An Ordering for Flexible and Finite Representation of Infinite Sequences of Proofs

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Recent work with Alexander Leitsch and Anela Lolic

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An Introduction

- This presentation is similar to the previous two in that:
  1) Motivated by the study of cut-elimination.
  2) Connected to our investigations of Herbrand's theorem in the presence of induction through schematic proof representation.
- Unlike the previous talks we are going to focus on schematic proof representation.
- We introduce an ordering developed specifically for finitely representing infinite sequences of proofs.
- Furthermore, we discuss uses beyond proof representation.
- But first let's cover some background material.

cut-elimination on inductive proofs?

- Induction is of crucial importance in mathematics.
- Tere exist forms of cut-elimination for inductive proofs.
- But Herbrand's theorem fails.
- solution: Replace inductive proofs by proof schemata and compute Herbrand systems.
- The analysis of Fürstenberg's prime proof by CERES was based on proof schemata.

Inductive proofs: extraction of Herbrand sequents?

Example: Let us consider the sequent S:

$$\forall x(P(x) \rightarrow P(f(x))) \vdash \forall n \forall x((P(\hat{g}(n,x)) \rightarrow Q(x)) \rightarrow (P(x) \rightarrow Q(x)))$$

where f is a unary function symbol and

$$\mathcal{E} = \{\hat{g}(0,x) = x, \hat{g}(s(n),x) = f(\hat{g}(n,x))\}.$$

The skolemized version:

$$S'\colon (\forall x)(P(x) \rightarrow P(f(x))) \vdash (P(\hat{g}(n_0,c)) \rightarrow Q(c)) \rightarrow (P(c) \rightarrow Q(c)).$$

Obviously, S' cannot be proven without induction - S' does not have a Herbrand sequent (w.r.t. the theory $\mathcal{E}$) .

- S' can be proven by induction - you prove the lemma

$$\psi\colon (\forall x)(P(x) \rightarrow P(f(x))) \vdash \forall n \forall x(P(x) \rightarrow P(\hat{g}(n,x)).$$

Proof Schemata Define Herbrand Schemata

$\psi(0) =$

$$\cfrac{\cfrac{\cfrac{\cfrac{P(\hat{g}(0,x_0)) \vdash P(\hat{g}(0,x_0))}{P(x_0) \vdash P(\hat{g}(0,x_0))}\ \mathcal{E}}{\vdash P(x_0) \to P(\hat{g}(0,x_0))}\ \to: r}{\vdash (\forall x)(P(x) \to P(\hat{g}(0,x)))}\ \forall: r}{(\forall x)(P(x) \to P(f(x))) \vdash (\forall x)(P(x) \to P(\hat{g}(0,x)))}\ w: l$$

and $\psi(k+1)$ is:

$$\cfrac{\dfrac{(\psi(k))}{(\forall x)(P(x) \to P(f(x))) \vdash (\forall x)(P(x) \to P(\hat{g}(\bar{k},x)))} \qquad (1)}{(\forall x)(P(x) \to P(f(x))) \vdash (\forall x)(P(x) \to P(\hat{g}(s(\bar{k}),x)))}\ cut, c: l$$

where (1) is:

$$
\begin{prooftree}
\AxiomC{$P(x_{k+1}) \vdash P(x_{k+1})$}
\AxiomC{$P(\hat{g}(\bar{k}, x_{k+1})) \vdash P(\hat{g}(\bar{k}, x_{k+1}))$}
\AxiomC{$P(\hat{g}(s(\bar{k}), x_{k+1})) \vdash P(\hat{g}(s(\bar{k}), x_{k+1}))$}
\RightLabel{$\mathcal{E}$}
\UnaryInfC{$P(f(\hat{g}(\bar{k}, x_{k+1}))) \vdash P(\hat{g}(s(\bar{k}), x_{k+1}))$}
\RightLabel{$\rightarrow: l$}
\BinaryInfC{$P(\hat{g}(\bar{k}, x_{k+1})), P(\hat{g}(\bar{k}, x_{k+1})) \rightarrow P(f(\hat{g}(\bar{k}, x_{k+1}))) \vdash P(\hat{g}(s(\bar{k}), x_{k+1}))$}
\RightLabel{$\forall: l$}
\UnaryInfC{$P(\hat{g}(\bar{k}, x_{k+1})), (\forall x)(P(x) \rightarrow P(f(x))) \vdash P(\hat{g}(s(\bar{k}), x_{k+1}))$}
\RightLabel{$\rightarrow: l$}
\BinaryInfC{$P(x_{k+1}), P(x_{k+1}) \rightarrow P(\hat{g}(\bar{k}, x_{k+1})), (\forall x)(P(x) \rightarrow P(f(x))) \vdash P(\hat{g}(s(\bar{k}), x_{k+1}))$}
\RightLabel{$\rightarrow: r$}
\UnaryInfC{$P(x_{k+1}) \rightarrow P(\hat{g}(\bar{k}, x_{k+1})), (\forall x)(P(x) \rightarrow P(f(x))) \vdash P(x_{k+1}) \rightarrow P(\hat{g}(s(\bar{k}), x_{k+1}))$}
\RightLabel{$\forall: l$}
\UnaryInfC{$(\forall x)(P(x) \rightarrow P(\hat{g}(\bar{k}, x))), (\forall x)(P(x) \rightarrow P(f(x))) \vdash P(x_{k+1}) \rightarrow P(\hat{g}(s(\bar{k}), x_{k+1}))$}
\RightLabel{$\forall: r$}
\UnaryInfC{$(\forall x)(P(x) \rightarrow P(\hat{g}(\bar{k}, x))), (\forall x)(P(x) \rightarrow P(f(x))) \vdash (\forall x)(P(x) \rightarrow P(\hat{g}(s(\bar{k}), x)))$}
\end{prooftree}
$$

Proof Schemata Define Herbrand Schemata

and define $\varphi(n) =$

$$\dfrac{\dfrac{(\psi(n))}{(\forall x)(P(x) \to P(f(x))) \vdash C(n)} \qquad \dfrac{(\chi(n))}{C(n) \vdash (P(\hat{g}(\bar{n}, c)) \to Q(c)) \to (P(c) \to Q(c))}}{(\forall x)(P(x) \to P(f(x))) \vdash (P(\hat{g}(\bar{n}, c)) \to Q(c)) \to (P(c) \to Q(c))}\ cut$$

where $C(n) = (\forall x)(P(x) \to P(\hat{g}(\bar{n}, x)))$ and $\chi(n)$ is:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(c) \vdash P(c) \qquad \dfrac{P(\hat{g}(\bar{n}, c)) \vdash P(\hat{g}(\bar{n}, c)) \qquad Q(c) \vdash Q(c)}{P(\hat{g}(\bar{n}, c)) \to Q(c), P(\hat{g}(\bar{n}, c)) \vdash Q(c)}\ \to: l}{P(c), P(\hat{g}(\bar{n}, c)) \to Q(c), P(c) \to P(\hat{g}(\bar{n}, c)) \vdash Q(c)}\ \to: l}{P(\hat{g}(\bar{n}, c)) \to Q(c), P(c) \to P(\hat{g}(\bar{n}, c)) \vdash P(c) \to Q(c)}\ \to: r}{P(c) \to P(\hat{g}(\bar{n}, c)) \vdash (P(\hat{g}(\bar{n}, c)) \to Q(c)) \to (P(c) \to Q(c))}\ \to: r}{(\forall x)(P(x) \to P(\hat{g}(\bar{n}, x))) \vdash (P(\hat{g}(\bar{n}, c)) \to Q(c)) \to (P(c) \to Q(c))}\ \forall: l$$

