When First-order Unification Calls itself

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Abstract. We present a unification problem based on recursively calling first-order syntactic unification on certain bindings of the resulting unifier when they occur. There are clearly cases when this procedure does not terminate. It remains an open question whether termination of this process may be decided.

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1 Introduction

In this short paper we present an interesting unification problem and an open question concerning its termination. We also provide a few examples and a conjecture concerning cyclic behavior of infinite instances of the problem. We assume knowledge of syntactic first-order unification (See [1]).

2 Problem Description

Let $V^x_{\mathbb{N}} = \{x_i \mid i \in \mathbb{N}\}$, $\sigma^x_{sh} = \{x_i \leftarrow x_{i+1} \mid i \in \mathbb{N}\}$, $\sigma^*_s = \{x^* \leftarrow s\}$, and $UNIF(\cdot, \cdot)$ be a unification algorithm which returns an m.g.u without renaming variables. Now let us consider first-order terms s and t such that $Var(s) \subset V^x_{\mathbb{N}} \cup \{x^*\}$ and $Var(t) \subset V^y_{\mathbb{N}}$. We refer to such terms as a *loop pair*, denoted by (s, t).

Question: Is termination of **LoopUnif**(s, t, s) is decidable?

```
function LoopUnif(s, t, s_b)
    if ∃r({x* ← r} ∈ UNIF(s, t) ∧ Var(r) ≠ ∅) then
        LoopUnif(s σ^x_sh σ*_{s_b}, t, s_b)
    end if
end function
```

3 Cyclicity Conjecture

Before we introduce the conjecture, consider the following non-triviality example:

Example 1.

$$s_0 = h(h(x_1, h(x_2, x_1)), x^*) \qquad t_0 = h(y_1, h(y_2, y_1))$$

Note that $\{x^* \leftarrow h(y_2, h(x_1, h(x_2, x_1)))\} \in \mathit{UNIF}(s_0, t_0)$ and thus we now need to consider the problem

$$s_1 = h(h(x_2, h(x_3, x_2)), h(h(x_1, h(x_2, x_1)), x^*)) \qquad t_1 = h(y_1, h(y_2, y_1))$$

where $s_b = h(h(x_1, h(x_2, x_1)), x^*)$. Note $\{x^* \leftarrow h(x_2, h(x_3, x_2))\} \in \mathit{UNIF}(s_1, t_1)$. In this case **LoopUnif**(s_0, t_0, s_0) will not terminate and on the $2n^{th}$ iteration we will have $\{x^* \leftarrow h(y_2, h(x_1, h(x_2, x_1)))\} \in \mathit{UNIF}(s_{2n}, t_{2n})$ and $2n + 1^{th}$ iteration we will have $\{x^* \leftarrow h(x_2, h(x_3, x_2))\} \in \mathit{UNIF}(s_{2n+1}, t_{2n+1})$.

For a loop pair (s, t) consider the following recursive definition:

$$(s_0, t_0) = (s, t) \qquad (s_{n+1}, t_{n+1}) = (s_n \sigma^x_{sh} \sigma^*_{s_0}, t_n \sigma_{sh})$$

Using this construction we can define the *loop sequence of* (s, t), denoted by $\mathit{loopSeq}_{s,t}$. The loop sequence is a list such that $\mathit{loopSeq}_{s,t}(0) = t$ and for $i > 0$, $\mathit{loopSeq}_{s,t}(i) = r_i$ where $\{x^* \leftarrow r_i\} \in \mathit{UNIF}(s_{i-1}, t_{i-1})$. If for some i, $\{x^* \leftarrow r_i\} \notin \mathit{UNIF}(s_{i-1}, t_{i-1})$, then for all $i \leq j$ $\mathit{loopSeq}_{s,t}(j) = \perp$ We conjecture the following:

Conjecture 1 (cyclicity). Let (s, t) be a loop pair such that for all $0 \leq i$ $\mathit{loopSeq}_{s,t}(j) \neq \perp$. Then there exists $i, k \in \mathbb{N}$,such that for all $i \leq j$, $|\mathit{loopSeq}_{s,t}(j)| = |\mathit{loopSeq}_{s,t}(j+k)|$

Note that the cyclicity property is not enough to prove decidability of termination, but if true it would imply that we do not need to deal with irrational sequences of term size.

