Schematic Cut Elimination and the Ordered Pigeonhole Principle

David M. Cerna
and
Alexander Leitsch

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- Though, a large portion of the proof analysis was worked out on pen and paper it lead to investigations towards the development of an inductive CERES method.
- These investigations lead to the development of the schematic CERES method [Dunchev et al. 2013], the method at the heart of the proof analysis we present today.

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- In cases when the induction introduces a new eigenvariable the transformation is not semantics preserving.
- A consequence of this issue is the loss of the sub-formula property.
- A Herbrand sequent cannot be extracted and the relationship between propositional logic and inductive first order is left obfuscated

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$$E_c \equiv \forall n \left(\left(P\left(f_I(n,c)\right) \to P\left(g(n,c)\right) \right) \to \left(P(c) \to P\left(g(n,c)\right) \right) \right)$$

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- Inductive Lemma:

$$\forall x \Big(P(x) \rightarrow P\big(f(x)\big) \Big) \vdash \forall n \forall x \Big(P(x) \rightarrow P\big(f_I(n,x)\big) \Big)$$

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$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \vdash \forall x\Big(P(x) \to P\big(f_I(0,x)\big)\Big)\end{array} \qquad \begin{array}{c}\vdots\\ \Gamma, \forall x\Big(P(x) \to P\big(f_I(\alpha,x)\big)\Big) \vdash \forall x\Big(P(x) \to P\big(f_I(s(\alpha),x)\big)\Big)\end{array}}{\forall x\Big(P(x) \to P\big(f(x)\big)\Big) \vdash \forall x\Big(P(x) \to P\big(f_I(\gamma,x)\big)\Big)}\, IND}{\color{red}{\forall x\Big(P(x) \to P\big(f(x)\big)\Big) \vdash C}}\, \forall_r$$

$$\dfrac{\color{red}{\forall x\Big(P(x) \to P\big(f(x)\big)\Big) \vdash C} \qquad \begin{array}{c}\vdots\\ C \vdash E_c\end{array}}{\forall x\Big(P(x) \to P\big(f(x)\big)\Big) \vdash E_c}\, cut$$

$C \equiv \forall n \forall x\Big(P(x) \to P\big(f_I(n,x)\big)\Big)$

- By reductive cut elimination we can eliminate the $\forall_r$.
- However, we get stuck at the induction due to the change in eigenvariables.

Avoiding Induction

- Instead of n being replaced by an eigenvariable we can consider a term t over the signature $\{0, 1, +, *\}$ s.t. $\vdash n = t$ in Peano arithmetic.
- Removing the induction and replacing it with a sequence of cuts allows reductive cut elimination to occur for any fixed term t.

$$
\dfrac{\dfrac{\begin{array}{c}\forall x\Big(P(f(x)) \rightarrow P\big(f(f_I(t,x))\big)\Big),\\ \forall x\Big(P(x) \rightarrow P\big(f(x)\big)\Big) \vdash \\ \forall x\Big(P(x) \rightarrow P\big(f_I(s(t),x)\big)\Big)\end{array} \qquad \begin{array}{c}\varphi(n) :: \forall x\Big(P(x) \rightarrow P\big(f(x)\big)\Big) \vdash \\ \forall x\Big(P(f(x)) \rightarrow P\big(f(f_I(t,x))\big)\Big)\end{array}}{\forall x\Big(P(x) \rightarrow P\big(f(x)\big)\Big), \forall x\Big(P(x) \rightarrow P\big(f(x)\big)\Big) \vdash \forall x\Big(P(x) \rightarrow P\big(f_I(s(t),x)\big)\Big)}\,cut}{\varphi(n+1) :: \forall x\Big(P(x) \rightarrow P\big(f(x)\big)\Big) \vdash \forall x\Big(P(x) \rightarrow P\big(f_I(s(t),x)\big)\Big)}\,c_I
$$

$$
\dfrac{\forall x\Big(P(x) \rightarrow P\big(f(x)\big)\Big) \vdash cut^{s(t)} \qquad cut^{s(t)} \vdash E_c^{s(t)}}{\forall x\Big(P(x) \rightarrow P\big(f(x)\big)\Big) \vdash E_c^{s(t)}}\,cut
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$$

- Though, this method constructs a proof admitting reductive cut-elimination it does not produce a finite representation of the reductive cut elimination for all possible t.