proof schemata define Herbrand schemata

$$\dfrac{\dfrac{\psi(n)}{\forall x(P(x) \rightarrow P(f(x))) \vdash C} \qquad \dfrac{\chi(n)}{C \vdash (P(\hat{g}(\bar{n}, c)) \rightarrow Q(c)) \rightarrow (P(c) \rightarrow Q(c))}}{S_n \colon \forall x(P(x) \rightarrow P(f(x))) \vdash (P(\hat{g}(\bar{n}, c)) \rightarrow Q(c)) \rightarrow (P(c) \rightarrow Q(c))}\ cut$$

for $C = \forall x(P(x) \rightarrow P(\hat{g}(\bar{n}, x)))$

For every n we get (by cut-elimination) a Herbrand sequent S_n^* (valid in $\mathcal{E} = \{\hat{g}(0, x) = x, \hat{g}(s(n), x) = f(\hat{g}(n, x))\}$):

$$\begin{aligned} S_0^* &= \vdash (P(\hat{g}(0, c)) \rightarrow Q(c)) \rightarrow (P(c) \rightarrow Q(c)), \\ S_{n+1}^* &= (P(x) \rightarrow P(f(x)))\theta_0, \ldots, (P(x) \rightarrow P(f(x))\theta_n \vdash \\ &\quad P(\hat{g}(n+1, c)) \rightarrow Q(c)) \rightarrow (P(c) \rightarrow Q(c)). \end{aligned}$$

for $\theta_0 = \{x \leftarrow \hat{g}(n, 0)\}, \ldots, \theta_n = \{x \leftarrow \hat{g}(n, c)\}$

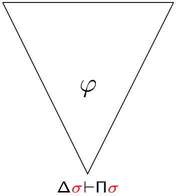
Herbrand substitution set $\rightarrow$ Herbrand system:

$$\Theta_0 = \emptyset, \quad \Theta_{n+1} = \Theta_n \cup \{\theta_n\}.$$

Beyond the Basic Example

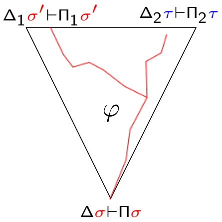
- The existing schematic version of CERES is defined for proof schema with a single free-parameter.
- Constructing proof schema which use more than one free-parameter is possible.
- It can also be shown that multi-parameter proof schemata are provability equivalent to the LK-calculus for Peano Arithmetic [Cerna,2018] (If we allow quantification of parameters).
- Though, there does not exists a corresponding CERES method. This is currently being investigated.

Quick Guide to Schematic Proof Construction



φ has free variables instantiated by σ to numerals.

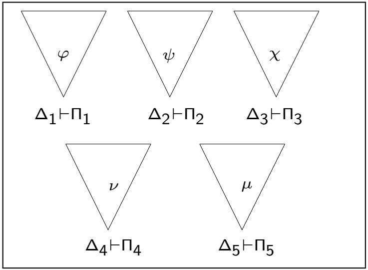
Quick Guide to Schematic Proof Construction

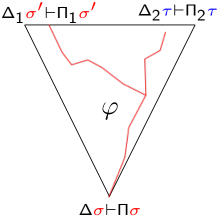


φ has free variables instantiated by σ to numerals.

$\sigma > \sigma'$ but no order relation between σ and τ.

Quick Guide to Schematic Proof Construction





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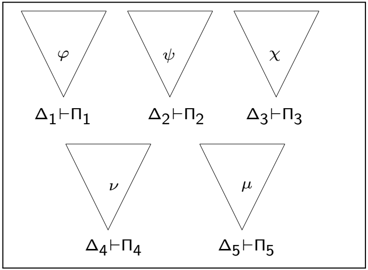

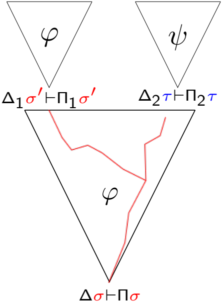

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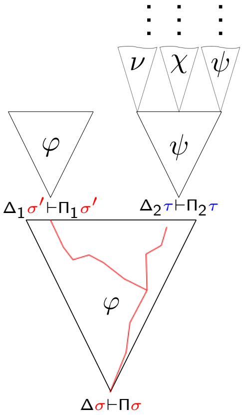
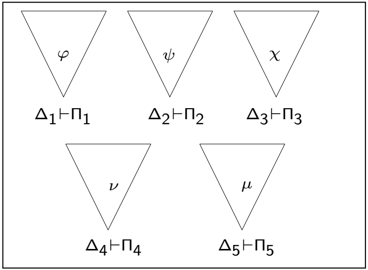


Nested proof calls allowed.



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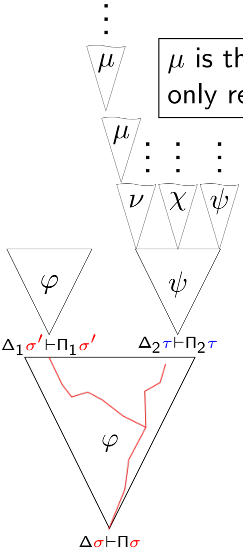
$\sigma > \sigma'$ but no order relation between σ and τ.

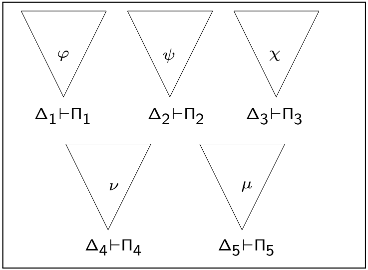
φ cannot be referenced in ψ, and ψ cannot be referenced in ν, etc.

Quick Guide to Schematic Proof Construction



μ is the least proof and only references itself.

Nested proof calls allowed.



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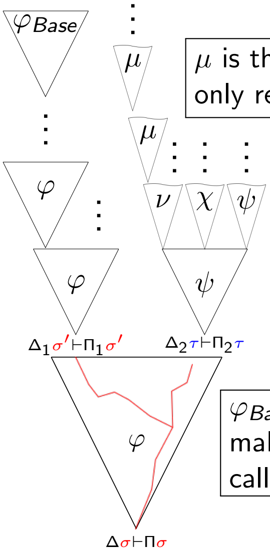

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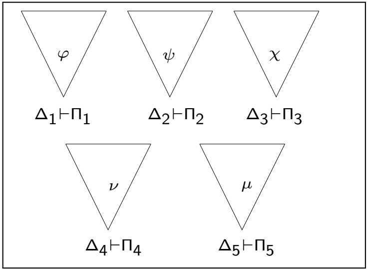

φ *Base* cannot make proof calls.

