Proof Schema and the Refutational Complexity of their Cut Structure

David M. Cerna

July 18, 2018

Rather than providing a formal definition of *proof schemata* [3] which are sets of **LK**-derivations linked together by matching end sequents of one proof to non-tautological leaves of another, we instead proceed by example and rely on an intuitive understanding of the concept. In particular, let us consider the following schematic sequent[1] where $\bigvee$ and $\bigwedge$ denote iterated disjunction and conjunction respectively and n is a numeric parameter defining the recursion, that is replacing n by a numeral and normalizing results in an **LK**-proof.

$$\Gamma, \neg f(a) < 0, \bigwedge_{i=0}^{n+1} \forall x\,(i = f(x) \wedge i = f(s(x))) \rightarrow f(x) = f(s(x))\ ,\ \forall x \bigvee_{i=0}^{n+1} i = f(x) \vdash \exists x(f(x) = f(s(x)))$$

$$\Gamma = \left\{ \begin{array}{ll} \forall x\left(\left(\bigvee_{i=0}^{n} i = f(x)\right) \rightarrow f(x) < (n+1)\right)\ , & \bigwedge_{i=0}^{n} \forall x\,(f(s(x)) < (i+1) \rightarrow (i = f(s(x)) \vee f(x) < i)) \\ \forall x\left(\left(\bigvee_{i=0}^{n} i = f(s(x))\right) \rightarrow f(x) < (n+1)\right) & \bigwedge_{i=0}^{n} \forall x\,(f(x) < (i+1) \rightarrow (i = f(x) \vee f(x) < i)) \end{array} \right\}$$

A proof schema of this statement can be constructed using the sequence of cuts $\exists x(0 = f(x) \wedge 0 = f(s(x))) \vee \forall x(f(x) < 0),\ \cdots\ , \exists x(n+1 = f(x) \wedge n+1 = f(s(x))) \vee \forall x(f(x) < n+1)$. We refer to this proof schema as the *Very Weak Pigeonhole Principle* (VWPHP)[2] and using previously developed methods for cut structure extraction a recursive NNF[2] representing the cut structure is constructible[3]:

$$\begin{array}{rcl} \varphi(0) & = & Next(0)\ \wedge\ (0 = f(a) \vee 0 = f(s(a))) \\ \chi(0) & = & (\neg f(a) < 0)\ \wedge\ \forall x((\neg 0 = f(x)) \vee (\neg 0 = f(s(x)))) \\ \varphi(n+1) & = & \forall x((n+1) = f(s(x)) \vee f(x) < (n+1))\ \wedge \forall x((n+1) = f(x) \vee f(x) < (n+1))\ \wedge \chi(n+1) \\ \chi(n+1) & = & \forall x((\neg(n+1) = f(x)) \vee (\neg(n+1) = f(s(x))))\ \wedge \forall x((\neg f(x) < (n+1)) \vee n = f(x) \vee f(x) < n) \\ & & \wedge \forall x((\neg f(s(x)) < (n+1)) \vee n = f(s(x)) \vee f(x) < n)\ \wedge \chi(n) \end{array}$$

Unlike previously studied examples of the same arithmetic complexity it is refutationally more difficult even though its refutations grow quadratically in terms of the numeric parameter n. A related proof schema referred to as the *Eventually Constant Schema* (ECS)[1] has a cut structure whose NNF representation grows exponentially in terms of the numeric parameter n.

$$\begin{array}{rcl} \psi(0) & = & \pi(0) \wedge \forall x\,(f(x) < 0 \vee 0 = f(x)) \\ \pi(0) & = & \forall x(\neg f(x) < 0) \wedge \forall x(x \leq x) \wedge \forall x(x \leq g(x)) \wedge \forall x(\neg 0 = f(x) \vee \neg 0 = f(g(x))) \\ \psi(n+1) & = & \pi(n+1) \wedge \forall x\,(f(x) < n+1 \vee n+1 = f(x)) \\ \pi(n+1) & = & \pi(n)\ \wedge\ \forall x\,(\neg n+1 = f(x)\ \vee\ \neg n+1 = f(g(x)))\ \wedge \\ & & \forall x \forall y\,(\neg f(x) < n+1 \vee \neg x \leq y \vee n = f(y) \vee f(y) < n) \end{array}$$

Yet construction of a recursively defined refutation is more complex for VWPHP. If we consider variations of a clause with respect to a particular numeral one will notice that the refutation of the VWPHP NNF requires $O(n)$ instantiations of the clause $\forall x((n+1) = f(x) \vee f(x) < (n+1))$ while the eventually constant schema requires $O(1)$ instances of the clause $\forall x\,(f(x) < n+1 \vee n+1 = f(x))$. This key difference seems to have more of an influence on recursive refutational complexity than refutation size. We can devise the following complexity classes which we refer to as $Const_{unsat}$, $k\text{-}Const_{unsat}$, and $Non\text{-}Const_{unsat}$ which separate these two problems by the number of instantiations of clausal variants. The class $Const_{unsat}$ differs $k\text{-}Const_{unsat}$ in that only a particular subset of the clauses need to have a constant number of instantiations. Our open questions concerning these refutational complexity classes concern further development of these ideas and whether it is possible to define the class of cut structures which result in recursive NNF formula of each complexity class.

References

[1] D. M. Cerna and A. Leitsch. Schematic cut elimination and the ordered pigeonhole principle. In *IJCAR*, pages 241–256, 2016.

[2] D. M. Cerna, A. Leitsch, and A. Lolic. A Resolution Calculus for Recursive Clause Sets [Preprint]. Risc report series, 2018. in review.

[3] A. Leitsch, N. Peltier, and D. Weller. CERES for first-order schemata. *J. Log. Comput.*, 27(7):1897–1954, 2017.

[1] A sequent which may contain recursively defined predicates and or function symbols.
[2] Negation Normal Form
[3] The construction is referred to as the *characteristic formula schema*