For the following example *LoopUnif* does not terminate and a cycle of length 3 is not found until the 9^{th} round.

Example 2.

$$s_0 = (h(x_1, h(x_4, h(x_1, x_4))), x^*)$$
$$t_0 = h(y_1, h(y_4, h(y_1, y_4)))$$

$$\begin{array}{ll}
|\mathit{loopSeq}_{s,t}(0)| = 5 & |\mathit{loopSeq}_{s,t}(1)| = 7 \\
|\mathit{loopSeq}_{s,t}(2)| = 6 & |\mathit{loopSeq}_{s,t}(3)| = 5 \\
|\mathit{loopSeq}_{s,t}(4)| = 7 & |\mathit{loopSeq}_{s,t}(5)| = 9 \\
|\mathit{loopSeq}_{s,t}(6)| = 8 & |\mathit{loopSeq}_{s,t}(7)| = 7 \\
|\mathit{loopSeq}_{s,t}(8)| = \mathbf{9} & |\mathit{loopSeq}_{s,t}(9)| = \mathbf{8} \\
|\mathit{loopSeq}_{s,t}(10)| = \mathbf{10} & |\mathit{loopSeq}_{s,t}(11)| = 9 \\
|\mathit{loopSeq}_{s,t}(12)| = 8 & |\mathit{loopSeq}_{s,t}(13)| = 10 \\
|\mathit{loopSeq}_{s,t}(14)| = 9 & |\mathit{loopSeq}_{s,t}(15)| = 8
\end{array}$$

While this example follows the cyclicity conjecture there exists very similar examples which fail and seem to exhibit cyclicity at some points:

Example 3.

$$s_0 = h(h(x_1, h(x_{16}, h(x_{32}, h(x_1, h(x_{16}, x_{32}))))), x^*)$$

$$t_0 = h(y_1, h(y_{16}, h(y_{32}, h(y_1, h(y_{16}, y_{32})))))$$

$$\begin{array}{ll}
|loopSeq_{s,t}(0)| = \mathbf{7} & |loopSeq_{s,t}(1)| = \mathbf{10} \\
|loopSeq_{s,t}(2)| = \mathbf{10} & |loopSeq_{s,t}(3)| = \mathbf{9} \\
|loopSeq_{s,t}(4)| = \mathbf{8} & |loopSeq_{s,t}(5)| = 7 \\
|loopSeq_{s,t}(6)| = 10 & |loopSeq_{s,t}(7)| = 10 \\
|loopSeq_{s,t}(8)| = 9 & |loopSeq_{s,t}(9)| = 8 \\
|loopSeq_{s,t}(10)| = 7 & |loopSeq_{s,t}(11)| = 10 \\
|loopSeq_{s,t}(12)| = 10 & |loopSeq_{s,t}(13)| = 9 \\
|loopSeq_{s,t}(14)| = 8 & |loopSeq_{s,t}(15)| = 7 \\
|loopSeq_{s,t}(16)| = 10 & |loopSeq_{s,t}(17)| = 10 \\
|loopSeq_{s,t}(18)| = \boxed{14} & |loopSeq_{s,t}(19)| = 13 \\
|loopSeq_{s,t}(20)| = 12 & |loopSeq_{s,t}(21)| = 11 \\
|loopSeq_{s,t}(22)| = 10 & |loopSeq_{s,t}(23)| = 18 \\
|loopSeq_{s,t}(24)| = 13 & |loopSeq_{s,t}(25)| = 12 \\
|loopSeq_{s,t}(26)| = 11 & |loopSeq_{s,t}(27)| = 15 \\
|loopSeq_{s,t}(28)| = \perp &
\end{array}$$

Notice that the cycle $7, 10, 10, 9, 8$ repeats 3 times before breaking.

While this last example does not refute the conjecture, it illustrates that even if the conjecture holds our question concerning termination remains non-trivial.

References

1. Franz Baader and Wayne Snyder. Unification theory. In John Alan Robinson and Andrei Voronkov, editors, *Handbook of Automated Reasoning (in 2 volumes)*, pages 445–532. Elsevier and MIT Press, 2001.