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Problems with Schematic proof Construction

- While the above schematic proof construction works and is provability equivalent to Peano arithmetic, it does not help much with proof analysis.
- A sequence of post-transformation proofs may require a more expressive language then the one outlined above to describe it.
- Let us consider cut-elimination for schematic proofs before looking into the properties of this more expressive language.

Global versus Local cut-elimination

Local cut-elimination reduces a cut formula's complexity or its distance from the leaves.

- Introduced by Gentzen as a method of proving consistency, the concept has been expanded well beyond the intended scope.

Global versus Local cut-elimination

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- Introduced by Gentzen as a method of proving consistency, the concept has been expanded well beyond the intended scope.

Global cut-elimination produces an intermediate representation of a formal proof's cut-structure.

- From this intermediate representation a new proof with a **trivial cut-structure** is produced.

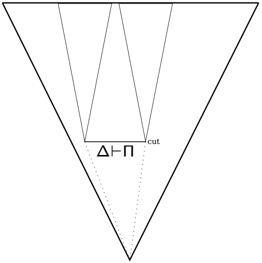
Local Cut-elimination and Schematic Proofs

- No cut reduction rules exists for links.

$$\dfrac{\dfrac{(\varphi, \cdots)}{C, \Delta \vdash \Gamma} \qquad \dfrac{(\varphi_j, \cdots)}{\Delta' \vdash \Gamma', C}}{\Delta, \Delta' \vdash \Gamma, \Gamma'}\ \text{cut}$$

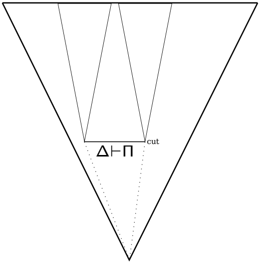
- When the call structure is non-recursive proof references can just be removed.
- However, recursive call structures block reduction of a cut formula's rank.
- Using a global approach we can extract the cut-structure as an unsatisfiable negation normal form inductive definition.

Global Cut-elimination Inductive NNF Definition Extraction



LK-Proof with cuts

Global Cut-elimination Inductive NNF Definition Extraction



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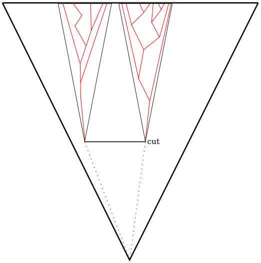
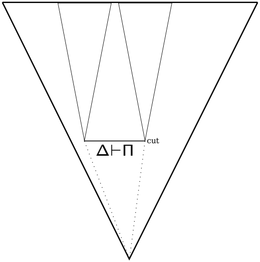


Paths to cut ancestors

Global Cut-elimination Inductive NNF Definition Extraction



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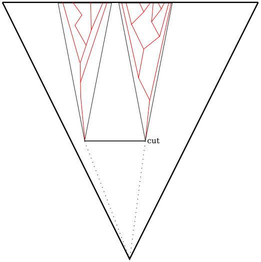
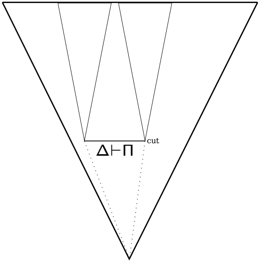


Paths to cut ancestors

$CL(A \vdash A) \equiv \{A\}$
$CL(A \vdash A) \equiv \{\neg A\}$
$CL(A \vdash A) \equiv \{\neg A \vee A\}$

- Proof references are denoted by defined symbols.
- Such an inductive NNF definition is always unsatisfiable.
- We need to refute this inductive NNF in a finitely representable way in-order to extract Herbrand information.

Global Cut-elimination Inductive NNF Definition Extraction



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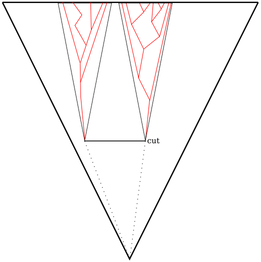

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$$CL(A \vdash A) \equiv \{\neg A \vee A\}$$
$$CL\left(\frac{\Delta \vdash \Pi}{\Delta' \vdash \Pi'}\ \rho\right) \equiv CL(\Delta \vdash \Pi)$$

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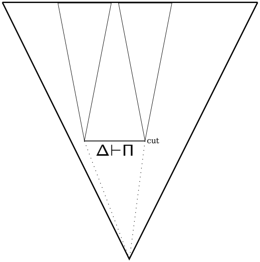

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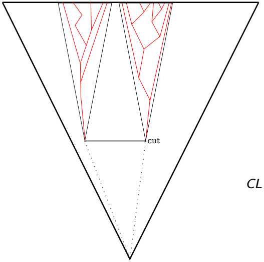

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$$CL\left(\frac{\Delta \vdash \Pi}{\Delta' \vdash \Pi'}\ \rho\right) \equiv CL(\Delta \vdash \Pi)$$

$$CL\left(\frac{\Delta \vdash \Pi \qquad \Delta' \vdash \Pi'}{\Delta'' \vdash \Pi''}\ \rho\right) \equiv$$

$$\begin{cases} CL(\Delta \vdash \Pi) \wedge CL(\Delta' \vdash \Pi') \\ CL(\Delta \vdash \Pi) \vee CL(\Delta' \vdash \Pi') \end{cases}$$

- Proof references are denoted by defined symbols.
- Such an inductive NNF definition is always unsatisfiable.
- We need to refute this inductive NNF in a finitely representable way in-order to extract Herbrand information.

Refuting the NNF

- Note that refuting this NNF for every instance is complexity-wise equivalent to proving the inductive statement using atomic cuts only.
- Given that cut-elimination for calculi with an induction inference is usually not possible or not possible while retaining analyticity this alludes to the difficulty of the problem.
- Let us look at an example before considering how to represent it's refutation.

Eventually Constant Statement (ECS)

- Consider a proof schema indexed by n of

$$\forall x \bigvee_{i=0}^{n} f(x) = i, \Delta \vdash \exists x \forall y (x \leq y \rightarrow f(x) = f(y))$$

- using a sequence of Σ_2-cuts

$$\exists x \forall y (((x \leq y) \Rightarrow n+1 = f(y)) \vee f(y) < n+1).$$

- This example was discussed in [Cerna and Leitsch 2016] where the proof was skolemized resulting in the end sequent

$$\forall x \bigvee_{i=0}^{n} f(x) = i, \Delta \vdash \exists x (x \leq g(x) \rightarrow f(x) = f(g(x))).$$

- Note that Δ contains additional assumptions concerning f.