Representing the Proof Sequence

- Let us consider a simple sequence of terms indexing a proof:

$$S \equiv 0, 0+1, (0+1)+1, ((0+1)+1)+1, \cdots$$

- If the proof for the $(n+1)^{\text{th}}$ term of the sequence S assumes the that there is a proof associated with the n^{th} term we have a structure similar to induction argument.

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— This can be written more formally in the LKS-calculus using proof links .

LKS-calculus Proof Structure

$$\frac{\Sigma \vdash P(0), \Delta \qquad \Pi, P(\alpha) \vdash P(s(\alpha)), \Gamma}{\Pi, \Sigma \vdash P(\beta), \Delta, \Gamma}$$

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\dfrac{\dfrac{\dfrac{\Psi}{\Pi' \vdash \Gamma'} \qquad \dfrac{\varphi(0)}{\Sigma \vdash P(0), \Delta}}{\vdots \qquad\qquad \vdots}}{\Pi'', \Sigma \vdash P(1), \Delta, \Gamma''}
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$$
\dfrac{\Sigma \vdash P(0), \Delta \qquad \Pi, \cancel{P(\alpha)} \vdash P(s(\alpha)), \Gamma}{\Pi, \Sigma \vdash \cancel{P(\beta)}, \Delta, \Gamma} \qquad \Rightarrow
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\qquad
\begin{array}{c} \\ P(s(\alpha)) \end{array}
$$

$$\Rightarrow$$

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$$\Downarrow$$

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$$\Downarrow \quad \alpha - 2 \ \textit{times}$$

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- As we increase the value of α, the length of the proof increases.
- We get a cyclic proof with internal structure we refer to as configurations .
- Configurations refer to the structure of the end sequents at proof links.

Primitive Recursive Proof Representation

- More formally, what we have outlined on the previous slide is a proof schema pair φ, i.e.

  $\varphi \equiv (\pi(0), \nu(k+1))$.

- Essentially a proof written primitive recursively using only one primitive recursion operator.
- Proof schemata are a sequence of ordered proof schema pairs.
- Let us consider a proof schema $\langle \varphi, \psi \rangle$.

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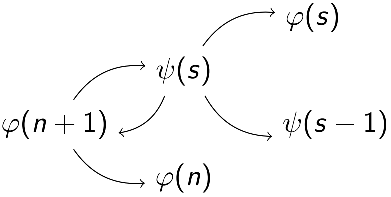

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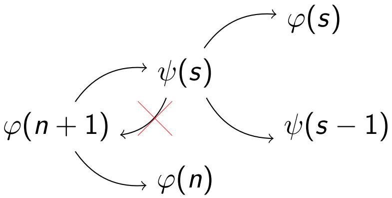

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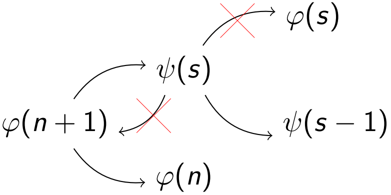

- No cyclic calls allowed. Order must be preserved $\varphi < \psi$.

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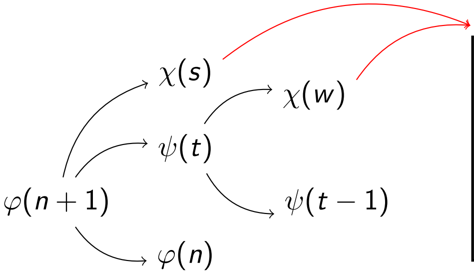

The two instances of χ were called by different proofs and can have different configurations concerning the ancestors of the cut formulae. The configurations must be denoted during the construction of the clause set.

