A Gentle Introduction to Mathematical Fuzzy Logic

3. Predicate Łukasiewicz and Gödel–Dummett logic

Petr Cintula[1] and Carles Noguera[2]

[1] Institute of Computer Science,
Czech Academy of Sciences, Prague, Czech Republic

[2] Institute of Information Theory and Automation,
Czech Academy of Sciences, Prague, Czech Republic

www.cs.cas.cz/cintula/MFL

Predicate language

Predicate language: $\mathcal{P} = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$: predicate and function symbols with arity

Object variables: denumerable set OV

$\mathcal{P}$-terms:

- if $v \in \mathrm{OV}$, then v is a $\mathcal{P}$-term
- if $f \in \mathbf{F}$, $\mathbf{ar}(\mathbf{F}) = n$, and $t_1, \dots, t_n$ are $\mathcal{P}$-terms, then so is $f(t_1, \dots, t_n)$

Formulas

Atomic $\mathcal{P}$-formulas: propositional constant $\overline{0}$ and expressions of the form $R(t_1, \ldots, t_n)$, where $R \in \mathbf{P}$, $\mathbf{ar}(\mathbf{R}) = n$, and $t_1, \ldots, t_n$ are $\mathcal{P}$-terms.

$\mathcal{P}$-formulas:

- the atomic $\mathcal{P}$-formulas are $\mathcal{P}$-formulas
- if α and β are $\mathcal{P}$-formulas, then so are $\alpha \wedge \beta$, $\alpha \vee \beta$, and $\alpha \to \beta$
- if $x \in \mathrm{OV}$ and α is a $\mathcal{P}$-formula, then so are $(\forall x)\alpha$ and $(\exists x)\alpha$

Basic syntactical notions

$\mathcal{P}$-theory: a set of $\mathcal{P}$-formulas

A closed $\mathcal{P}$-term is a $\mathcal{P}$-term without variables.

An occurrence of a variable x in a formula φ is bound if it is in the scope of some quantifier over x; otherwise it is called a free occurrence.

A variable is free in a formula φ if it has a free occurrence in φ.

A $\mathcal{P}$-sentence is a $\mathcal{P}$-formula with no free variables.

A term t is substitutable for the object variable x in a formula $\varphi(x, \vec{z})$ if no occurrence of any variable occurring in t is bound in $\varphi(t, \vec{z})$ unless it was already bound in $\varphi(x, \vec{z})$.

Axiomatic system

A Hilbert-style proof system for $\mathrm{CL}\forall$ can be obtained as:

(P)	axioms of CL substituting propositional variables by $\mathcal{P}$-formulas	
$(\forall 1)$	$(\forall x)\varphi(x, \vec{z}) \to \varphi(t, \vec{z})$	t substitutable for x in φ
$(\forall 2)$	$(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$	x not free in χ
(MP)	*modus ponens* for $\mathcal{P}$-formulas	
(gen)	from φ infer $(\forall x)\varphi$.	

Let us denote as $\vdash_{\mathrm{CL}\forall}$ the provability relation.

Semantics

Classical $\mathcal{P}$-structure: a tuple $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in \mathbf{P}}, \langle f_{\mathbf{M}} \rangle_{f \in \mathbf{F}} \rangle$ where

- $M \neq \emptyset$
- $P_{\mathbf{M}} \subseteq M^n$, for each n-ary $P \in \mathbf{P}$
- $f_{\mathbf{M}} \colon M^n \to M$ for each n-ary $f \in \mathbf{F}$.

$\mathbf{M}$-evaluation v: a mapping $\mathrm{v} \colon \mathrm{OV} \to M$

For $x \in \mathrm{OV}, m \in M$, and $\mathbf{M}$-evaluation v, we define $\mathrm{v}[x{:}m]$ as

$$\mathrm{v}[x{:}m](y) = \begin{cases} m & \text{if } y = x \\ \mathrm{v}(y) & \text{otherwise} \end{cases}$$

Tarski truth definition

Interpretation of $\mathcal{P}$-terms

$$\begin{array}{rcll} \|x\|^{\mathbf{M}}_{\mathrm{v}} & = & \mathrm{v}(x) & \text{for } x \in \mathrm{OV} \\ \|f(t_1,\dots,t_n)\|^{\mathbf{M}}_{\mathrm{v}} & = & f_{\mathbf{M}}(\|t_1\|^{\mathbf{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathbf{M}}_{\mathrm{v}}) & \text{for } n\text{-ary } f \in \mathbf{F} \end{array}$$

Truth-values of $\mathcal{P}$-formulas

$$\begin{array}{rll} \|P(t_1,\dots,t_n)\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \text{iff} & \langle \|t_1\|^{\mathbf{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathbf{M}}_{\mathrm{v}} \rangle \in P_{\mathbf{M}} \qquad \text{for } P \in \mathbf{P} \\ \|\overline{0}\|^{\mathbf{M}}_{\mathrm{v}} = 0 & & \\ \|\alpha \wedge \beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \text{iff} & \|\alpha\|^{\mathbf{M}}_{\mathrm{v}} = 1 \text{ and } \|\beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 \\ \|\alpha \vee \beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \text{iff} & \|\alpha\|^{\mathbf{M}}_{\mathrm{v}} = 1 \text{ or } \|\beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 \\ \|\alpha \to \beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \text{iff} & \|\alpha\|^{\mathbf{M}}_{\mathrm{v}} = 0 \text{ or } \|\beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 \\ \|(\forall x)\varphi\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \text{iff} & \text{for each } m \in M \text{ we have } \|\varphi\|^{\mathbf{M}}_{\mathrm{v}[x:m]} = 1 \\ \|(\exists x)\varphi\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \text{iff} & \text{there is } m \in M \text{ such that } \|\varphi\|^{\mathbf{M}}_{\mathrm{v}[x:m]} = 1 \end{array}$$

Model and semantical consequence

We write $\mathbf{M} \models \varphi$ if $\|\varphi\|^{\mathbf{M}}_{\mathrm{v}} = 1$ for each $\mathbf{M}$-evaluation v.

Model: We say that a $\mathcal{P}$-structure $\mathbf{M}$ is a $\mathcal{P}$-model of a $\mathcal{P}$-theory T, $\mathbf{M} \models T$ in symbols, if $\mathbf{M} \models \varphi$ for each $\varphi \in T$.

Consequence: A $\mathcal{P}$-formula φ is a semantical consequence of a $\mathcal{P}$-theory T, $T \models_{\mathrm{CL}\forall} \varphi$, if each $\mathcal{P}$-model of T is also a model of φ.

The completeness theorem

Problem of completeness of $\mathrm{CL}\forall$: formulated by Hilbert and Ackermann (1928) and solved by Gödel (1929):

Theorem 3.1 (Gödel's completeness theorem)

For every predicate language $\mathcal{P}$ and for every set $T \cup \{\varphi\}$ of $\mathcal{P}$-formulas :

$$T \vdash_{\mathrm{CL}\forall} \varphi \qquad \textit{iff} \qquad T \models_{\mathrm{CL}\forall} \varphi$$

Some history

1947 Henkin: alternative proof of Gödel's completeness theorem

1961 Mostowski: interpretation of existential (resp. universal) quantifiers as suprema (resp. infima)

1963 Rasiowa, Sikorski: first-order intuitionistic logic

1963 Hay: infinitary standard Łukasiewicz first-order logic

1969 Horn: first-order Gödel–Dummett logic

1974 Rasiowa: first-order implicative logics

1990 Novák: first-order Pavelka logics

1992 Takeuti, Titani: first-order Gödel–Dummett logic with additional connectives

1998 Hájek: first-order axiomatic extensions of HL

2005 Cintula, Hájek: first-order core fuzzy logics

2011 Cintula, Noguera: first-order semilinear logics

Basic syntax is the again the same

Let L be G or Ł and $\mathbb{L}$ be $\mathbb{G}$ or $\mathbb{MV}$ correspondingly

Predicate language: $\mathcal{P} = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$

Object variables: denumerable set OV

$\mathcal{P}$-terms, (atomic) $\mathcal{P}$-formulas, $\mathcal{P}$-theories: as in $\mathrm{CL}\forall$

free/bounded variables, substitutable terms, sentences: as in $\mathrm{CL}\forall$

Recall classical semantics

Classical $\mathcal{P}$-structure: a tuple $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in \mathbf{P}}, \langle f_{\mathbf{M}} \rangle_{f \in \mathbf{F}} \rangle$ where

- $M \neq \emptyset$
- $P_{\mathbf{M}} \subseteq M^n$, for each n-ary $P \in \mathbf{P}$
- $f_{\mathbf{M}} \colon M^n \to M$ for each n-ary $f \in \mathbf{F}$.

$\mathbf{M}$-evaluation v: a mapping $\mathrm{v} \colon \mathrm{OV} \to M$

For $x \in \mathrm{OV}, m \in M$, and $\mathbf{M}$-evaluation v, we define $\mathrm{v}[x{:}m]$ as

$$\mathrm{v}[x{:}m](y) = \begin{cases} m & \text{if } y = x \\ \mathrm{v}(y) & \text{otherwise} \end{cases}$$

Reformulating classical semantics

Classical $\mathcal{P}$-structure: a tuple $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in \mathbf{P}}, \langle f_{\mathbf{M}} \rangle_{f \in \mathbf{F}} \rangle$ where

- $M \neq \emptyset$
- $P_{\mathbf{M}} \colon M^n \to \{0, 1\}$, for each n-ary $P \in \mathbf{P}$
- $f_{\mathbf{M}} \colon M^n \to M$ for each n-ary $f \in \mathbf{F}$.