The 2-repetition Statement

- As an exercise we can consider what happens when we interpret g as the successor function

$$\forall x \bigvee_{i=0}^{n} f(x) = i, \Delta' \vdash \exists x(f(x) = f(suc(x))),$$

- i.e. what cut formula and which axioms are needed for this statement to hold. The cut is weaker,

$$\exists x(f(x) = k \wedge f(suc(x)) = k) \vee \forall x(f(x) < k)$$

- and the axioms concerning f are simpler

$$f(suc(x)) < s(k) \vdash f(suc(x)) = k, f(x) < k$$

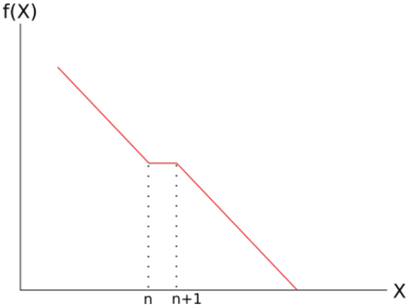
$$f(x) < s(k) \vdash f(x) = k, f(x) < k$$

The 2-repetition Statement

- A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is said to be 2-repeating if there exists at least two consecutive values $x, y \in \mathbb{N}$ s.t. $f(x) = f(y)$.

Assertion (2-repetition)

Every total monotonically decreasing function $f : \mathbb{N} \rightarrow \mathbb{N}$ is at least 2-repeating.



Analysis of the 2-repetition Statement

- Analysis of ECS was already performed in [Cerna and Leitsch 2016], the 2-repetition Statement is substantially weaker.
- While the inductive superposition prover of N. Peltier cannot find an invariant for ECS, it can find one for 2-repetition. [Leitsch et al. 2017] can be used for analysis.

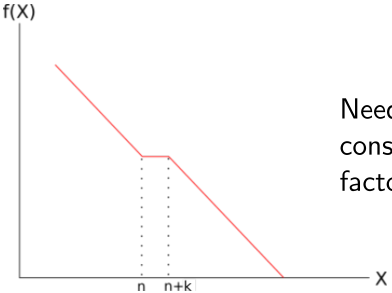
```
=============================== PROOF ==================================

% Proof 1 at 0.017 (+ 0.000) seconds.
% Given clauses 73.

% number of calls to fixpoint : 3
 S_init  :
 (51:  [ EQ(v0,f(h(v1))) | LE(f(v1),v0) if  n = s(v0) ].
50:  [ EQ(v0,f(v1)) | LE(f(v1),v0) if  n = s(v0) ].
33:  [ PHI(v0,v1) if  n = s(v0) ].
)
 S_loop  :
 (82:  [ EQ(v0,f(h(v1))) | LE(f(v1),v0) if  n = s(s(v0)) ].
80:  [ EQ(v0,f(v1)) | LE(f(v1),v0) if  n = s(s(v0)) ].
53:  [ PHI(v0,v1) if  n = s(s(v0)) ].
)
 The empty clauses  :
 (45:  [  n = 0 ].
81:  [  n = s(0) ].
112:  [  n = s(s(0)) ].
) max_rank 2
```

The k-repetition Statement

- A function $f : \mathbb{N} \to \mathbb{N}$ is said to be k-repeating if there exists at least k consecutive values $x_1, \cdots, x_k \in \mathbb{N}$ s.t. $f(x_1) = \cdots = f(x_k)$.

Assertion (k-repetition)

Every total monotonically decreasing function $f : \mathbb{N} \to \mathbb{N}$ is at least k-repeating.



Need to check all possible function constructions of which there are factorially many.

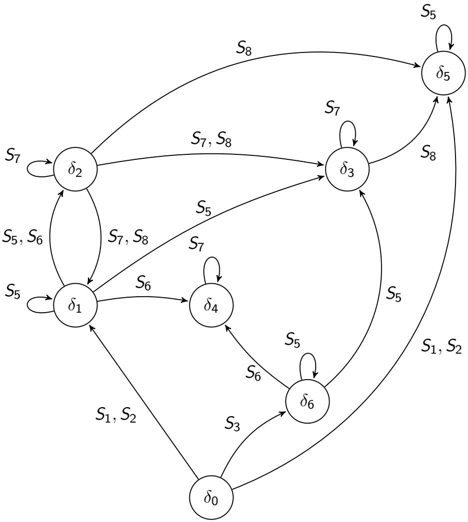
Cut Structure of k-Repetition as an Inductive Definition

$$
\begin{aligned}
\hat{O}(n,m) &\Longrightarrow \hat{D}(n,m) \wedge \ \hat{P}(n,m)\\
\hat{D}(n,0) &\Longrightarrow \forall x(f(x) = \hat{S}(n,a) \ \vee \ f(x) < \hat{S}(n,a))\\
\hat{D}(n,s(m)) &\Longrightarrow \forall x(f(\hat{S}(s(m),x)) = \hat{S}(n,a) \ \vee \ f(x) < \hat{S}(n,a)) \wedge \ \hat{D}(n,m)\\
\hat{P}(0,m) &\Longrightarrow \forall x(\hat{C}(x,0,m)) \ \wedge \ f(a) \not< 0\\
\hat{P}(s(n),m) &\Longrightarrow (\forall x(\hat{C}(x,s(n),m)) \ \wedge (\hat{T}(n,m)) \ \wedge \ \hat{P}(n,m)\\
\hat{C}(y,n,0) &\Longrightarrow f(y) \neq \hat{S}(n,a)\\
\hat{C}(y,n,s(m)) &\Longrightarrow f(\hat{S}(s(m),y)) \neq \hat{S}(n,a) \ \vee \ \hat{C}(y,n,m)\\
\hat{T}(n,0) &\Longrightarrow \forall x(f(x) \not< \hat{S}(s(n),a) \ \vee \ f(x) = \hat{S}(n,a) \ \vee \ f(x) < \hat{S}(n,a))\\
\hat{T}(n,s(m)) &\Longrightarrow \forall x(f(\hat{S}(s(m),x)) \not< \hat{S}(s(n),a) \ \vee \ f(\hat{S}(s(m),x)) = \hat{S}(n,a) \ \vee\\
&\qquad f(x) < \hat{S}(n,a)) \wedge \ \hat{T}(n,m)\\
\hat{S}(0,y) &\Longrightarrow y\\
\hat{S}(s(n),y) &\Longrightarrow suc(\hat{S}(s(n),y))
\end{aligned}
$$

Analysis of the k-repetition Statement

- The k-repetition statement has two independent parameters which is beyond the capabilities of N. Peltier's superposition prover and thus beyond the analysis method of [Leitsch et al. 2017]
- While two parameters need not be problematic, in the case of the k-repetition nested loops are needed to build the refutation.
- The schematic proof construction method discussed earlier cannot describe such a refutation.
- A more flexible proof construction mechanism is needed.

Call Structure of the Refutation



A more flexible formalism

- Rather than developing a formalism for proof schema in particular we provide a general scaffolding for recursive object construction.
  - For example, Proof schemata, Resolution Refutations, Schematic Unifiers, Herbrand Systems, etc.
- The scaffolding provides a "structural semantics" for finite representability.
- This scaffolding may be decorated by objects which match its structure.
- So what is this scaffolding made of?