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P(f(x_1)) \vdash P(f^2(x_1)) \\
\vdots \\
P(f^n(x_n)) \vdash P(f^{n+1}(x_n)) \\
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$\vdots$

$P(f^n(x_n)) \vdash P(f^{n+1}(x_n))$

$P(f^{n+1}(x_{n+1})) \vdash$

$$\frac{\vdash P(c) \qquad P(x_0) \vdash P(f(x_0))}{\vdash P(f(c))}$$

Schematic CERES: Clause set Extraction & Refutation

- As with the CERES method, a clause set must be extracted, though, as expected, it is recursively defined.
- Essentially, proof schema have an infinite sequence of finite clause sets whose infinite sequence of refutations need to be described finitely.

Example:

$$\vdash P(c)$$
$$P(x_0) \vdash P(x_0)$$
$$P(x_0) \vdash P(f(x_0))$$
$$P(f(x_1)) \vdash P(f^2(x_1))$$
$$\vdots$$
$$P(f^n(x_n)) \vdash P(f^{n+1}(x_n))$$
$$P(f^{n+1}(x_{n+1})) \vdash$$

$$\frac{\vdash P(c) \qquad P(x_0) \vdash P(f(x_0))}{\vdash P(f(c))}$$

$$\vdots$$

$$\frac{P(f^{n+1}(x_{n+1})) \vdash \qquad \dfrac{\vdash P(f^n(c)) \qquad P(f^n(x_n)) \vdash P(f^{n+1}(x_n))}{\vdash P(f^{n+1}(c))}}{\vdash}$$

Schematic Resolution Calculus

- The resolution refutation on the previous slide can be written in the schematic resolution calculus as following:

$$\rho(n+1) \Rightarrow r(P(f^{n+2}(x_{n+2}))\vdash;r(\chi(n+1);P(f^{n+1}(x_{n+1}))\vdash P(f^{n+2}(x_{n+1}));P(f^{n}(c)));P(f^{n+1}(c)))$$

$$\rho(0) \Rightarrow \chi(0)$$

$$\chi(n+1) \Rightarrow r(\chi(n);P(f^{n}(x_n))\vdash P(f^{n+1}(x_n));P(f^{n}(c)))$$

$$\chi(0) \Rightarrow r(P(c);P(x_0)\vdash P(f(x_0));P(c))$$

- Unification is not described in the resolution terms nor in the resolution schema, but rather as a substitution schema. In our case it would be $[x \setminus \lambda k.c]$, where $i \in [0, n]$.

End Result of Schematic Cut-elimination

- Unlike the CERES method for FOL, the schematic CERES method does not produce an ACNF.
  - The projections are not constructed.
- The “ACNF” is just a pairing of a resolution refutation schema with a substitution schema.
- Rather then returning a cut-free **LKS**-proof, the final result is a Herbrand system.
- To the best of our knowledge, other methods dealing with cut elimination in the presence of Induction cannot produce a Herbrand system being that they do not preserve the sub-formula property.

Weakening the Pigeonhole Principle

- Proof analysis of the full pigeonhole principle (NiA-schema) was attempted but, refuting the clause set was problematic.
- The NiA-schema considers every permutation of pigeon assignment. This translates to the clause set.

Let $f : \mathbb{N} \rightarrow \mathbb{N}_n$*, where* $n \in \mathbb{N}$*, be total, then there exists* $i, j \in \mathbb{N}$ *such that* $i < j$ *and* $f(i) = f(j)$*.*

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- Our formal proof of this statement uses the following sequence of lemmata.

$$\exists p \exists q \left(p, q \in \mathbb{N} \wedge p < q \wedge f(p) = f(q)\right) \vee$$
$$\forall x \exists y \left(x, y \in \mathbb{N} \wedge x \leq y \wedge f(y) \in \mathbb{N}_{n-1}\right)$$

- Problematically, the clause set extract from our formal proof using this lemma was too complex to analyse using our current techniques.

