \|r_m^{(N)}\|$ is possible with any nonzero spectrum of the input matrix, where it is assumed that the total number of inner GMRES iterations Nm is smaller than the system size.

Our work can be seen as an extension of the results in [10]. In the talk we would show that it is possible to prescribe all the residual norms generated by N cycles of GMRES(m), including the norms *inside* cycles, under two conditions: The last residual norm of every cycle does not stagnate, i.e. $\|r_m^{(k)}\| > \|r_{m-1}^{(k)}\|$ for $k = 1, 2, \ldots, N$ and Nm is smaller than the system size. We show that this is possible with any nonzero eigenvalues of the system matrix. In addition, *all* Ritz values generated inside all cycles, can take arbitrary nonzero values.

If we allow stagnation at the end of cycles, the admissible residual norms and Ritz values satisfy an additional, interesting restriction. It is known that stagnation in the GMRES process takes place at and only at iterations where the corresponding Hessenberg matrix is singular, that is, if and only if there is a zero Ritz value. If at some cycle k, $\|r_{m-j}^{(k)}\| = \cdots = \|r_m^{(k)}\|$ or, equivalently, the last j iterations generate a zero Ritz value, then we show that there must be stagnation (or a zero

Ritz value) during the *first* j iterations of the $(k+1)$st cycle. In other words, stagnation at the end of one cycle is literally mirrored at the beginning of the next cycle and residual norms cannot be prescribed at the beginning of that cycle.

The mentioned results are based on a construction of linear systems which generate, when the restarted Arnoldi orthogonalization process is applied and leaving aside the case of stagnation at the end of cycles, subsequent size $(m+1) \times m$ Hessenberg matrices whose entries can be fully prescribed. It turns out that the constructed linear systems have a fascinating property: Besides the fact that they generate Nm prescribed residual norms of GMRES(m), they generate exactly the same Nm residual norms when full GMRES is applied. Moreover, this property can be used to modify the linear systems such, that GMRES(m) converges faster than GMRES($m+i$) for some positive integers i. It thus offers some insight in the causes of the very counterintuitive behavior observed sometimes in practice, where convergence speed decreases when the restart parameter m is increased.

If time allows it, our talk would also comment on some consequences of our results for restarted Arnoldi methods to solve non-Hermitian eigenproblems.

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