Junctions

$$\left(\delta, \overrightarrow{t}_n\right)$$

- The above object is a junction, δ is a symbol from a set of symbols Δ, and $\overrightarrow{t}_n$ is a tuple of numeric terms which may contain parameters and primitive recursive functions.
- Junctions whose tuples of numeric terms are of the same length are well ordered by $<_n$.
- Junctions differing in tuple length must have different symbols.
- Junctions can also be well-ordered by the symbols $<_\Delta$ (by tuple size).
- We say $p \lhd q$ if either $q <_n p$ or $p <_\Delta q$.
- Note that $\lhd$ is not well founded.

Junctions

$$(\delta, n, m)$$

$$(\delta_2, n, s(m))$$

$$(\delta_2, n, s(m), k, 0)$$

$$(\delta_3, n, s(m), k, 0)$$

- The first two are valid Junctions.
- The third is not, same symbol as the second but different tuple length
- The fourth is also valid.
- Note, have an infinite set of parameters $\mathcal{P}$, a parameter assignment σ assigns a numeral to each parameter.
- The set of all parameter assignments is $\mathcal{S}$.

Flows

$$P_{S^*} = \left\{ \begin{array}{c} (A_1, S_1) \\ (A_2, S_2) \\ \dots \\ (A_n, S_n) \end{array} \right\}$$

- The A_i are sets of junctions.
- Let $\mathcal{S}$ be the set of parameter assignments, S^* is a partitioning of $\mathcal{S}$ into disjoint sets S_1 through S_n.
- If $\sigma \in S_i$ then $P_{S^*}(\sigma) = A_i$
- There is a is special junction $[P_{S^*}]$ s.t. $\forall \sigma \in \mathcal{S},\ [P_{S^*}] \in P_{S^*}(\sigma)$, The Source of the flow.
- Note, $\forall q \in A_i,\ q \neq [P_{S^*}]$ then $[P_{S^*}] \lhd q$

Example Flow

$$P_{S^*} = \left\{ \begin{array}{c} \left(\left\{ \begin{array}{c} (\delta_2, n, k), \\ (\delta_1, \hat{p}(n), s(k)), \\ (\delta_2, \hat{p}(n), s(k)), \\ (\delta_3, \hat{p}(n), s(k)) \end{array} \right\}, S_1 \right) \\ \\ \left(\left\{ \begin{array}{c} (\delta_2, n, k), \\ (\delta_3, \hat{p}(n), s(k)) \end{array} \right\}, S_2 \right) \end{array} \right.$$

- $S^* = S_1 \cup S_2$ where $S_1 = \{\sigma \in \mathcal{S} \ \& \ \sigma(n) \downarrow_\omega > 0\}$ and $S_2 = \{\sigma \in \mathcal{S} \ \& \ \sigma(n) \downarrow_\omega = 0\}$.
- If $\sigma \in S_1$ then $P_{S^*}(\sigma) = \{(\delta_2, n, k), (\delta_1, \hat{p}(n), s(k)), (\delta_2, \hat{p}(n), s(k)), (\delta_3, \hat{p}(n), s(k))\}$
- $[P_{S^*}] = (\delta_2, n, k)$
- $(\delta_2, n, k) \lhd (\delta_3, \hat{p}(n), s(k))$ where $\hat{p}$ is the predecessor function and $<_2$ is the lexicographical ordering.

Connecting flows

- Let $P_{S_1^*}$ and $P_{S_2^*}$ be flows and σ a parameter assignment such that
  - there exists $q \in P_{S_1^*}(\sigma)$ such that $\left[P_{S_1^*}\right] \neq q$ and
  - there exists a parameter assignment θ such that
  - $q\sigma = \left[P_{S_2^*}\right]\theta$ then

  we refer to the flows as linked.
- If a set of flows is completely linked together the result is a call graph.

Call graph definition

Definition

A finite set of flows $\mathcal{G}$ is referred to as a call graph if for every $P_{\mathcal{S}_1^*} \in \mathcal{G}$, $S \in \mathcal{S}^*$, $\sigma \in S$, $j \in P_{\mathcal{S}_1^*}$ there exists a unique $P_{\mathcal{S}_2^*} \in \mathcal{G}$ and $\theta \in \mathcal{S}$ s.t. $\theta([P_{\mathcal{S}_2^*}]) \downarrow_\omega = \sigma(j) \downarrow_\omega$. We write $\mathit{flow}(j, \sigma) = P_{\mathcal{S}_2^*}$ and $\mathit{subst}(j, \sigma) = \theta$.

- Essentially in a call graph every Junction under every parameter assignment is a the source of some flow.

Example Call Graph

- $\mathcal{G} = \{P_1, P_2\}$ is a call graph , where

$$P_1 = \{(\{(\delta, n), (\delta', n, p(n), n, 0)\}, \mathcal{S})\}$$

$$P_2 = \left\{ \begin{array}{c} (\{(\delta', n, m, k, w), (\delta', n, m, p(k), s(w))\}, S_1) \ , \\ (\{(\delta', n, m, k, w), (\delta', n, p(m), n, w)\}, S_2) \ , \\ (\{(\delta', n, m, k, w)\}, S_3) \end{array} \right\}$$

- $S_1 = \{\sigma | \sigma \in \mathcal{S}, \sigma(k) \downarrow_\omega > 0\}$,
- $S_2 = \{\sigma | \sigma \in \mathcal{S}, \sigma(k) \downarrow_\omega = 0 \ \& \ \sigma(m) \downarrow_\omega > 0\}$,
- $S_3 = \{\sigma | \sigma \in \mathcal{S}, \sigma(k) \downarrow_\omega = 0 \ \& \ \sigma(m) \downarrow_\omega = 0\}$
- Note that for $\sigma \in S_3$ $|P_2(\sigma)|$ such parameter assignments are referred to as sinks.

What does a call Graph define

- Starting from any parameter assignment a call graph provides a way to compute new parameter assignments until one reaches a sink.

$$[\sigma, subst(j_1,\sigma)]\,,[subst(j_1,\sigma), subst(j_2,subst(j_1,\sigma))]\,,$$

$$[subst(j_2,subst(j_1,\sigma)), subst(j_3, subst(j_2,subst(j_1,\sigma)))]\,,\cdots$$

- Let us consider the above call graph and how we may apply it to a given parameter assignment.

$$\left[\overbrace{\{n \leftarrow \alpha_1\}}^{\sigma}, \overbrace{\left\{\begin{array}{c} n \leftarrow \alpha_1\ ,\ m \leftarrow p(\alpha_1), \\ k \leftarrow \alpha_1\ ,\ w \leftarrow 0 \end{array}\right\}}^{\theta_1}\right]$$

where $j_1 = (\delta', n, p(n), n, 0)$, $flow(j_1,\sigma) = P_2$, and $Subst(j_1,\sigma) = \theta_1$.