$\mathbf{M}$-evaluation v: a mapping $\mathrm{v} \colon \mathrm{OV} \to M$

For $x \in \mathrm{OV}, m \in M$, and $\mathbf{M}$-evaluation v, we define $\mathrm{v}[x{:}m]$ as

$$\mathrm{v}[x{:}m](y) = \begin{cases} m & \text{if } y = x \\ \mathrm{v}(y) & \text{otherwise} \end{cases}$$

And now the ‘fuzzy’ semantics for logic L . . .

$\boldsymbol{A}$-$\mathcal{P}$-structure ($A \in \mathbb{L}$): a tuple $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in \mathbf{P}}, \langle f_{\mathbf{M}} \rangle_{f \in \mathbf{F}} \rangle$ where

- $M \neq \emptyset$
- $P_{\mathbf{M}} \colon M^n \to A$, for each n-ary $P \in \mathbf{P}$
- $f_{\mathbf{M}} \colon M^n \to M$ for each n-ary $f \in \mathbf{F}$.

$\mathbf{M}$-evaluation v: a mapping $\mathrm{v} \colon \mathrm{OV} \to M$

For $x \in \mathrm{OV}, m \in M$, and $\mathbf{M}$-evaluation v, we define $\mathrm{v}[x{:}m]$ as

$$\mathrm{v}[x{:}m](y) = \begin{cases} m & \text{if } y = x \\ \mathrm{v}(y) & \text{otherwise} \end{cases}$$

Recall classical Tarski truth definition

Interpretation of $\mathcal{P}$-terms

$$\begin{array}{rcll} \|x\|^{\mathbf{M}}_{\mathrm{v}} & = & \mathrm{v}(x) & \text{for } x \in \mathrm{OV} \\ \|f(t_1,\dots,t_n)\|^{\mathbf{M}}_{\mathrm{v}} & = & f_{\mathbf{M}}(\|t_1\|^{\mathbf{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathbf{M}}_{\mathrm{v}}) & \text{for } n\text{-ary } f \in \mathbf{F} \end{array}$$

Truth-values of $\mathcal{P}$-formulas

$$\begin{array}{rcll} \|P(t_1,\dots,t_n)\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \text{iff} & \langle \|t_1\|^{\mathbf{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathbf{M}}_{\mathrm{v}} \rangle \in P_{\mathbf{M}} & \text{for } n\text{-ary } P \in \mathbf{P} \\ \|\overline{0}\|^{\mathbf{M}}_{\mathrm{v}} = 0 & & & \\ \|\alpha \wedge \beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \text{iff} & \|\alpha\|^{\mathbf{M}}_{\mathrm{v}} = 1 \text{ and } \|\beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \\ \|\alpha \vee \beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \text{iff} & \|\alpha\|^{\mathbf{M}}_{\mathrm{v}} = 1 \text{ or } \|\beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \\ \|\alpha \to \beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \text{iff} & \|\alpha\|^{\mathbf{M}}_{\mathrm{v}} = 0 \text{ or } \|\beta\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \\ \|(\forall x)\varphi\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \text{iff} & \text{for each } m \in M \text{ we have } \|\varphi\|^{\mathbf{M}}_{\mathrm{v}[x:m]} = 1 & \\ \|(\exists x)\varphi\|^{\mathbf{M}}_{\mathrm{v}} = 1 & \text{iff} & \text{there is } m \in M \text{ such that } \|\varphi\|^{\mathbf{M}}_{\mathrm{v}[x:m]} = 1 & \end{array}$$

Reformulating classical Tarski truth definition

Interpretation of $\mathcal{P}$-terms

$$\begin{aligned} \|x\|^{\mathbf{M}}_{\mathrm{v}} &= \mathrm{v}(x) && \text{for } x \in \mathrm{OV} \\ \|f(t_1,\dots,t_n)\|^{\mathbf{M}}_{\mathrm{v}} &= f_{\mathbf{M}}(\|t_1\|^{\mathbf{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathbf{M}}_{\mathrm{v}}) && \text{for } n\text{-ary } f \in \mathbf{F} \end{aligned}$$

Truth-values of $\mathcal{P}$-formulas

$$\begin{aligned} \|P(t_1,\dots,t_n)\|^{\mathbf{M}}_{\mathrm{v}} &= P_{\mathbf{M}}(\|t_1\|^{\mathbf{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathbf{M}}_{\mathrm{v}}) && \text{for } n\text{-ary } P \in \mathbf{P} \\ \|\overline{0}\|^{\mathbf{M}}_{\mathrm{v}} &= \overline{0}^{\mathbf{2}} \\ \|\alpha \wedge \beta\|^{\mathbf{M}}_{\mathrm{v}} &= \min_{\leq_{\mathbf{2}}}\{\|\alpha\|^{\mathbf{M}}_{\mathrm{v}}, \|\beta\|^{\mathbf{M}}_{\mathrm{v}}\} \\ \|\alpha \vee \beta\|^{\mathbf{M}}_{\mathrm{v}} &= \max_{\leq_{\mathbf{2}}}\{\|\alpha\|^{\mathbf{M}}_{\mathrm{v}}, \|\beta\|^{\mathbf{M}}_{\mathrm{v}}\} \\ \|\alpha \to \beta\|^{\mathbf{M}}_{\mathrm{v}} &= \|\alpha\|^{\mathbf{M}}_{\mathrm{v}} \to^{\mathbf{2}} \|\beta\|^{\mathbf{M}}_{\mathrm{v}} \\ \|(\forall x)\varphi\|^{\mathbf{M}}_{\mathrm{v}} &= \inf_{\leq_{\mathbf{2}}}\{\|\varphi\|^{\mathbf{M}}_{\mathrm{v}[x:m]} \mid m \in M\} \\ \|(\exists x)\varphi\|^{\mathbf{M}}_{\mathrm{v}} &= \sup_{\leq_{\mathbf{2}}}\{\|\varphi\|^{\mathbf{M}}_{\mathrm{v}[x:m]} \mid m \in M\} \end{aligned}$$

And now the Tarski truth definition for 'fuzzy' semantics

Interpretation of $\mathcal{P}$-terms

$$
\begin{aligned}
\|x\|^{\mathbf{M}}_{\mathrm{v}} &= \mathrm{v}(x) && \text{for } x \in \mathrm{OV} \\
\|f(t_1, \dots, t_n)\|^{\mathbf{M}}_{\mathrm{v}} &= f_{\mathbf{M}}(\|t_1\|^{\mathbf{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathbf{M}}_{\mathrm{v}}) && \text{for } n\text{-ary } f \in \mathbf{F}
\end{aligned}
$$

Truth-values of $\mathcal{P}$-formulas

$$
\begin{aligned}
\|P(t_1, \dots, t_n)\|^{\mathbf{M}}_{\mathrm{v}} &= P_{\mathbf{M}}(\|t_1\|^{\mathbf{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathbf{M}}_{\mathrm{v}}) && \text{for } n\text{-ary } P \in \mathbf{P} \\
\|\overline{0}\|^{\mathbf{M}}_{\mathrm{v}} &= \overline{0}^{\boldsymbol{A}} \\
\|\alpha \wedge \beta\|^{\mathbf{M}}_{\mathrm{v}} &= \min_{\leq_{\boldsymbol{A}}}\{\|\alpha\|^{\mathbf{M}}_{\mathrm{v}}, \|\beta\|^{\mathbf{M}}_{\mathrm{v}}\} \\
\|\alpha \vee \beta\|^{\mathbf{M}}_{\mathrm{v}} &= \max_{\leq_{\boldsymbol{A}}}\{\|\alpha\|^{\mathbf{M}}_{\mathrm{v}}, \|\beta\|^{\mathbf{M}}_{\mathrm{v}}\} \\
\|\alpha \to \beta\|^{\mathbf{M}}_{\mathrm{v}} &= \|\alpha\|^{\mathbf{M}}_{\mathrm{v}} \to^{\boldsymbol{A}} \|\beta\|^{\mathbf{M}}_{\mathrm{v}} \\
\|(\forall x)\varphi\|^{\mathbf{M}}_{\mathrm{v}} &= \inf_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathbf{M}}_{\mathrm{v}[x:m]} \mid m \in M\} \\
\|(\exists x)\varphi\|^{\mathbf{M}}_{\mathrm{v}} &= \sup_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathbf{M}}_{\mathrm{v}[x:m]} \mid m \in M\}
\end{aligned}
$$

Model and semantical consequence

Problem: the infimum/supremum need not exist! In such case we take its value (and values of all its superformulas) as undefined

Definition 3.2 (Model)

A tuple $\mathfrak{M} = \langle \mathbf{A}, \mathbf{M} \rangle$ is a $\mathbb{K}$-$\mathcal{P}$-model of T, $\mathfrak{M} \models T$ in symbols, if

- $\mathfrak{M}$ is $\boldsymbol{A}$-$\mathcal{P}$-structure for some $\boldsymbol{A} \in \mathbb{K} \subseteq \mathbb{L}$
- $\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}}$ is defined $\mathbf{M}$-evaluation v and each formula φ
- $\|\psi\|^{\mathfrak{M}}_{\mathrm{v}} = \overline{1}^{\boldsymbol{A}}$ for each $\mathbf{M}$-evaluation v and each $\psi \in T$

Definition 3.3 (Semantical consequence)

A $\mathcal{P}$-formula φ is a semantical consequence of a $\mathcal{P}$-theory T w.r.t. the class $\mathbb{K}$ of L-algebras, $T \models_{\mathbb{K}} \varphi$ in symbols, if for each $\mathbb{K}$-$\mathcal{P}$-model $\mathfrak{M}$ of T we have $\mathfrak{M} \models \varphi$.