Adding order to PHP(ECA-schema)

- What made analysis of the NiA-schema hard was the complexity of the cut formula.
  - Essentially, we have a sequence of Π_2 cuts.
- Our solution to this problem is to modify the cut formulae to Σ_2 cuts and see how much of the NiA-schema can still be proven in the LKS-calculus.

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- Our solution to this problem is to modify the cut formulae to Σ_2 cuts and see how much of the NiA-schema can still be proven in the LKS-calculus.
- We ended up with the following statement:

Given a total monotonically decreasing function $f : \mathbb{N} \rightarrow \{0, \cdots, n\}$, for $n \in \mathbb{N}$, there exists an $x \in \mathbb{N}$ such that for all $y \in \mathbb{N}$, where $x \leq y$, it is the case that $f(x) = f(y)$.

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- Notice that the end sequent contains a Σ_2 formula and needs to be skolemized.
- The cut formula is as follows:

$$\exists x \forall y \left(\left((x \leq y) \rightarrow n + 1 = f(y) \right) \vee f(y) < n + 1 \right)$$

Schematic Clause Set Terms

$$cl^{\varphi,\Omega(0)}(0) \oplus \left(\{\vdash f(\alpha) < 0\} \otimes \left(\{\vdash 0 = f(\alpha)\} \otimes \{0 \leq \beta \vdash\} \right) \right) = CL_{ECA}(0)$$

$$\{f(\alpha) < 0 \vdash\} \oplus \left(\{f(g(\alpha)) < 0 \vdash\} \oplus \left(\{\vdash \alpha \leq \alpha\} \oplus \left(\{\vdash \alpha \leq g(\alpha)\} \oplus \{0 = f(\alpha), 0 = f(g(\alpha)) \vdash\} \right) \right) \right) = cl^{\varphi,\Omega(0)}(0)$$

Schematic Clause Set Terms

$$
\begin{tikzcd}[column sep=small]
 & \{0 \leq \beta \vdash\} \arrow[d] \\
\{\vdash n+1 = f(\alpha)\} \arrow[r] & \otimes \arrow[d] \\
\{\vdash f(\alpha) < n+1\} \arrow[r] & \otimes \arrow[d] \\
cl^{\varphi,\Omega(n+1)}(n+1) \arrow[r] & \oplus \arrow[d, phantom, "\shortparallel"] \\
 & {\color{red} CL_{ECA}(n+1)}
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & \{f(\alpha) < n+1, \alpha \leq \beta \vdash n = f(\beta), f(\beta) < n\} \arrow[d] \\
\{f(\beta) < n+1 \vdash f(\beta) < n+1\} \arrow[r] & \oplus \arrow[d] \\
\{\alpha \leq \beta \vdash \alpha \leq \beta\} \arrow[r] & \oplus \arrow[d] \\
\{n+1 = f(\beta) \vdash n+1 = f(\beta)\} \arrow[r] & \oplus \arrow[d] \\
\{\alpha \leq g(\alpha)\} \arrow[r] & \oplus \arrow[d] \\
\{\vdash \alpha \leq \alpha\} \arrow[r] & \oplus \arrow[d] \\
\{n+1 = f(\alpha), n+1 = f(g(\alpha)) \vdash\} \arrow[r] & \oplus \arrow[d] \\
cl^{\varphi,\Omega(n)}(n) \arrow[r] & \oplus \arrow[d, phantom, "\shortparallel"] \\
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\end{tikzcd}
$$

— Note that $g(\cdot)$ is the skolem symbol. Also,

$$\Omega(n) \equiv \exists x \forall y \left(((x \leq y) \rightarrow n+1 = f(y)) \vee f(y) < n+1\right).$$