What does a call Graph define

$$\left[\overbrace{\left\{\begin{array}{c} n \leftarrow \alpha_1 \ , \ m \leftarrow p(\alpha_1), \\ k \leftarrow \alpha_1, w \leftarrow 0 \end{array}\right\}}^{\theta_1}, \overbrace{\left\{\begin{array}{c} n \leftarrow \alpha_1 \ , \ m \leftarrow p(\alpha_1), \\ k \leftarrow p(\alpha_1), w \leftarrow s(0) \end{array}\right\}}^{\theta_2}\right]$$

where $j_2 = (\delta', n, m, p(k), s(w))$, $flow(j_2, \theta_1) = P_2$, and $Subst(j_2, \theta_1) = \theta_2$.

$$\left[\overbrace{\left\{\begin{array}{c} n \leftarrow \alpha_1 \ , \ m \leftarrow p(\alpha_1), \\ k \leftarrow p(\alpha_1) \ , \ w \leftarrow s(0) \end{array}\right\}}^{\theta_2}, \overbrace{\left\{\begin{array}{c} n \leftarrow \alpha_1 \ , \ m \leftarrow p(\alpha_1), \\ k \leftarrow p(p(\alpha_1)) \ , \ w \leftarrow s(s(0)) \end{array}\right\}}^{\theta_3}\right]$$

where $j_3 = (\delta', n, m, p(k), s(w))$, $flow(j_3, \theta_2) = P_2$, and $Subst(j_3, \theta_2) = \theta_3$.

$$\vdots$$

What does a call Graph define

$$\left[\overbrace{\left\{\begin{array}{c} n \leftarrow \alpha_1 \ , \ m \leftarrow p(\alpha_1), \\ k \leftarrow p^{\alpha_1}(\alpha_1) \ , \ w \leftarrow s^{\alpha_1}(0) \end{array}\right\}}^{\theta_3}, \overbrace{\left\{\begin{array}{c} n \leftarrow \alpha_1 \ , \ m \leftarrow p(p(\alpha_1)), \\ k \leftarrow \alpha_1 \ , \ w \leftarrow s^{\alpha_1}(0) \end{array}\right\}}^{\theta_4}\right]$$

$j_4 = (\delta', n, p(m), n, w)$, $flow(j_4, \theta_4) = P_2$, and $Subst(j_4, \theta_3) = \theta_4$.

$$\left[\overbrace{\left\{\begin{array}{c} n \leftarrow \alpha_1 \ , \ m \leftarrow p_1^{\alpha}(\alpha_1), \\ k \leftarrow 0 \ , \ w \leftarrow s^{(\alpha_1)^2}(0) \end{array}\right\}}^{\theta_4}\right]$$

Being that we have reached a sink at this Junction *flow* and *Subst* are only defined for the source of P_2.

Call Graph Traces

- The Call graph discussed above results in a simple linear structure when applied to a parameter assignment.
- In general the application of a call graph to a parameter assignment may result in branching.
- To capture this structure we defined Call Graph Traces.
- A trace is just a tree where each node is labeled by the junction the call graph passed during application to a parameter assignment.

Example: Branching Call Graph

- Consider the call graph $\mathcal{G} = \{P_1, P_2\}$ where the flows are defined as follows:

$$P_1 = \{ \ (\{(\delta, n), (\delta, p(n)), (\delta', n, n)\}, S_1) \ , \ (\{(\delta, n)\}, S_2) \ \}$$

$$P_2 = \{ \ (\{(\delta', n, m), (\delta', n, p(m))\}, S'_1) \ , \ (\{(\delta', n, m)\}, S'_2) \ \}$$

- $S_1 = \{\sigma \in \mathcal{S} \ \& \ \sigma(n) \downarrow_\omega > 0\}$
- $S_2 = \{\sigma \in \mathcal{S} \ \& \ \sigma(n) \downarrow_\omega = 0\}$
- $S'_1 = \{\sigma \in \mathcal{S} \ \& \ \sigma(m) \downarrow_\omega > 0\}$
- $S'_2 = \{\sigma \in \mathcal{S} \ \& \ \sigma(m) \downarrow_\omega = 0\}$

Trace for non-branching Call Graph

$$T(\mathcal{G}, P_1, \{n \leftarrow 2\}) = [(\delta, 2), T(\mathcal{G}, P_2, \{n \leftarrow 2, m \leftarrow 1, k \leftarrow 2, w \leftarrow 0\})]$$

$$T\left(\mathcal{G}, P_2, \left\{ \begin{array}{c} n \leftarrow 2, m \leftarrow 1, \\ k \leftarrow 2, w \leftarrow 0 \end{array} \right\}\right) = \left[(\delta', 2, 1, 2, 0), T\left(\mathcal{G}, P_2, \left\{ \begin{array}{c} n \leftarrow 2, m \leftarrow 1, \\ k \leftarrow 1, w \leftarrow 1 \end{array} \right\}\right)\right]$$

$$T\left(\mathcal{G}, P_2, \left\{ \begin{array}{c} n \leftarrow 2, m \leftarrow 1, \\ k \leftarrow 1, w \leftarrow 1 \end{array} \right\}\right) = \left[(\delta', 2, 1, 1, 1), T\left(\mathcal{G}, P_2, \left\{ \begin{array}{c} n \leftarrow 2, m \leftarrow 1, \\ k \leftarrow 0, w \leftarrow 2 \end{array} \right\}\right)\right]$$

$$T\left(\mathcal{G}, P_2, \left\{ \begin{array}{c} n \leftarrow 2, m \leftarrow 1, \\ k \leftarrow 0, w \leftarrow 2 \end{array} \right\}\right) = \left[(\delta', 2, 1, 0, 2), T\left(\mathcal{G}, P_2, \left\{ \begin{array}{c} n \leftarrow 2, m \leftarrow 0, \\ k \leftarrow 2, w \leftarrow 2 \end{array} \right\}\right)\right]$$

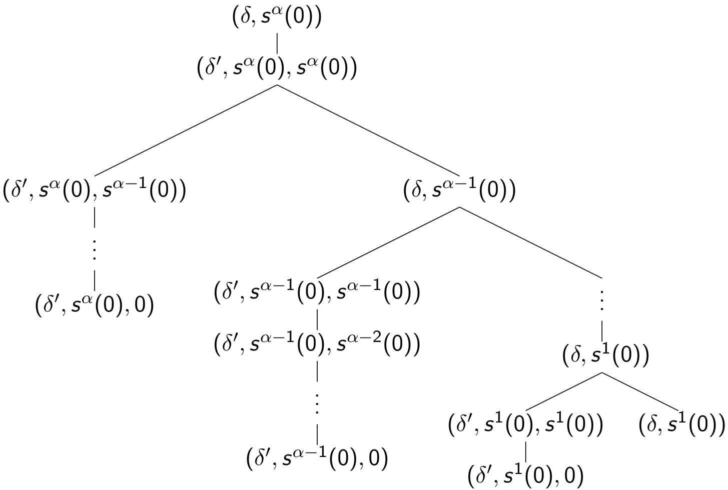
$$T\left(\mathcal{G}, P_2, \left\{ \begin{array}{c} n \leftarrow 2, m \leftarrow 0, \\ k \leftarrow 2, w \leftarrow 2 \end{array} \right\}\right) = \left[(\delta', 2, 0, 2, 2), T\left(\mathcal{G}, P_2, \left\{ \begin{array}{c} n \leftarrow 2, m \leftarrow 0, \\ k \leftarrow 1, w \leftarrow 3 \end{array} \right\}\right)\right]$$

