David Cerna :The OEIS, Combinatorics, and Schematic Lemma Elimination in Formal Proofs

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October 6, 2014

The focus of our work is lemma elimination in recursively defined first-order logic proofs by resolution. By lemma, we mean an argument used to prove a theorem, of which is not needed to state the theorem. When we state that the proofs are recursively defined we mean that the proofs have a primitive recursive structure, i.e.

$$P(n+1) \Rightarrow Q\left[P(n)\right] \tag{1a}$$

$$P(0) \Rightarrow D \tag{1b}$$

where D is a first-order logic proof, $Q\left[\cdot\right]$ is a first order logic proof containing at least one instance of the proof in the brackets, and n is a natural number. The idea behind this work is to take a recursively defined first-order logic proof with lemmata (arguments not needed to state the theorem) and transform the proof into a proof without any lemmata (every argument in the proof is essential to the statement of the final theorem). The method of transforming proofs that we use is called the CERES method [1, 3, 8] which collects the arguments used to prove the lemmata in the form of a first-order *clause set* (recursively defined of course) and using the resolution method [9], we build a backbone for the proof without lemmata. One can think of the lemma-free proof as a proof without the use of other mathematical theories, i.e. a proof about prime numbers which does not use topology [2]. We will refer to recursively defined clause sets as schematic clause sets. Also, a clause can be thought of as a set of first-order formulae with a specific form.

Though, this area seems far removed from enumerative combinatorics and integer sequences, it happens to be the case that the clause sets, being recursive in nature, end up with useful and interesting combinatoric structure. The combinatoric structure arises from the way resolution constructs the backbone of the proof without lemmata. others have taken note of this necessary structure within resolution proofs [4, 5], however, a system which heavily relies on this structure and contains a language which can partially describe it has not been investigated prior to our work. Also, the combinatorial structure, in our case, is derived from the mathematical arguments used, rather than analysing a specific problem like the *pigeon-hole principle.* Essentially, if one changes the lemmata used to prove the theorem, the output of the CERES method after clausal analysis will differ. Though, in our work to date, the underlying mathematical argument was indeed the pigeon-hole principle, there is no reason for this to be the case in general.

Using resolution and a schematic clause set S, we can derive a new schematic clause set S' which is an extension of S by deriving new clauses derived from the old ones. The construction of the backbone of the lemma-free proof is not a completely automated process and requires analysis of the clauses derivable from the clause set (the process is automated in the non-recursive case). We have found that analysing the combinatorial structure of the derivable clauses helps with the process of constructing the backbone, it also provides us with information regarding the complexity of the output proof. In [7], we where able to find a few patterns in the derivable clauses allowing us to construct the backbone of every instance of the recursive proof (i.e. instantiation of the variable n in Eq. 1a). Namely, the patterns which were found were A000142, A007318, and A093964. We also discovered a previously unknown recurrence enumerating A093964. Currently we are applying the same method to a generalization of [7], which we call the two parameter Non-injectivity Assertion [6]. For certain instances we have found that certain derived clauses have patterns consistent with the central binomial coefficient A000984 and a quite newly constructed combinatorial set and integer sequence, that of the *atomic permutations with 3 runs of equal length n* A241193.

It can easily be the case (we expect it to happen with our current analysis), that a new integer sequence pertaining to a new combinatoric structure can emerge, however, this has not been the case as of this date. Thus not only is it possible to apply (as we have done) current integer sequences to schematic clause set analysis, we can possibly use the schematic clause set analysis to find new integer sequences through the analysis of the combinatorial structure of the derived clauses. As of this date, the method of extracting combinatorial structure from the clause set is still in its infancy, though we provide an outline how the above work was carried out with the hopes of constructing a more general method in future work.

References

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