The semantics of chains

Proposition 3.4 (Assume that x is not free in ψ ...)

$$\varphi \models_{\mathbb{L}} (\forall x)\varphi \quad \textit{thus} \quad \varphi \models_{\mathbb{K}} (\forall x)\varphi$$

$$\varphi \vee \psi \models_{\mathbb{L}_{\mathrm{lin}}} ((\forall x)\varphi) \vee \psi \quad \textit{BUT} \quad \varphi \vee \psi \not\models_{\mathbb{G}} ((\forall x)\varphi) \vee \psi$$

Observation

Thus $\models_{\mathbb{L}} \subsetneq \models_{\mathbb{L}_{\mathrm{lin}}}$ even though in propositional logic $\models_{\mathbb{L}} = \models_{\mathbb{L}_{\mathrm{lin}}}$

Axiomatization: two first-order logics over L

Minimal predicate logic $\mathrm{L}\forall^{\mathrm{m}}$:

(P)	first-order substitutions of axioms and the rule of L	
$(\forall 1)$	$(\forall x)\varphi(x, \vec{z}) \to \varphi(t, \vec{z})$	t substitutable for x in φ
$(\exists 1)$	$\varphi(t, \vec{z}) \to (\exists x)\varphi(x, \vec{z})$	t substitutable for x in φ
$(\forall 2)$	$(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$	x not free in χ
$(\exists 2)$	$(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$	x not free in χ
(gen)	from φ infer $(\forall x)\varphi$	

Predicate logic $\mathrm{L}\forall$: an the extension of $\mathrm{L}\forall^{\mathrm{m}}$ by:

$(\forall 3)$	$(\forall x)(\varphi \vee \chi) \to ((\forall x)\varphi) \vee \chi$	x not free in χ

Theorems (for x not free in χ)

The logic $\mathrm{L}\forall^{\mathrm{m}}$ proves:

1. $\chi \leftrightarrow (\forall x)\chi$
2. $(\exists x)\chi \leftrightarrow \chi$
3. $(\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to (\forall x)\psi)$
4. $(\forall x)(\forall y)\varphi \leftrightarrow (\forall y)(\forall x)\varphi$
5. $(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to (\exists x)\psi)$
6. $(\exists x)(\exists y)\varphi \leftrightarrow (\exists y)(\exists x)\varphi$
7. $(\forall x)(\chi \to \varphi) \leftrightarrow (\chi \to (\forall x)\varphi)$
8. $(\forall x)(\varphi \to \chi) \leftrightarrow ((\exists x)\varphi \to \chi)$
9. $(\exists x)(\chi \to \varphi) \to (\chi \to (\exists x)\varphi)$
10. $(\exists x)(\varphi \to \chi) \to ((\forall x)\varphi \to \chi)$
11. $(\exists x)(\varphi \vee \psi) \leftrightarrow (\exists x)\varphi \vee (\exists x)\psi$
12. $(\exists x)(\varphi \,\&\, \chi) \leftrightarrow (\exists x)\varphi \,\&\, \chi$
13. $(\exists x)(\varphi^n) \leftrightarrow ((\exists x)\varphi)^n$

The logic $\mathrm{L}\forall$ furthermore proves:

14. $(\forall x)\varphi \vee \chi \leftrightarrow (\forall x)(\varphi \vee \chi)$
15. $(\exists x)(\varphi \wedge \chi) \leftrightarrow (\exists x)\varphi \wedge \chi$

Exercise 13

Prove these theorems.

$Ł\forall = Ł\forall^m$

Proposition 3.5

$Ł\forall = Ł\forall^m$.

Proof.

It is enough to show that $Ł\forall^m$ proves $(\forall 3)$. From $(\alpha \vee \beta) \leftrightarrow ((\alpha \to \beta) \to \beta)$ and (3) we obtain $(\forall x)(\varphi \vee \psi) \to (\forall x)((\psi \to \varphi) \to \varphi)$. Now, again by (3), we have $(\forall x)((\psi \to \varphi) \to \varphi) \to ((\forall x)(\psi \to \varphi) \to (\forall x)\varphi)$. By (7) and suffixing, $((\forall x)(\psi \to \varphi) \to (\forall x)\varphi) \to ((\psi \to (\forall x)\varphi) \to (\forall x)\varphi)$, and finally we have $((\psi \to (\forall x)\varphi) \to (\forall x)\varphi) \to (\forall x)\varphi \vee \psi$. Transitivity ends the proof. □

Syntactical properties of $\vdash_{\mathrm{L}\forall^{\mathrm{m}}}$ and $\vdash_{\mathrm{L}\forall}$

Let $\vdash$ be either $\vdash_{\mathrm{L}\forall^{\mathrm{m}}}$ or $\vdash_{\mathrm{L}\forall}$.

Theorem 3.6 (Congruence Property)

Let φ, ψ be sentences, χ a formula, and $\hat{\chi}$ a formula resulting from χ by replacing some occurrences of φ by ψ. Then

$$\vdash \varphi \leftrightarrow \varphi \qquad \varphi \leftrightarrow \psi \vdash \psi \leftrightarrow \varphi$$

$$\varphi \leftrightarrow \psi \vdash \chi \leftrightarrow \hat{\chi} \qquad \varphi \leftrightarrow \delta, \delta \leftrightarrow \psi \vdash \varphi \leftrightarrow \psi.$$

Theorem 3.7 (Constants Theorem)

Let $\Sigma \cup \{\varphi(x, \vec{z})\}$ be a theory and c a constant not occurring there. Then $\Sigma \vdash \varphi(c, \vec{z})$ iff $\Sigma \vdash \varphi(x, \vec{z})$.

Exercise 14

Prove the Constants Theorem for $\vdash_{\mathrm{G}\forall^{\mathrm{m}}}$.

Deduction theorems

Theorem 3.8

For each $\mathcal{P}$-theory $T \cup \{\varphi, \psi\}$:

- $T, \varphi \vdash_{\mathrm{G}\forall^{\mathrm{m}}} \psi$ *iff* $T \vdash_{\mathrm{G}\forall^{\mathrm{m}}} \varphi \to \psi$.
- $T, \varphi \vdash_{\mathrm{G}\forall} \psi$ *iff* $T \vdash_{\mathrm{G}\forall} \varphi \to \psi$.
- $T, \varphi \vdash_{\mathrm{Ł}\forall} \psi$ *iff* $T \vdash_{\mathrm{Ł}\forall} \varphi^n \to \psi$ *for some* $n \in \mathrm{N}$.

Syntactical properties of $\vdash_{\mathrm{L}\forall}$

Theorem 3.9 (Proof by Cases Property)

For a $\mathcal{P}$-theory T and $\mathcal{P}$-sentences φ, ψ, χ:

$$\frac{T, \varphi \vdash_{\mathrm{L}\forall} \chi \qquad T, \psi \vdash_{\mathrm{L}\forall} \chi}{T, \varphi \vee \psi \vdash_{\mathrm{L}\forall} \chi} \tag{PCP}$$

Proof.

We show by induction $T \vee \chi \vdash \varphi \vee \chi$ whenever $T \vdash \varphi$ and χ is a sentence; the rest is the same as in the propositional case.
Let δ be an element of the proof of φ from T: the claim is

- trivial if $\delta \in T$ or δ is an axiom;
- proved as in the propositional case if δ is obtained using (MP)
- easy if $\delta = (\forall x)\psi$ is obtained using (gen): from the IH we get $T \vee \chi \vdash \psi \vee \chi$ and using (gen), $(\forall 3)$, and (MP) we obtain $T \vee \chi \vdash ((\forall x)\psi) \vee \chi$. □

Syntactical properties of $\vdash_{\mathrm{L}\forall}$

Theorem 3.9 (Proof by Cases Property)

For a $\mathcal{P}$-theory T and $\mathcal{P}$-sentences φ, ψ, χ:

$$\frac{T, \varphi \vdash_{\mathrm{L}\forall} \chi \qquad T, \psi \vdash_{\mathrm{L}\forall} \chi}{T, \varphi \vee \psi \vdash_{\mathrm{L}\forall} \chi} \tag{PCP}$$

Theorem 3.10 (Semilinearity Property)

For a $\mathcal{P}$-theory T and $\mathcal{P}$-sentences φ, ψ, χ:

$$\frac{T, \varphi \to \psi \vdash_{\mathrm{L}\forall} \chi \qquad T, \psi \to \varphi \vdash_{\mathrm{L}\forall} \chi}{T \vdash_{\mathrm{L}\forall} \chi} \tag{SLP}$$

Proof.