$$T\left(\mathcal{G}, P_2, \left\{ \begin{array}{c} n \leftarrow 2, m \leftarrow 0, \\ k \leftarrow 1, w \leftarrow 3 \end{array} \right\}\right) = \left[(\delta', 2, 0, 1, 3), T\left(\mathcal{G}, P_2, \left\{ \begin{array}{c} n \leftarrow 2, m \leftarrow 0, \\ k \leftarrow 0, w \leftarrow 4 \end{array} \right\}\right)\right]$$

$$T\left(\mathcal{G}, P_2, \left\{ \begin{array}{c} n \leftarrow 2, m \leftarrow 0, \\ k \leftarrow 0, w \leftarrow 4 \end{array} \right\}\right) = [(\delta', 2, 0, 0, 4), \emptyset]$$

$$[(\delta, 2), [(\delta', 2, 1, 2, 0), [(\delta', 2, 1, 1, 1), [(\delta', 2, 1, 0, 2), [(\delta', 2, 0, 2, 2), [(\delta', 2, 0, 1, 3), [(\delta', 2, 0, 0, 4), \emptyset]]]]]]]$$

Trace for branching Call Graph



Trace Normalization

- Note that Traces seem to result in finite trees but we also mentioned that the order $\lhd$ is not well founded.
- Fortunately finite call graphs use a well founded suborder of $\lhd$

Theorem

Let $\mathcal{G}$ be a finite call graph, $P_{S^*} \in \mathcal{G}$ a flow, and $\sigma \in \mathcal{S}$ a parameter assignment. Then $T(\mathcal{G}, P_{S^*}, \sigma)$ always produces a finite trace.

- while this isn't too surprising from what we have discussed so far call graphs may contain mutually recursive calls.
- This feature is essential for the refutation of the cut structure of the k-repetition statement.

Mutual Calls

$$
C_2 = \left\{ \begin{array}{c} \left(\left\{ \begin{array}{c} (\delta_1, n, m, w, k, r, 0), \\ (\delta_1, n, m, p(w), s(k), r, 0), \\ (\delta_2, n, m, w, k, p(p(r)), 0), \\ (\delta_3, n, m, w, k, p(r), 0) \end{array} \right\}, S_5 \right) \\ \left(\left\{ \begin{array}{c} (\delta_1, n, m, w, k, r, 0), \\ (\delta_4, n, m, 0, k, p(r), 0), \\ (\delta_2, n, m, 0, k, p(p(r)), 0) \end{array} \right\}, S_6 \right) \end{array} \right\}
C_3 = \left\{ \begin{array}{c} \left(\left\{ \begin{array}{c} (\delta_2, n, m, w, k, r, q), \\ (\delta_2, n, m, w, , p(r), s(q)), \\ (\delta_1, n, m, p(w), s(k), s(r), 0), \\ (\delta_3, n, m, w, k, p(r), s(q)) \end{array} \right\}, S_7 \right) \\ \left(\left\{ \begin{array}{c} (\delta_2, n, m, w, k, r, q), \\ (\delta_1, n, m, p(w), s(k), s(m), 0), \\ (\delta_3, n, m, w, k, 0, s(q)), \\ (\delta_5, n, m, w, k, 0, s(q)) \end{array} \right\}, S_8 \right) \end{array} \right\}
$$

- $S_5 = \{\sigma | \sigma \in \mathcal{S} \ , \ \ w\sigma > 0\}$
- $S_6 = \{\sigma | \sigma \in \mathcal{S} \ , \ \ w\sigma = 0\}$
- $S_7 = \{\sigma | \sigma \in \mathcal{S} \ , \ \ r\sigma > 0\}$
- $S_8 = \{\sigma | \sigma \in \mathcal{S} \ , \ \ r\sigma = 0\}$

Mutual Calls

$$C_2 = \left\{ \begin{array}{c} \left(\left\{ \begin{array}{c} (\delta_1, n, m, w, k, r, 0), \\ (\delta_1, n, m, p(w), s(k), r, 0), \\ (\delta_2, n, m, w, k, p(p(r)), 0), \\ (\delta_3, n, m, w, k, p(r), 0) \end{array} \right\}, S_5 \right) \\ \left(\left\{ \begin{array}{c} (\delta_1, n, m, w, k, r, 0), \\ (\delta_4, n, m, 0, k, p(r), 0), \\ (\delta_2, n, m, 0, k, p(p(r)), 0) \end{array} \right\}, S_6 \right) \end{array} \right\} \quad C_3 = \left\{ \begin{array}{c} \left(\left\{ \begin{array}{c} (\delta_2, n, m, w, k, r, q), \\ (\delta_2, n, m, w, , p(r), s(q)), \\ (\delta_1, n, m, p(w), s(k), s(r), 0), \\ (\delta_3, n, m, w, k, p(r), s(q)) \end{array} \right\}, S_7 \right) \\ \left(\left\{ \begin{array}{c} (\delta_2, n, m, w, k, r, q), \\ (\delta_1, n, m, p(w), s(k), s(m), 0), \\ (\delta_3, n, m, w, k, 0, s(q)), \\ (\delta_5, n, m, w, k, 0, s(q)) \end{array} \right\}, S_8 \right) \end{array} \right\}$$

- $S_5 = \{\sigma | \sigma \in \mathcal{S} \ , \ \ w\sigma > 0\}$
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- $S_7 = \{\sigma | \sigma \in \mathcal{S} \ , \ \ r\sigma > 0\}$
- $S_8 = \{\sigma | \sigma \in \mathcal{S} \ , \ \ r\sigma = 0\}$

Mutual Calls

$$C_2 = \left\{ \begin{array}{c} \left(\left\{ \begin{array}{c} (\delta_1, n, m, w, k, r, 0), \\ (\delta_1, n, m, p(w), s(k), r, 0), \\ (\delta_2, n, m, w, k, p(p(r)), 0), \\ (\delta_3, n, m, w, k, p(r), 0) \end{array} \right\}, S_5 \right) \\ \left(\left\{ \begin{array}{c} (\delta_1, n, m, w, k, r, 0), \\ (\delta_4, n, m, 0, k, p(r), 0), \\ (\delta_2, n, m, 0, k, p(p(r)), 0) \end{array} \right\}, S_6 \right) \end{array} \right\} \quad C_3 = \left\{ \begin{array}{c} \left(\left\{ \begin{array}{c} (\delta_2, n, m, w, k, r, q), \\ (\delta_2, n, m, w, , p(r), s(q)), \\ (\delta_1, n, m, p(w), s(k), s(r), 0), \\ (\delta_3, n, m, w, k, p(r), s(q)) \end{array} \right\}, S_7 \right) \\ \left(\left\{ \begin{array}{c} (\delta_2, n, m, w, k, r, q), \\ (\delta_1, n, m, p(w), s(k), s(m), 0), \\ (\delta_3, n, m, w, k, 0, s(q)), \\ (\delta_5, n, m, w, k, 0, s(q)) \end{array} \right\}, S_8 \right) \end{array} \right\}$$