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\{\vdash n+1 = f(\alpha)\} \arrow[r] & \otimes \arrow[d] \\
\{\vdash f(\alpha) < n+1\} \arrow[r] & \otimes \arrow[d] \\
cl^{\varphi,\Omega(n+1)}(n+1) \arrow[r] & \oplus \arrow[d, equal] \\
 & CL_{ECA}(n+1)
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \{f(\alpha) < n+1, \alpha \leq \beta \vdash n = f(\beta), f(\beta) < n\} \arrow[d] \\
\{f(\beta) < n+1 \vdash f(\beta) < n+1\} \arrow[r] & \oplus \arrow[d] \\
\{\alpha \leq \beta \vdash \alpha \leq \beta\} \arrow[r] & \oplus \arrow[d] \\
\{n+1 = f(\beta) \vdash n+1 = f(\beta)\} \arrow[r] & \oplus \arrow[d] \\
\{\alpha \leq g(\alpha)\} \arrow[r] & \oplus \arrow[d] \\
\{\vdash \alpha \leq \alpha\} \arrow[r] & \oplus \arrow[d] \\
\{n+1 = f(\alpha), n+1 = f(g(\alpha)) \vdash\} \arrow[r] & \oplus \arrow[d] \\
\mathit{Recursion} \Rightarrow cl^{\varphi,\Omega(n)}(n) \arrow[r] & \oplus \arrow[d, equal] \\
 & cl^{\varphi,\Omega(n+1)}(n+1)
\end{tikzcd}
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schematic clause

- After extracting the schematic clause set terms, we perform normalization and tautology elimination.
- This results in the following schematic clause set:

$$C1(x,k) \equiv \ \vdash x(k) \leq x(k)$$

$$C2(x,k) \equiv \ \vdash x(k) \leq g(x(k))$$

$$C3(x,i,k) \equiv \quad i = f(x(k)), i = f(g(x(k))) \vdash$$

$$\begin{aligned} C4(x,y,i,k) \equiv \quad & y(k) \leq x(k), f(y(k)) < i+1 \vdash \\ & f(x(k)) < i, i = f(x(k)) \end{aligned}$$

$$\begin{aligned} C4'(x,y,i,k) \equiv \quad & y(k) \leq x(k+1), f(y(k)) < i+1 \vdash \\ & f(x(k+1)) < i, i = f(x(k+1)) \end{aligned}$$

$$C5(x,k) \equiv \quad f(x(k)) < 0 \vdash$$

$$C6(x,k) \equiv \quad f(g(x(k))) < 0 \vdash$$

$$C7(x,k) \equiv \quad 0 \leq x(k) \vdash f(x(k)) < n, f(x(k)) = n$$

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- Note that x is what is referred to as a **schematic variable**, a variable of type $\omega \rightarrow \iota$.
- This is similar to a second order variable, however, instantiation of k produces a first order variable.

Refutation of the Clause set

— Once we extract the clause set we need a refutation in order to construct the ACNF.

— A problems with Schematic CERES is that both finding the substitution schema and a refutation are undecidable problems.

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- Once we extract the clause set we need a refutation in order to construct the ACNF.
- A problems with Schematic CERES is that both finding the substitution schema and a refutation are undecidable problems.
- However, instantiating the parameter n gives us a first order clause set.
- For example, the following is the clause set for $n = 3$.