Easy using PCP and $\vdash_{\mathrm{L}\forall} (\varphi \to \psi) \vee (\psi \to \varphi)$. □

Soundness

Exercise 15

Prove for L be either Ł of G that

- $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} \subseteq \models_{\mathbb{L}}$
- $\vdash_{\mathrm{L}\forall} \subseteq \models_{\mathbb{L}_{\mathrm{lin}}}$
- $\vdash_{Ł\forall} \subseteq \models_{\mathbb{MV}}$

Recall that $\vdash_{\mathrm{G}\forall} \not\subseteq \models_{\mathbb{G}}$

Failure of certain classical theorems (for x not free in χ)

Recall:

$\vdash_{\mathrm{L}\forall} (\forall x)\varphi \vee \chi \leftrightarrow (\forall x)(\varphi \vee \chi)$ $\quad$ $\vdash_{\mathrm{L}\forall} (\exists x)(\varphi \wedge \chi) \leftrightarrow (\exists x)\varphi \wedge \chi$

$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)(\chi \to \varphi) \leftrightarrow (\chi \to (\forall x)\varphi)$ $\quad$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)(\varphi \to \chi) \leftrightarrow ((\exists x)\varphi \to \chi)$

$\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)(\chi \to \varphi) \to (\chi \to (\exists x)\varphi)$ $\quad$ $\vdash_{\mathrm{L}\forall^{\mathrm{m}}} (\exists x)(\varphi \to \chi) \to ((\forall x)\varphi \to \chi)$

Proposition 3.11

The formulas in the first row are not provable in $\mathrm{G}\forall^{\mathrm{m}}$ *and the converse directions of formulas in the last row are provable* $\mathrm{Ł}\forall^{\mathrm{m}}$ *but not in* $\mathrm{G}\forall$*.*

Exercise 16

Prove the second part of the previous proposition.

Towards completeness: Lindenbaum–Tarski algebra

Let L be G or $\mathrm{Ł}$ and $\vdash$ be either $\vdash_{\mathrm{L}\forall^{\mathrm{m}}}$ or $\vdash_{\mathrm{L}\forall}$. Let T be a $\mathcal{P}$-theory.

Lindenbaum–Tarski algebra of T ($\mathbf{LindT}_T$):

- domain $L_T = \{[\varphi]_T \mid \varphi$ a $\mathcal{P}$-sentence$\}$ where

$$[\varphi]_T = \{\psi \mid \psi \text{ a } \mathcal{P}\text{-sentence and } T \vdash \varphi \leftrightarrow \psi\}.$$

- operations:

$$\circ^{\mathbf{LindT}_T}([\varphi_1]_T, \dots, [\varphi_n]_T) = [\circ(\varphi_1, \dots, \varphi_n)]_T$$

Exercise 17

- $\mathbf{LindT}_T \in \mathbb{L}$
- $[\varphi]_T \leq_{\mathbf{LindT}_T} [\psi]_T$ iff $T \vdash \varphi \to \psi$
- $\mathbf{LindT}_T \in \mathbb{L}_{\mathrm{lin}}$ if, and only if, T is linear.

Towards completeness: Canonical model

Canonical model ($\mathfrak{CM}_T$) of a $\mathcal{P}$-theory T (in $\vdash$): $\mathcal{P}$-structure $\langle \mathbf{LindT}_T, \mathbf{M}\rangle$ such that

- domain of $\mathbf{M}$: the set CT of closed $\mathcal{P}$-terms
- $f_{\mathbf{M}}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$ for each n-ary $f \in \mathbf{F}$, and
- $P_{\mathbf{M}}(t_1, \ldots, t_n) = [P(t_1, \ldots, t_n)]_T$ for each n-ary $P \in \mathbf{P}$.

A $\mathcal{P}$-theory T is $\forall$-Henkin if for each $\mathcal{P}$-formula ψ such that $T \nvdash (\forall x)\psi(x)$ there is a constant c in $\mathcal{P}$ such that $T \nvdash \psi(c)$.

Towards completeness: Canonical model

$\forall$-Henkin: $T \nvdash (\forall x)\psi(x)$ implies $T \nvdash \psi(c)$ for some constant c

Proposition 3.12

Let T be a $\forall$-Henkin $\mathcal{P}$-theory. Then for each $\mathcal{P}$-sentence φ we have $\|\varphi\|^{\mathfrak{CM}_T} = [\varphi]_T$ and so $\mathfrak{CM}_T \models \varphi$ iff $T \vdash \varphi$.

Proof.

Let v be evaluation s.t. $\mathrm{v}(x) = t^x$ for some $t^x \in CT$. We show by induction that $\|\varphi(x_1, \ldots, x_n)\|^{\mathfrak{CM}_T}_{\mathrm{v}} = [\varphi(t^x_1, \ldots, t^x_n)]_T$.

Towards completeness: Canonical model

$\forall$-Henkin: $T \nvdash (\forall x)\psi(x)$ implies $T \nvdash \psi(c)$ for some constant c

Proposition 3.12

Let T be a $\forall$-Henkin $\mathcal{P}$-theory. Then for each $\mathcal{P}$-sentence φ we have $\|\varphi\|^{\mathfrak{CM}_T} = [\varphi]_T$ and so $\mathfrak{CM}_T \models \varphi$ iff $T \vdash \varphi$.

Proof.

Let v be evaluation s.t. $\mathrm{v}(x) = t^x$ for some $t^x \in CT$. We show by induction that $\|\varphi(x_1, \ldots, x_n)\|^{\mathfrak{CM}_T}_{\mathrm{v}} = [\varphi(t^x_1, \ldots, t^x_n)]_T$.

The base case and the induction step for connectives is just the definition.

Towards completeness: Canonical model

$\forall$-Henkin: $T \nvdash (\forall x)\psi(x)$ implies $T \nvdash \psi(c)$ for some constant c

Proposition 3.12

Let T be a $\forall$-Henkin $\mathcal{P}$-theory. Then for each $\mathcal{P}$-sentence φ we have $\|\varphi\|^{\mathfrak{CM}_T} = [\varphi]_T$ and so $\mathfrak{CM}_T \models \varphi$ iff $T \vdash \varphi$.

Proof.

Let v be evaluation s.t. $\mathrm{v}(x) = t^x$ for some $t^x \in CT$. We show by induction that $\|\varphi(x_1, \ldots, x_n)\|^{\mathfrak{CM}_T}_{\mathrm{v}} = [\varphi(t^x_1, \ldots, t^x_n)]_T$.

Quantifiers: $[(\forall x)\varphi]_T \stackrel{?}{=} \|(\forall x)\varphi\|^{\mathfrak{CM}_T} = \inf_{\leq_{\mathbf{LindT}_T}}\{[\varphi(t)]_T \mid t \in CT\}$

From $T \vdash (\forall x)\varphi \to \varphi(t)$ we get that $[(\forall x)\varphi]_T$ is a lower bound.

We show it is the largest one: take any χ s.t. $[\chi]_T \not\leq_{\mathbf{LindT}_T} [(\forall x)\varphi]_T$; thus $T \nvdash \chi \to (\forall x)\varphi$, and so $T \nvdash (\forall x)(\chi \to \varphi)$. So there is $c \in CT$ s.t. $T \nvdash (\chi \to \varphi(c))$, i.e., $[\chi]_T \not\leq_{\mathbf{LindT}_T} [\varphi(c)]_T$. □

Completeness theorem for $\mathrm{L}\forall^{\mathrm{m}}$

Theorem 3.13 (Completeness theorem for $\mathrm{L}\forall^{\mathrm{m}}$)

Let L be either Ł or G and $T \cup \{\varphi\}$ a $\mathcal{P}$-theory. Then:
$T \vdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi$ iff $T \models_{\mathbb{L}} \varphi$.

All we need to prove this theorem is to show that:

Lemma 3.14 (Extension lemma for $\mathrm{L}\forall^{\mathrm{m}}$)

Let $T \cup \{\varphi\}$ be a $\mathcal{P}$-theory such that $T \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi$. Then there is $\mathcal{P}' \supseteq \mathcal{P}$ and a $\forall$-Henkin $\mathcal{P}'$-theory $T' \supseteq T$ such that $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \varphi$.

Proof.

$\mathcal{P}' = \mathcal{P}$ + countably many new object constants. Let T' be T as $\mathcal{P}'$-theory. Take any $\mathcal{P}'$-formula $\psi(x)$, such that $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} (\forall x)\psi(x)$. Thus $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \psi(x)$ and so $T' \nvdash_{\mathrm{L}\forall^{\mathrm{m}}} \psi(c)$ for some $c \in \mathcal{P}'$ not occurring in $T' \cup \{\psi\}$ (by Constants Theorem). □

Completeness theorem for $\mathrm{L}\forall$

Theorem 3.15 (Completeness theorem for $\mathrm{L}\forall$)

Let L *be either* Ł *or* G *and* $T \cup \{\varphi\}$ *a* $\mathcal{P}$*-theory. Then*

$$T \vdash_{\mathrm{L}\forall} \varphi \quad \textit{iff} \quad T \models_{\mathbb{L}_{\mathrm{lin}}} \varphi.$$

All we need to prove this theorem is to show that:

Lemma 3.16 (Extension lemma for $\mathrm{L}\forall$)

Let $T \cup \{\varphi\}$ *be a* $\mathcal{P}$*-theory such that* $T \nvdash_{\mathrm{L}\forall} \varphi$*. Then there is a predicate language* $\mathcal{P}' \supseteq \mathcal{P}$ *and a linear* $\forall$*-Henkin* $\mathcal{P}'$*-theory* $T' \supseteq T$ *such that* $T' \nvdash_{\mathrm{L}\forall} \varphi$*.*

Initializing the construction

Let $\mathcal{P}'$ be the expansion of $\mathcal{P}$ by countably many new constants.

We enumerate all $\mathcal{P}'$-formulas with one free variable: $\{\chi_i(x) \mid i \in \mathrm{N}\}$.

We construct a sequence of $\mathcal{P}'$-sentences φ_i and an increasing chain of $\mathcal{P}'$-theories T_i such that $T_i \nvdash \varphi_j$ for each $j \leq i$.

Take $T_0 = T$ and $\varphi_0 = \varphi$, which fulfils our conditions.