- $S_5 = \{\sigma | \sigma \in \mathcal{S} \ , \ w\sigma > 0\}$
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- $S_7 = \{\sigma | \sigma \in \mathcal{S} \ , \ r\sigma > 0\}$
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Mutual Calls

$$C_2 = \left\{ \begin{array}{c} \left(\left\{ \begin{array}{c} (\delta_1, n, m, w, k, r, 0), \\ (\delta_1, n, m, p(w), s(k), r, 0), \\ (\delta_2, n, m, w, k, p(p(r)), 0), \\ (\delta_3, n, m, w, k, p(r), 0) \end{array} \right\}, S_5 \right) \\ \left(\left\{ \begin{array}{c} (\delta_1, n, m, w, k, r, 0), \\ (\delta_4, n, m, 0, k, p(r), 0), \\ (\delta_2, n, m, 0, k, p(p(r)), 0) \end{array} \right\}, S_6 \right) \end{array} \right\} \quad C_3 = \left\{ \begin{array}{c} \left(\left\{ \begin{array}{c} (\delta_2, n, m, w, k, r, q), \\ (\delta_2, n, m, w, , p(r), s(q)), \\ (\delta_1, n, m, p(w), s(k), s(r), 0), \\ (\delta_3, n, m, w, k, p(r), s(q)) \end{array} \right\}, S_7 \right) \\ \left(\left\{ \begin{array}{c} (\delta_2, n, m, w, k, r, q), \\ (\delta_1, n, m, p(w), s(k), s(m), 0), \\ (\delta_3, n, m, w, k, 0, s(q)), \\ (\delta_5, n, m, w, k, 0, s(q)) \end{array} \right\}, S_8 \right) \end{array} \right\}$$

- $S_5 = \{\sigma | \sigma \in \mathcal{S} \ , \ \ w\sigma > 0\}$
- $S_6 = \{\sigma | \sigma \in \mathcal{S} \ , \ \ w\sigma = 0\}$
- $S_7 = \{\sigma | \sigma \in \mathcal{S} \ , \ \ r\sigma > 0\}$
- $S_8 = \{\sigma | \sigma \in \mathcal{S} \ , \ \ r\sigma = 0\}$

Mutual Calls

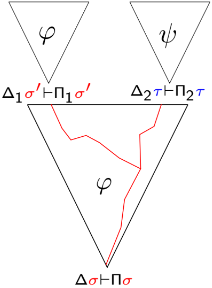
$$C_2 = \left\{ \begin{array}{c} \left(\left\{ \begin{array}{c} (\delta_1, n, m, w, k, r, 0), \\ (\delta_1, n, m, p(w), s(k), r, 0), \\ (\delta_2, n, m, w, k, p(p(r)), 0), \\ (\delta_3, n, m, w, k, p(r), 0) \end{array} \right\}, S_5 \right) \\ \left(\left\{ \begin{array}{c} (\delta_1, n, m, w, k, r, 0), \\ (\delta_4, n, m, 0, k, p(r), 0), \\ (\delta_2, n, m, 0, k, p(p(r)), 0) \end{array} \right\}, S_6 \right) \end{array} \right\} C_3 = \left\{ \begin{array}{c} \left(\left\{ \begin{array}{c} (\delta_2, n, m, w, k, r, q), \\ (\delta_2, n, m, w, , p(r), s(q)), \\ (\delta_1, n, m, p(w), s(k), s(r), 0), \\ (\delta_3, n, m, w, k, p(r), s(q)) \end{array} \right\}, S_7 \right) \\ \left(\left\{ \begin{array}{c} (\delta_2, n, m, w, k, r, q), \\ (\delta_1, n, m, p(w), s(k), s(m), 0), \\ (\delta_3, n, m, w, k, 0, s(q)), \\ (\delta_5, n, m, w, k, 0, s(q)) \end{array} \right\}, S_8 \right) \end{array} \right\}$$

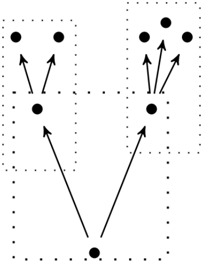
- $S_5 = \{\sigma | \sigma \in \mathcal{S} \ , \ w\sigma > 0\}$
- $S_6 = \{\sigma | \sigma \in \mathcal{S} \ , \ w\sigma = 0\}$
- $S_7 = \{\sigma | \sigma \in \mathcal{S} \ , \ r\sigma > 0\}$
- $S_8 = \{\sigma | \sigma \in \mathcal{S} \ , \ r\sigma = 0\}$

Decorating Traces with Proof Schemata

- Every derivation in a proof schema is associated with a proof symbol.
- Every derivation is associated with a set of free parameters.
- Every derivation has a set of non-trivial leaves(possibly empty)
- can use these properties to match derivations to flows and junctions contained in the flows.

Decorating Traces with Proof Schemata





Decoration of C_2 for partition S_5

$$(\delta_3, n, m, w, k, pred(r), 0)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash \hat{F}_4(\bar{\mathbf{X}}, k, 0)}{\vdash f(\hat{S}(X_3(k,0)) \not< s(k) \vee f(X_3(k,0)) < k \vee f(X_3(k,0)) = k}\; B\hat{F}_4 r}{\vdash f(X_3(k,0)) \not< s(k), f(X_3(k,0)) < k \vee f(X_3(k,0)) = k}\; \vee : r}{\vdash f(X_3(k,0)) \not< s(k), f(X_3(k,0)) < k, f(X_3(k,0)) = k}\; \vee : r}{f(X_3(k,0)) < s(k) \vdash f(X_3(k,0)) < k, f(X_3(k,0)) = k}\; \neg : r$$

(2)

$$(\delta_1, n, m, p(w), s(k), r, 0)$$

$$\dfrac{\vdash f(\alpha) < s(k) \qquad\qquad (2)}{\vdash f(\alpha) < k, f(\alpha) = k}\; \mathrm{Res}(\{\ X_3(k,0) \leftarrow \alpha\ \})$$

(1)

$$(\delta_2, n, m, w, k, p(p(r)), 0)$$

$$\dfrac{\dfrac{(1)}{\vdash f(\alpha) < k, f(\alpha) = k} \qquad \dfrac{\dfrac{\dfrac{\vdash f(\alpha) < k, \hat{F}_5(\alpha, \bar{\mathbf{X}}, k, 0)}{\vdash f(\alpha) < k, f(\hat{S}(0,\alpha)) \neq k}\; B\hat{F}_5 r}{\vdash f(\alpha) < k, f(\alpha) \neq k}\; B\hat{S} r}{f(\alpha) = k \vdash f(\alpha) < k}\; \neg : r}{\vdash f(\alpha) < k}\; \mathrm{Res}(\emptyset)$$

$$(\delta_1, n, m, w, k, r, 0)$$

Conclusion & Future work

- The formalism presented above can be used for finite representation of the resolution refutation need for proof analysis using CERES.
- Additionally, we can imagine a reverse resolution calculus
  - Given an unsatisfiable recursive clause set does there exists a refutation of the clause set using the given Call Graph.

  This is currently being investigated.

Thank you for your time.