$$\vdash x(k) \leq x(k)$$
$$\vdash x(k) \leq g(x(k))$$
$$0 = f(x(k)), 0 = f(g(x(k))) \vdash$$
$$1 = f(x(k)), 1 = f(g(x(k))) \vdash$$
$$2 = f(x(k)), 2 = f(g(x(k))) \vdash$$
$$3 = f(x(k)), 3 = f(g(x(k))) \vdash$$
$$y(k) \leq x(k), f(y(k)) < 1 \vdash$$
$$f(x(k)) < 0, 0 = f(x(k))$$
$$y(k) \leq x(k), f(y(k)) < 2 \vdash$$
$$f(x(k)) < 1, 1 = f(x(k))$$
$$y(k) \leq x(k), f(y(k)) < 3 \vdash$$
$$f(x(k)) < 2, 2 = f(x(k))$$

$$y(k) \leq x(k+1), f(y(k)) < 1 \vdash$$
$$f(x(k+1)) < 0, 0 = f(x(k+1))$$
$$y(k) \leq x(k+1), f(y(k)) < 2 \vdash$$
$$f(x(k+1)) < 1, 1 = f(x(k+1))$$
$$y(k) \leq x(k+1), f(y(k)) < 3 \vdash$$
$$f(x(k+1)) < 2, 2 = f(x(k+1))$$
$$f(x(k)) < 0 \vdash$$
$$f(g(x(k))) < 0 \vdash$$
$$0 \leq x(k) \vdash f(x(k)) < 3, f(x(k)) = 3$$

Refutation of the Clause set

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310[0:MRR:309.0,306.1]	$\vdash\ f(\alpha) < 3$
311[0:MRR:10.1,310.0]	$\alpha \leq \beta \vdash\ 2 = f(\beta)\quad f(\beta) < 2$
312[0:Res:2.0,311.0]	$\vdash\ 2 = f(\alpha)\quad f(\alpha) < 2$
314[0:Res:312.0,6.1]	$2 = f(\alpha)\ \vdash\ f(g(\beta)) < 2$
315[0:Res:314.1,11.1]	$2 = f(\alpha)\quad g(\alpha) \leq \beta\ \vdash\ 1 = f(\beta)\quad f(\beta) < 1$
316[0:Res:312.0,315.0]	$g(\alpha) \leq \beta\ \vdash\ f(\alpha) < 2\ \ 1 = f(\beta)\quad f(\beta) < 1$
317[0:Res:2.0,316.0]	$\vdash\ f(\alpha) < 2\ \ 1 = f(g(\alpha))\quad f(g(\alpha)) < 1$
318[0:Res:3.0,316.0]	$\vdash\ f(\alpha) < 2\ \ 1 = f(g(g(\alpha)))\quad f(g(g(\alpha))) < 1$
321[0:Res:318.1,7.1]	$1 = f(g(\alpha))\ \vdash\ f(\alpha) < 2\ \ f(g(g(\alpha))) < 1$
322[0:Res:321.2,14.1]	$1 = f(g(\alpha))\quad g(g(\alpha)) \leq \beta\ \vdash\ f(\alpha) < 2\ \ 0 = f(\beta)$
325[0:Res:317.1,322.0]	$g(g(\alpha)) \leq \beta\ \vdash\ f(\alpha) < 2\ \ f(g(\alpha)) < 1\ \ f(\alpha) < 2\ \ 0 = f(\beta)$
327[0:Obv:325.1]	$g(g(\alpha)) \leq \beta\ \vdash\ f(g(\alpha)) < 1\ \ f(\alpha) < 2\ \ 0 = f(\beta)$
328[0:Res:2.0,327.0]	$\vdash\ f(g(\alpha)) < 1\ \ f(\alpha) < 2\ \ 0 = f(g(g(\alpha)))$
329[0:Res:3.0,327.0]	$\vdash\ f(g(\alpha)) < 1\ \ f(\alpha) < 2\ \ 0 = f(g(g(g(\alpha))))$
335[0:Res:329.2,8.1]	$0 = f(g(g(\alpha)))\ \vdash\ f(g(\alpha)) < 1\ \ f(\alpha) < 2$
336[0:MRR:335.0,328.2]	$\vdash\ f(g(\alpha)) < 1\ \ f(\alpha) < 2$
337[0:Res:336.0,14.1]	$g(\alpha) \leq \beta\ \vdash\ f(\alpha) < 2\ \ 0 = f(\beta)$
338[0:Res:2.0,337.0]	$\vdash\ f(\alpha) < 2\ \ 0 = f(g(\alpha))$
339[0:Res:3.0,337.0]	$\vdash\ f(\alpha) < 2\ \ 0 = f(g(g(\alpha)))$
344[0:Res:339.1,8.1]	$0 = f(g(\alpha))\ \vdash\ f(\alpha) < 2$
345[0:MRR:344.0,338.1]	$\vdash\ f(\alpha) < 2$

Spass output

n=5

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Notice the invariant used by Spass.