In the induction step we distinguish two possibilities and show that the required conditions are met:

The induction step

(H1) If $T_i \vdash \varphi_i \vee (\forall x)\chi_{i+1}(x)$: then we define $\varphi_{i+1} = \varphi_i$ and
$$T_{i+1} = T_i \cup \{(\forall x)\chi_{i+1}(x)\}.$$

(H2) If $T_i \nvdash \varphi_i \vee (\forall x)\chi_{i+1}(x)$, then we define $T_{i+1} = T_i$ and
$\varphi_{i+1} = \varphi_i \vee \chi_{i+1}(c)$ for some c not occurring in $T_i \cup \{\varphi_j \mid j \leq i\}$.

Assume, for a contradiction, that $T_{i+1} \vdash \varphi_j$ for some $j \leq i+1$. Then also $T_{i+1} \vdash \varphi_{i+1}$.

Thus in case (H1) we have $T_i \cup \{(\forall x)\chi_{i+1}(x)\} \vdash \varphi_i$. Since, trivially, $T_i \cup \{\varphi_i\} \vdash \varphi_i$ we obtain by Proof by Cases Property that
$T_i \cup \{\varphi_i \vee (\forall x)\chi_{i+1}(x)\} \vdash \varphi_i$ and so $T_i \vdash \varphi_i$; a contradiction!

The induction step

(H1) If $T_i \vdash \varphi_i \vee (\forall x)\chi_{i+1}(x)$: then we define $\varphi_{i+1} = \varphi_i$ and
$T_{i+1} = T_i \cup \{(\forall x)\chi_{i+1}(x)\}$.

(H2) If $T_i \nvdash \varphi_i \vee (\forall x)\chi_{i+1}(x)$, then we define $T_{i+1} = T_i$ and
$\varphi_{i+1} = \varphi_i \vee \chi_{i+1}(c)$ for some c not occurring in $T_i \cup \{\varphi_j \mid j \leq i\}$.

Assume, for a contradiction, that $T_{i+1} \vdash \varphi_j$ for some $j \leq i+1$. Then also $T_{i+1} \vdash \varphi_{i+1}$.

Thus in case (H2) we have $T_i \vdash \varphi_i \vee \chi_{i+1}(c)$. Using Constants Theorem we obtain $T_i \vdash \varphi_i \vee \chi_{i+1}(x)$ and thus by (gen), $(\forall 3)$, and (MP) we obtain $T_i \vdash \varphi_i \vee (\forall x)\chi_{i+1}(x)$; a contradiction!

Final touches . . .

Let T' be a maximal theory extending $\bigcup T_i$ s.t. $T' \nvdash \varphi_i$ for each i. Such T' exists thanks to Zorn's Lemma: let $\mathcal{T}$ be a chain of such theories then clearly so is $\bigcup \mathcal{T}$.

T' is linear: assume that $T' \nvdash \psi \to \chi$ and $T' \nvdash \chi \to \psi$. Then there are i, j such that $T', \psi \to \chi \vdash \varphi_i$ and $T', \chi \to \psi \vdash \varphi_j$. Thus also

$$T', \psi \to \chi \vdash \varphi_{\max\{i,j\}} \text{ and } T', \chi \to \psi \vdash \varphi_{\max\{i,j\}}.$$

Thus by Semilinearity Property also $T' \vdash \varphi_{\max\{i,j\}}$; a contradiction!

T' is $\forall$-Henkin: if $T' \nvdash (\forall x)\chi_{i+1}(x)$, then we must have used case (H2); since $T' \nvdash \varphi_{i+1}$ and $\varphi_{i+1} = \varphi_i \vee \chi_{i+1}(c))$ we also have $T' \nvdash \chi_{i+1}(c)$.

It works in Gödel–Dummett logic

Theorem 3.17

The following are equivalent for every set of $\mathcal{P}$-formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$:

1. $\Gamma \vdash_{\mathrm{G}\forall} \varphi$
2. $\Gamma \models_{\mathbb{G}_{\mathrm{lin}}} \varphi$
3. $\Gamma \models_{[0,1]_{\mathrm{G}}} \varphi$

Recall the proof in the propositional case

Contrapositively: assume that $T \not\vdash_{\mathrm{G}} \varphi$. Let $\boldsymbol{B}$ be a countable G-chain and e a $\boldsymbol{B}$-evaluation such that $e[T] \subseteq \{\overline{1}^{\boldsymbol{B}}\}$ and $e(\varphi) \neq \overline{1}^{\boldsymbol{B}}$.

There has to be (because every countable order can be monotonously embedded into a dense one) a mapping $f\colon B \to [0,1]$ such that $f(\overline{0}) = 0, f(\overline{1}) = 1$, and for each $a, b \in B$ we have:

$$a \leq b \quad \text{iff} \quad f(a) \leq f(a)$$

We define a mapping $\overline{e}\colon Fm_{\mathcal{L}} \to [0,1]$ as

$$\overline{e}(\psi) = f(e(\psi))$$

and prove (by induction) that it is $[0,1]_{\mathrm{G}}$-evaluation.

Then $\overline{e}(\psi) = 1$ iff $e(\psi) = \overline{1}^{\boldsymbol{B}}$ and so $\overline{e}[T] \subseteq \{1\}$ and $\overline{e}(\varphi) \neq 1$.

Would it work in the first-order level?

Contrapositively: assume that $T \not\vdash_{G\forall} \varphi$. Let $\boldsymbol{B}$ be a countable G-chain and $\mathfrak{M} = \langle \boldsymbol{B}, \mathbf{M} \rangle$ a model of T such that $\|\varphi\|^{\mathbf{M}}_{\mathrm{v}} \neq \overline{1}^{\boldsymbol{B}}$.

There has to be (because every countable order can be monotonously embedded into a dense one) a mapping $f \colon B \to [0,1]$ such that $f(\overline{0}) = 0, f(\overline{1}) = 1$, and for each $a, b \in B$ we have:

$$a \leq b \quad \text{iff} \quad f(a) \leq f(a)$$

Would it work in the first-order level?

Contrapositively: assume that $T \nvdash_{\mathrm{G}\forall} \varphi$. Let $\boldsymbol{B}$ be a countable G-chain and $\mathfrak{M} = \langle \boldsymbol{B}, \mathbf{M} \rangle$ a model of T such that $\|\varphi\|^{\mathbf{M}}_{\mathrm{v}} \neq \overline{1}^{\boldsymbol{B}}$.

There has to be (because every countable order can be monotonously embedded into a dense one) a mapping $f \colon B \to [0,1]$ such that $f(\overline{0}) = 0, f(\overline{1}) = 1$, and for each $a, b \in B$ we have:

$$f(a \wedge b) = f(a) \wedge f(b) \text{ and } f^{-1}(a \wedge b) = f^{-1}(a) \wedge f^{-1}(b)$$

Would it work in the first-order level?

Contrapositively: assume that $T \not\vdash_{\mathrm{G}\forall} \varphi$. Let $\boldsymbol{B}$ be a countable G-chain and $\mathfrak{M} = \langle \boldsymbol{B}, \mathbf{M} \rangle$ a model of T such that $\|\varphi\|^{\mathbf{M}}_{\mathrm{v}} \neq \overline{1}^{\boldsymbol{B}}$.

There has to be (because every countable order can be monotonously embedded into a dense one) a mapping $f\colon B \to [0,1]$ such that $f(\overline{0}) = 0, f(\overline{1}) = 1$, and for each $a, b \in B$ we have:

$$f(\bigwedge_{a\in X} a) = \bigwedge_{a\in X} f(a) \text{ and } f^{-1}(\bigwedge_{a\in X} a) = \bigwedge_{a\in X} f^{-1}(a)$$

We define a $[0,1]_{\mathrm{G}}$-structure $\bar{\mathbf{M}}$ with the same domain, functions and

$$P_{\bar{\mathbf{M}}}(x_1, \dots, x_n) = f(P_{\mathbf{M}}(x_1, \dots, x_n))$$

and prove (by induction) that $\|\psi\|^{\bar{\mathbf{M}}}_{\mathrm{v}} = f(\|\psi\|^{\mathbf{M}}_{\mathrm{v}})$. Then $\|\psi\|^{\bar{\mathbf{M}}}_{\mathrm{v}} = 1$ iff $\|\psi\|^{\mathbf{M}}_{\mathrm{v}} = \overline{1}^{\boldsymbol{B}}$ and so $\langle [0,1]_{\mathrm{G}}, \bar{\mathbf{M}} \rangle$ is model of T and $\|\varphi\|^{\bar{\mathbf{M}}}_{\mathrm{v}} \neq 1$.

What about the case of Łukasiewicz logic?

Theorem 3.18

There is a formula φ such that $\models_{[0,1]_Ł} \varphi$ and $\nvdash_{Ł\forall} \varphi$.

Neither the set of theorems nor the set of satisfiable formulas w.r.t. the models of standard MV-algebra $[0, 1]_Ł$ are recursively enumerable. In fact we have:

Theorem 3.19 (Ragaz, Goldstern, Hájek)

The set $\mathrm{stTAUT}(Ł\forall)$ is Π_2-complete and $\mathrm{stSAT}(Ł\forall)$ is Π_1-complete.

Finite model property: The classical case

- Valid sentences of $\mathrm{CL}\forall$ (in any predicate language) are recursively enumerable thanks to the completeness theorem.
- Löwenheim (1915): Monadic classical logic (the fragment of $\mathrm{CL}\forall$ only with unary predicates and no functional symbols) has the finite model property, and hence it is decidable.
- Church (1936) and Turing (1937): if the predicate language contains at least a binary predicate, then $\mathrm{CL}\forall$ is undecidable.
- Surány (1959): The fragment of $\mathrm{CL}\forall$ with three variables is undecidable.
- Mortimer (1975): The fragment of $\mathrm{CL}\forall$ with two variables has the finite model property, and hence it is decidable.

Finite model property: the fuzzy case

In Gödel–Dummett logic the FMP does not even hold for formulas with one variable (a model is finite if it has a finite domain).

Example in $G\forall = \models_{[0,1]_G}$

$$\varphi = \neg(\forall x)P(x) \wedge \neg(\exists x)\neg P(x).$$

Evidently φ has no finite model and so $\varphi \models^{\mathrm{fin}}_{[0,1]_G} \overline{0}$. But consider $[0,1]_G$-model $\mathfrak{M}$ with domain N, where $P_{\mathbf{M}}(n) = \frac{1}{n+1}$. Then clearly for each $n \in$ N: $\|P(n)\| > 0$ and $\inf_{n \in \mathrm{N}} \|P(n)\| = 0$, i.e., $\mathfrak{M} \models \varphi$, and so $\varphi \not\models_{[0,1]_G} \overline{0}$.

The infimum is not the minimum, it is not *witnessed*.

Exercise 18

Show that $\models_{[0,1]_Ł}$ does not have the FMP (hint: use the formula $(\exists x)(P(x) \leftrightarrow \neg P(x))\ \&\ (\forall x)(\exists y)(P(x) \leftrightarrow P(y)\ \&\ P(y))$).

Witnessed models

Definition 3.20

A $\mathcal{P}$-model $\mathfrak{M}$ is witnessed if for each $\mathcal{P}$-formula $\varphi(x, \vec{y})$ and for each $\vec{a} \in M$ there are $b_s, b_i \in M$ such that:

$$\|(\forall x)\varphi(x, \vec{a})\|^{\mathfrak{M}} = \|\varphi(b_i, \vec{a})\|^{\mathfrak{M}} \qquad \|(\exists x)\varphi(x, \vec{a})\|^{\mathfrak{M}} = \|\varphi(b_s, \vec{a})\|^{\mathfrak{M}}.$$

Exercise 19

Consider formulas

$$(W\exists)\ \ (\exists x)((\exists y)\psi(y, \vec{z}) \to \psi(x, \vec{z})) \qquad (W\forall)\ \ (\exists x)(\psi(x, \vec{z}) \to (\forall y)\psi(y, \vec{z}))$$

Show that not all models of these formulas are witnessed and these formulas are

- true in all witnessed models of $\mathrm{G}\forall$
- not provable in $\mathrm{G}\forall$
- provable in (true in all models of) $\text{Ł}\forall$

Witnessed logic and witnessed completeness

Theorem 3.21 (Witnessed completeness theorem for $Ł\forall$)

Let $T \cup \{\varphi\}$ a theory. Then $T \vdash_{Ł\forall} \varphi$ iff for each witnessed $\mathbb{MV}_{\mathrm{lin}}$-model $\mathfrak{M}$ of T we have $\mathfrak{M} \models \varphi$.

Definition 3.22

The logic $G\forall^{w}$ is the extension of $G\forall$ by the axioms $(W\exists)$ and $(W\forall)$.

(note that the analogous definition for L would yield $Ł\forall^{w} = Ł\forall$)

Theorem 3.23 (Witnessed completeness theorem for $G\forall^{w}$)

Let $T \cup \{\varphi\}$ be a theory. Then $T \vdash_{G\forall^{w}} \varphi$ iff for each witnessed $\mathbb{G}_{\mathrm{lin}}$-model $\mathfrak{M}$ of T we have $\mathfrak{M} \models \varphi$.

A proof

A theory T is Henkin if it is $\forall$-Henkin and for each φ such that $T \vdash (\exists x)\varphi(x)$ there is a constant such that $T \vdash \varphi(c)$.

Assume that we can prove:

Lemma 3.24 (Full Extension lemma for $\mathrm{L}\forall$)

Let $T \cup \{\varphi\}$ be a $\mathcal{P}$-theory such that $T \nvdash_{\mathrm{L}\forall^{\mathrm{w}}} \varphi$. Then there is a predicate language $\mathcal{P}' \supseteq \mathcal{P}$ and a linear Henkin $\mathcal{P}'$-theory $T' \supseteq T$ such that $T' \nvdash_{\mathrm{L}\forall^{\mathrm{w}}} \varphi$.

Then the proof of the witnessed completeness is an easy corollary of the following straightforward proposition

Proposition 3.25

Let T be a Henkin $\mathcal{P}$-theory. Then $\mathfrak{CM}_T$ is a witnessed model.

Before we prove the full extension lemma ...

Definition 3.26

Let $\mathcal{P}_1 \subseteq \mathcal{P}_2$. A $\mathcal{P}_2$-theory T_2 is a conservative expansion of a $\mathcal{P}_1$-theory T_1 if for each $\mathcal{P}_1$-formula φ, $T_2 \vdash \varphi$ iff $T_1 \vdash \varphi$.

Proposition 3.27

For each predicate language $\mathcal{P}$, each $\mathcal{P}$-theory T, each $\mathcal{P}$-formula $\varphi(x)$, and any constant $c \notin \mathcal{P}$ holds that $T \cup \{\varphi(c)\}$ is a conservative expansion (in the logic $\mathrm{L}\forall$) of $T \cup \{(\exists x)\varphi(x)\}$.

Proof.

Assume that $T \cup \{\varphi(c)\} \vdash_{\mathrm{L}\forall} \psi$. Then, by Deduction Theorem, there is n such that $T \vdash_{\mathrm{L}\forall} \varphi(c)^n \to \psi$. Thus by the Constants Theorem and $(\exists 2)$ we obtain $T \vdash_{\mathrm{L}\forall} (\exists x)(\varphi(x)^n) \to \psi$. Using (13) we obtain $T \vdash_{\mathrm{L}\forall} ((\exists x)\varphi(x))^n \to \psi$. Deduction Theorem completes the proof. □

A proof of full extension lemma

Modify the proof of the extension lemma, s.t. after going through options (H1) and (H2) on the i-th step we construct theories T'_{i+1}. Then we distinguish two new options:

(W1) If $T'_{i+1}, (\exists x)\chi_{i+1} \nvdash \varphi_{i+1}$: then we define $T_{i+1} = T'_{i+1} \cup \{\chi_{i+1}(c)\}$.
for some c not occurring in $T'_i \cup \{\varphi_j \mid j \leq i\}$.

(W2) If $T'_{i+1}, (\exists x)\chi_{i+1} \vdash \varphi_{i+1}$: then we define $T_{i+1} = T'_{i+1}$

The induction assumption $T_{i+1} \nvdash \varphi_{i+1}$ holds: in (W2) trivially, in case of (W1) we use the fact that $T'_{i+1} \cup \{\chi_{i+1}(c)\}$ is a conservative expansion of $T'_{i+1} \cup \{(\exists x)\chi_{i+1}(x)\}$.

The rest is the same as the proof of the extension lemma, we only show that T' is Henkin: it $T' \vdash (\exists x)\chi_{i_1}(x)$ then we used case (W1) (from $T', (\exists x)\chi_{i+1}(x) \vdash \varphi_{i+1}$, a contradiction). Thus $T' \vdash \chi_{i+1}(c)$.

Skolemization

Theorem 3.28

For Gödel–Dummett logic we have: $T \cup \{(\forall \vec{y})\varphi(f_\varphi(\vec{y}), \vec{y})\}$ is a conservative expansion of $T \cup \{(\forall \vec{y})(\exists x)\varphi(x, \vec{y})\}$ for each $\mathcal{P}$-theory $T \cup \{\varphi(x, \vec{y})\}$, and a functional symbol $f_\varphi \notin \mathcal{P}$ of the proper arity.

A hint of the proof.

Take $\mathcal{P}$-formula χ s.t. $T \cup \{(\forall y)(\exists x)\varphi(x, y)\} \nvdash \chi$. Let T' be a Henkin $\mathcal{P}'$-theory $T' \supseteq T \cup \{(\forall y)(\exists x)\varphi(x, y)\}$ s.t. $T' \nvdash \chi$, and hence $\mathfrak{CM}_{T'} \not\models \chi$.
For each closed $\mathcal{P}'$-term t we have $T' \vdash (\exists x)\varphi(x, t)$ (by $(\forall 1)$) and hence there is a $\mathcal{P}'$-constant c_t such that $T' \vdash \varphi(c_t, t)$.
We define a model $\mathfrak{M}$ by expanding $\mathfrak{CM}_{T'}$ with one functional symbol defined as: $(f_\varphi)_{\mathfrak{M}}(t) = c_t$
Observe that for each $\mathcal{P}'$-formula: $\mathfrak{M} \models \psi$ iff $\mathfrak{CM}'_T \models \psi$
Thus $\mathfrak{M} \models T$ and $\mathfrak{M} \not\models \chi$ and so clearly $\mathfrak{M} \models (\forall y)\varphi(f_\varphi(y), y)$
And so we have established $T \cup \{(\forall y)\varphi(f_\varphi(y), y)\} \nvdash \chi$. □

Important sets of sentences

Definition 3.29

Let L be G or $\mathrm{Ł}$ and $\mathbb{K}$ a non-empty class of L-chains. We define:

$$\mathrm{TAUT}(\mathbb{K}) = \{\varphi \mid \text{ for every } \mathbb{K}\text{-model } \mathfrak{M}, \|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}\}.$$

$$\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K}) = \{\varphi \mid \text{ for every } \mathbb{K}\text{-model } \mathfrak{M}, \|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} > \overline{0}^{\boldsymbol{A}}\}.$$

$$\mathrm{SAT}(\mathbb{K}) = \{\varphi \mid \text{ there exist } \mathbb{K}\text{-model } \mathfrak{M} \text{ s.t. } \|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}.$$

$$\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K}) = \{\varphi \mid \text{ there exist } \mathbb{K}\text{-model } \mathfrak{M} \text{ s.t. } \|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} > \overline{0}^{\boldsymbol{A}}.$$

Instead of $\mathrm{TAUT}(\mathbb{K})$ we write

- $\mathrm{genTAUT}(\mathrm{L}\forall)$ if $\mathbb{K}$ is the class of all L-chains (general semantics).
- $\mathrm{stTAUT}(\mathrm{L}\forall)$ if $\mathbb{K}$ contains only the standard L-chain on $[0, 1]$ (standard semantics).