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A resolution proof schema $\mathcal{R}(n)$ is a structure $(\varrho_1, \cdots, \varrho_\alpha)$ together with a set of rewrite rules $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_\alpha$, where the $\mathcal{R}_i$ (*for* $1 \leq i \leq \alpha$) are pairs of rewrite rules

$$\varrho_i(0, \overline{w}, \overline{u}, \overline{X}) \rightarrow \eta_i \text{ and } \varrho_i(k+1, \overline{w}, \overline{u}, \overline{X}) \rightarrow \eta_i'$$

where, $\overline{w}$, $\overline{u}$, and $\overline{X}$ are vectors of ω, schematic, and clause variables respectively, η_i is a resolution term over terms of the form $\varrho_j(a_j, \overline{m}, \overline{t}, \overline{C})$ for $i < j \leq \alpha$, and η_i' is a resolution term over terms of the form $\varrho_j(a_j, \overline{m}, \overline{t}, \overline{C})$ and $\varrho_i(k, \overline{m}, \overline{t}, \overline{C})$ for $i < j \leq \alpha$; by a_j, we denote a term of the ω sort.

Refutation of the Clause set

— These scratch pad variables are used in our refutation as follows:

$$\varrho_4(n+1,k,x,y,Y)\Rightarrow$$

$$r(\varrho_5(n,k+1,x,y,Y\circ f(x(k+1))<n+1\vdash);r(C2(x,k);C7(x,k+1);f(x(k+1))<n+1)$$

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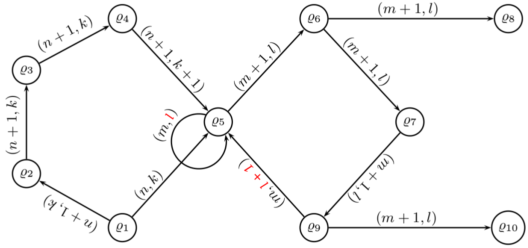

From Refutation to Herbrand System

- The schematic refutation is represented by the following term:

$$\rho_{ECA}(\gamma)=\varrho_1(n,k,x,y,Y)\theta\nu\vartheta[n\leftarrow\gamma]=\varrho_1(\gamma,\overline{\mu},\lambda_k.(i_s(k)),\lambda_k.(h(k)),\vdash)$$

where $\gamma \in \mathbb{N}$, ϑ is the substitution schema, the clause substitution is $\theta = \{Y \leftarrow \vdash\}$, the ω-variable substitution is $\nu = \{k \leftarrow \overline{\mu}\}$ for any $\overline{\mu} \in \mathbb{N}$, and $i_s(0) = 0,\ i_s(s(k)) = s(i_s(k))$.

From Refutation to Herbrand System

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- The pair $(\rho_{ECA}(\gamma), \vartheta)$ is the ACNF of the proof schema
- Our next step is to derive a Herbrand system based on the following sps-schema derived from the end sequent:

$$S(n) \equiv \left(\forall x \bigvee_{i=0}^{n} i = f(x), \forall y\, (0 \leq y \rightarrow f(y) \leq f(0))\right) \vdash \exists x\, (x \leq g(x) \rightarrow f(x) = f(g(x)))$$

Herbrand System Extraction

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- Only clauses C2, C3 and C7 have corresponding end sequent formulae.