And analogously for $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$, $\mathrm{SAT}(\mathbb{K})$ and $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ ...

Relations between sets

Lemma 3.30

1. $\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ *iff* $\neg\varphi \notin \mathrm{SAT}(\mathbb{K})$,
2. $\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ *iff* $\neg\varphi \notin \mathrm{TAUT}(\mathbb{K})$.

Lemma 3.31

If $\mathrm{L} = \text{Ł}$, *then for every* φ*:*

1. $\varphi \in \mathrm{SAT}(\mathbb{K})$ *iff* $\neg\varphi \notin \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$,
2. $\varphi \in \mathrm{TAUT}(\mathbb{K})$ *iff* $\neg\varphi \notin \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$.

Arithmetical hierarchy

- Let $\Phi(x)$ be an arithmetical formula with one free variable; we say $\Phi(x)$ defines a set $A \subseteq \mathrm{N}$ iff for any $n \in \mathrm{N}$ we have $n \in A$ iff $\mathrm{N} \models \Phi(n)$.
- An arithmetical formula is bounded iff all its quantifiers are bounded (i.e., are of the form $\forall x \leq t$ or $\exists x \leq t$ for some term t).
- An arithmetical formula is a Σ_1-formula (Π_1-formula) iff it has the form $\exists x \Phi$ ($\forall x \Phi$ respectively) where Φ is a bounded formula.
- A formula is Σ_2 (Π_2) iff it has the form $\exists x \Phi$ ($\forall x \Phi$ respectively) where Φ is a Π_1-formula (Σ_1-formula respectively).
- Inductively, one defines Σ_n- and Π_n-formulas for any natural number $n \geq 1$.

Arithmetical hierarchy

- A set $A \subseteq \mathbf{N}$ is in the class Σ_n iff there is a Σ_n-formula that defines A in $\mathbf{N}$; analogously for the class Π_n.
- Any set that is in Σ_n is also in Σ_m and Π_m for $m > n$.
- If $A \subseteq \mathbf{N}$ is a Σ_n-set, then $\overline{A}$ is a Π_n-set.
- Σ_1-sets are exactly recursively enumerable sets, while recursive sets are $\Sigma_1 \cap \Pi_1$.
- A problem P_1 is reducible to a problem P_2 ($P_1 \preceq P_2$) iff there is a deterministic Turing machine such that, for any pair of input x and its output y, we have $x \in P_1$ iff $y \in P_2$.
- A problem P is Σ_n-hard iff $P' \preceq_{\mathrm{m}} P$ for any Σ_n-problem P'.
- A problem P is Σ_n-complete iff it is Σ_n-hard and at the same time it is a Σ_n-problem. Analogously for Π_n.

Lower bounds

Proposition 3.32

For every class $\mathbb{K}$ of chains, $\mathrm{TAUT}(\mathbb{K})$ and $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ are Σ_1-hard. and the sets $\mathrm{SAT}(\mathbb{K})$ and $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ are Π_1-hard.

Proof (for $\mathrm{SAT}(\mathbb{K})$, the others are much harder).

Let φ be a sentence with predicate symbols $\{P_i \mid 1 \leq i \leq n\}$. Observe that

$$\varphi \in \mathrm{SAT}(\mathbf{2}) \quad \text{iff} \quad \varphi \wedge \bigwedge_{1 \leq i \leq n} (\forall \overrightarrow{x})(P_i(\overrightarrow{x}) \vee \neg P_i(\overrightarrow{x})) \in \mathrm{SAT}(\mathbb{K})$$

Since the satisfiability problem in classical logic is Π_1-hard so it must be $\mathrm{SAT}(\mathbb{K})$. □

Upper bounds

Proposition 3.33

If $\mathrm{L}\forall$ *is complete w.r.t. models over* $\mathbb{K}$*, then* $\mathrm{TAUT}(\mathbb{K})$ *and* $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ *are* Σ_1*, while* $\mathrm{SAT}(\mathbb{K})$ *and* $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ *are* Π_1*.*

Proof.

$\mathrm{TAUT}(\mathbb{K})$ is Σ_1 because it is the set of theorems of a recursively axiomatizable logic. As regards to $\mathrm{SAT}(\mathbb{K})$, notice that for every φ we have: $\varphi \in \mathrm{SAT}(\mathbb{K})$ iff $\varphi \not\models_{\mathbb{K}} \overline{0}$ iff $\varphi \nvdash_{\mathrm{L}\forall} \overline{0}$. Thus $\mathrm{SAT}(\mathbb{K})$ is in Π_1.
The other two claim follows from Lemma 3.30. □

Complexity of general semantics and undecidability

Theorem 3.34

$\mathrm{genTAUT}(\mathrm{L}\forall)$ *and* $\mathrm{genTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are* Σ_1*-complete,* $\mathrm{genSAT}(\mathrm{L}\forall)$ *and* $\mathrm{genSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are* Π_1*-complete.*

Corollary 3.35

$\mathrm{G}\forall$ *and* $\text{Ł}\forall$ *are undecidable.*

Complexity of standard semantics

Due to the standard completeness of $G\forall$ we know

Theorem 3.36

$\mathrm{stTAUT}(L\forall)$ *and* $\mathrm{stTAUT}_{\mathrm{pos}}(L\forall)$ *are* Σ_1*-complete,* $\mathrm{stSAT}(L\forall)$ *and* $\mathrm{stSAT}_{\mathrm{pos}}(L\forall)$ *are* Π_1*-complete.*

Actually we have:

$$\mathrm{stTAUT}(G\forall) = \mathrm{genTAUT}(G\forall) \qquad \mathrm{stTAUT}_{\mathrm{pos}}(G\forall) = \mathrm{genTAUT}_{\mathrm{pos}}(G\forall)$$
$$\mathrm{stSAT}(G\forall) = \mathrm{genSAT}(G\forall) \qquad \mathrm{stSAT}_{\mathrm{pos}}(G\forall) = \mathrm{genSAT}_{\mathrm{pos}}(G\forall).$$

Due to the failure of standard completeness of Ł$\forall$ we know

$$\mathrm{stTAUT}(\text{Ł}\forall) \neq \mathrm{genTAUT}(\text{Ł}\forall) \qquad \mathrm{stSAT}_{\mathrm{pos}}(\text{Ł}\forall) \neq \mathrm{genSAT}_{\mathrm{pos}}(\text{Ł}\forall).$$

Complexity of standard semantics of Łukasiwicz logic

Proposition 3.37

$\mathrm{stTAUT}_{\mathrm{pos}}(Ł\forall) = \mathrm{genTAUT}(Ł\forall)$ *and* $\mathrm{stSAT}(Ł\forall) = \mathrm{genSAT}(Ł\forall)$.

Complexity of standard semantics of Łukasiwicz logic

Proposition 3.37

$\mathrm{stTAUT}_{\mathrm{pos}}(\text{Ł}\forall) = \mathrm{genTAUT}(\text{Ł}\forall)$ *and* $\mathrm{stSAT}(\text{Ł}\forall) = \mathrm{genSAT}(\text{Ł}\forall)$*.*

Corollary 3.38

The set $\mathrm{stTAUT}_{\mathrm{pos}}(\text{Ł}\forall)$ *is* Σ_1*-complete and* $\mathrm{stSAT}(\text{Ł}\forall)$ *is* Π_1*-complete.*

Complexity of standard semantics of Łukasiwicz logic

Proposition 3.37

$\mathrm{stTAUT}_{\mathrm{pos}}(\text{Ł}\forall) = \mathrm{genTAUT}(\text{Ł}\forall)$ *and* $\mathrm{stSAT}(\text{Ł}\forall) = \mathrm{genSAT}(\text{Ł}\forall)$.

Corollary 3.38

The set $\mathrm{stTAUT}_{\mathrm{pos}}(\text{Ł}\forall)$ *is* Σ_1*-complete and* $\mathrm{stSAT}(\text{Ł}\forall)$ *is* Π_1*-complete.*

Theorem 3.39 (Ragaz, Goldstern, Hájek)

The set $\mathrm{stTAUT}(\text{Ł}\forall)$ *is* Π_2*-complete and* $\mathrm{stSAT}_{\mathrm{pos}}(\text{Ł}\forall)$ *is* Σ_2*-complete.*

Formal fuzzy mathematics

First-order fuzzy logic is strong enough to support non-trivial formal mathematical theories

Mathematical concepts in such theories show gradual rather than bivalent structure

Examples:

- Skolem, Hájek (1960, 2005): naïve set theory over Ł
- Takeuti–Titani (1994): ZF-style fuzzy set theory in a system close to Gödel logic ($\Rightarrow$ contractive)
- Restall (1995), Hájek–Paris–Shepherdson (2000): arithmetic with the truth predicate over Ł
- Hájek–Haniková (2003): ZF-style set theory over $\mathrm{HL}_\triangle$
- Novák (2004): Church-style fuzzy type theory over $\mathrm{IMTL}_\triangle$
- Běhounek–Cintula (2005): higher-order fuzzy logic

Hájek–Haniková fuzzy set theory

Logic: First-order $\mathrm{HL}_{\triangle}$ with identity

Language: $\in$

Axioms (z not free in φ):

- $\triangle(\forall u)(u \in x \leftrightarrow u \in y) \to x = y$ (extensionality)
- $(\exists z)\triangle(\forall y)\neg(y \in z)$ (empty set $\emptyset$)
- $(\exists z)\triangle(\forall u)(u \in z \leftrightarrow (u = x \vee u = y))$ (pair $\{x, y\}$)
- $(\exists z)\triangle(\forall u)(u \in z \leftrightarrow (\exists y)(u \in y \,\&\, y \in x))$ (union $\bigcup$)
- $(\exists z)\triangle(\forall u)(u \in z \leftrightarrow \triangle(\forall x \in u)(x \in y))$ (weak power)
- $(\exists z)\triangle(\emptyset \in z \,\&\, (\forall x \in z)(x \cup \{x\} \in z))$ (infinity)
- $(\exists z)\triangle(\forall u)(u \in z \leftrightarrow (u \in x \,\&\, \varphi(u, x))$ (separation)
- $(\exists z)\triangle[(\forall u \in x)(\exists v)\varphi(u, v) \to (\forall u \in x)(\exists v \in z)\varphi(u, v)]$ (collection)
- $\triangle(\forall x)((\forall y \in x)\varphi(y) \to \varphi(x)) \to \triangle(\forall x)\varphi(x)$ ($\in$-induction)
- $(\exists z)\triangle((\forall u)(u \in z \vee \neg(u \in z)) \,\&\, (\forall u \in x)(u \in z))$ (support)

Properties

Semantics: A cumulative hierarchy of $\mathbf{HL}$-valued fuzzy sets

Features:

- Contains an inner model of classical ZF:
  (as the subuniverse of hereditarily crisp sets)
- Conservatively extends classical ZF with fuzzy sets
- Generalizes Takeuti–Titani's construction
  in a non-contractive fuzzy logic

Cantor–Łukasiewicz set theory

Logic: First-order Łukasiewicz logic Ł$\forall$

Language: $\in$, set comprehension terms $\{x \mid \varphi\}$

Axioms:

- $y \in \{x \mid \varphi\} \leftrightarrow \varphi(y)$ (unrestricted comprehension)

Features:

- Non-contractivity of Ł blocks Russell's paradox
- Consistency conjectured by Skolem (1960—still open: in 2010 a gap found by Terui in White's 1979 consistency proof)
- Adding extensionality is inconsistent with CŁ
- Open problem: define a reasonable arithmetic in CŁ (some negative results by Hájek, 2005)

Fuzzy class theory = (Henkin-style) higher-order fuzzy logic

Logic: Any first-order deductive fuzzy logic with Δ and $=$

Originally: ŁΠ for its expressive power

Language:

- Sorts of variables for atoms, classes, classes of classes, etc.
- Subsorts for k-tuples of objects at each level
- $\in$ between successive sorts
- At all levels: $\{x \mid \dots\}$ for classes, $\langle \dots \rangle$ for tuples

Axioms (for all sorts):

- $\langle x_1, \dots, x_k \rangle = \langle y_1, \dots, y_k \rangle \to x_1 = y_1 \,\&\, \dots \,\&\, x_k = y_k$ (tuple identity)
- $(\forall x)\triangle(x \in A \leftrightarrow x \in B) \to A = B$ (extensionality)
- $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$ (class comprehension)

Properties

Semantics:

Fuzzy sets and relations of all orders over a crisp ground set
(Henkin-style $\Rightarrow$ non-standard models exist,
full higher-order fuzzy logic is not axiomatizable)

Features:

- Suitable for the reconstruction and graded generalization of large parts of traditional fuzzy mathematics
- Several mathematical disciplines have been developed within its framework, using it as a foundational theory:
  (e.g. fuzzy relations, fuzzy numbers, fuzzy topology)
- The results obtained trivialize initial parts of traditional fuzzy set theory

Counterfactual conditionals

Counterfactuals are conditionals with false antecedents:

If it were the case that A, it would be the case that C

Their logical analysis is notoriously problematic:

- If interpreted as material implications, they come out always true due to the false antecedent
- However, some counterfactuals are obviously false

$\Rightarrow$ a simple logical analysis does not work

Properties of counterfactuals

Counterfactual conditionals do not obey standard inference rules of the material implication:

Weakening: $$\frac{A \mathrel{\Box\!\!\to} C}{A \wedge B \mathrel{\Box\!\!\to} C}$$

If I won the lottery, I would go for a trip around the globe.
If I won the lottery and then WW3 started, I would go for a trip around the globe. (!)

Contraposition: $$\frac{A \mathrel{\Box\!\!\to} C}{\neg C \mathrel{\Box\!\!\to} \neg A}$$

If I won the lottery, I would still live in the Prague.
If I left Prague, I would not win the lottery (!)

Properties of counterfactuals

Transitivity: $$\frac{A \mathrel{\Box\!\!\rightarrow} B, B \mathrel{\Box\!\!\rightarrow} C}{A \mathrel{\Box\!\!\rightarrow} C}$$

If I quitted teaching in the university, I would try to teach in some high school.
If I became a millionaire, I would quit teaching in the university.
If I became a millionaire, I would try to teach in some high school. (!)

Lewis' semantics of counterfactuals

Lewis' semantics is based on a *similarity relation* which orders possible worlds with respect to their similarity to the actual world:

The counterfactual conditional $A \mathrel{\Box\!\!\rightarrow} C$ is true at a world w w.r.t. a similarity ordering if (very roughly) in the closest possible word to w where A holds also C holds.

Why a fuzzy semantics for counterfactuals?

Lewis' semantics is based on the notion of similarity of possible worlds

Similarity relations are prominently studied in fuzzy mathematics (formalized as axiomatic theories over fuzzy logic)

$\Rightarrow$ Let us see if fuzzy logic can provide a viable semantics for counterfactuals

Advantages and disadvantages

Advantages

- Automatic accommodation of gradual counterfactuals
  "If ants were *large,* they would be *heavy.*"
- Accommodation of graduality of counterfactuals
  (some counterfactual conditionals seem to hold
  to larger degrees than others)
  "If ants were *large,* they would be *heavy*" vs.
  "If ants were *large,* they would *rule the earth*"
- Standard fuzzy handling of the similarity of worlds

Disadvantages

- Needs non-classical logic for semantic reasoning
  (but a well-developed one $\Rightarrow$ a low cost for experts)

Similarity relations = fuzzy equivalence relations

Axioms: Sxx, $Sxy \to Syx$, $Sxy \,\&\, Syz \to Sxz$

(interpreted in fuzzy logic!)

Notice: Similarities are *transitive* (in the sense of fuzzy logic), but avoid Poincaré's paradox:

$$x_1 \approx x_2 \approx x_3 \approx \cdots \approx x_n, \text{ though } x_1 \not\approx x_n,$$

since the degree of $x_1 \approx x_n$ can decrease with n,

due to the non-idempotent $\&$ of fuzzy logic

Ordering of worlds by similarity

Σxy ... the world x is similar to the world y

$x \preccurlyeq_w y$ ... x is *more or roughly as* similar to w as y

Define: $x \preccurlyeq_w y \ \equiv\ \Sigma wy \lesssim \Sigma wx$

The closest A-worlds: $\mathrm{Min}_{\preccurlyeq_w} A = \{x \mid x \in A \wedge (\forall a \in A)(x \preccurlyeq_w a)\}$

(the properties of minima in fuzzy orderings are well known)

Define: $\|A \mathrel{\Box\!\!\rightarrow} B\|_w \ \equiv\ (\mathrm{Min}_{\preccurlyeq_w} A) \subseteq B$

... the closest A-worlds are B-worlds (fuzzily!)

Properties of fuzzy counterfactuals

Non-triviality: $(A \mathrel{\Box\!\!\rightarrow} B) = 1$ for all B only if $A = \emptyset$

Non-desirable properties are invalid:

$\not\models (A \mathrel{\Box\!\!\rightarrow} B) \mathbin{\&} (B \mathrel{\Box\!\!\rightarrow} C) \rightarrow (A \mathrel{\Box\!\!\rightarrow} C)$

$\not\models (A \mathrel{\Box\!\!\rightarrow} C) \rightarrow (A \mathbin{\&} B \mathrel{\Box\!\!\rightarrow} C)$

$\not\models (A \mathrel{\Box\!\!\rightarrow} C) \rightarrow (\neg C \mathrel{\Box\!\!\rightarrow} \neg A)$

Desirable properties are valid, eg:

$\models \Box(A \rightarrow B) \;\rightarrow\; (A \mathrel{\Box\!\!\rightarrow} B) \;\rightarrow\; (A \rightarrow B)$

+ many more theorems on $\Box\!\!\rightarrow$ easily derivable in higher-order fuzzy logic

However, some of Lewis' tautologies only hold for full degrees