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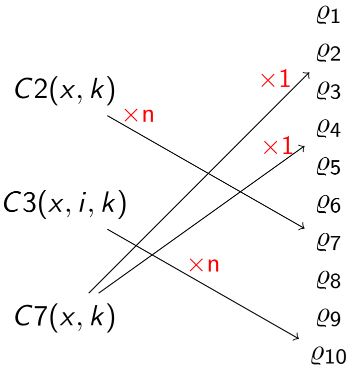

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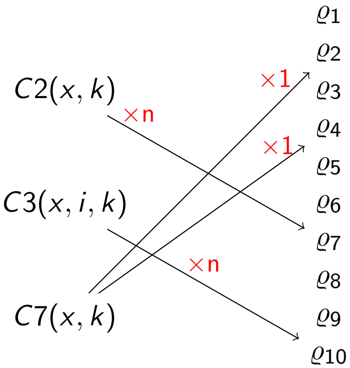

- This results in the following substitution rewrite system.

$$\mathcal{R} = \left\{ \begin{array}{l} w_1^{\varphi}(k+1) \Rightarrow [[0];[g(0)]] \\ w_1^{\varphi}(0) \Rightarrow [[0];[g(0)]] \\ \\ w_2^{\varphi}(k+1) \Rightarrow [[0];[g(0)]] \\ w_2^{\varphi}(0) \Rightarrow [[0];[g(0)]] \\ \\ w_1^{\psi}(k+1) \Rightarrow [[h(k+1)]; w_1^{\psi}(k)] \\ w_1^{\psi}(0) \Rightarrow [0] \end{array} \right.$$

Herbrand System

— Putting everything together we get the following Herbrand sequent.

$$\bigvee_{i=0}^{n} i = f(0), \bigvee_{i=0}^{n} i = f(g(0)), (0 \leq 0 \rightarrow f(0) \leq f(0)),$$

$$(0 \leq g(0) \rightarrow f(g(0)) \leq f(0)) \vdash \bigvee_{i=0}^{n} (h(i) \leq g(h(i)) \rightarrow f(h(i)) = f(g(h(i))))$$

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$$f(\alpha) < n+1, \alpha \leq \beta \vdash n = f(\beta), f(\beta) < n$$

- We started from a specific axiom set that enforces the constraints placed on $f()$ and $g()$

Axioms for our Theory

— The axiom set is as follows:

$$
\begin{aligned}
&A1(i):\ \bigvee_{j=0}^{i-1} j = f(\alpha), i = f(g(\alpha)), f(g(\alpha)) < f(\alpha) \vdash \\
&A2(i):\ i = f(\alpha), \bigvee_{j=0}^{i-1} j = f(g(\alpha)), \alpha \leq g(\alpha) \vdash \\
&A3(i):\ i = f(\alpha), i = f(g(\alpha)) \vdash f(\alpha) = f(g(\alpha)) \\
&A4(i): f(g(\alpha)) = f(\alpha) \vdash f(\alpha) = f(g(\alpha)) \\
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- Interestingly enough a minimal Herbrand sequent can have the induction completely removed.

$$
\bigvee_{i=0}^{n} i = f(0), \bigvee_{i=0}^{n} i = f(g(0)), (0 \leq g(0) \rightarrow f(g(0)) \leq f(0)),
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(0 \leq 0 \rightarrow f(0) \leq f(0)) \vdash 0 \leq g(0) \rightarrow f(0) = f(g(0)).
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Conclusion

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  1) Our goal was to take the proof of a simple mathematical statement, a proof containing cuts, and analyse it using the method.
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  3) We weakened the cuts of the NIA-schema and derived a new proof, the ECA-schema, which we can handle.
- We then carried out the proof analysis of the ECA-schema and algorithmically derived a Herbrand system.

Open Problems and Future Work

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- Is there a universal resolution calculus for all primitive recursive proof schemata?
- If not, what class of proof schema is the current method complete for?

Thank you for your time.