Complexity in BL

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A decision problem is a $P \subseteq \Sigma^*$

Some basic decision problems in propositional logic:
- theoremhood of a $L$ (tautologousness/satisfiability in a class of algebras $\mathbb{K}$)
- the relation of provability from finite sets of premises in $L$
- ...

For a given (decidable) decision problem $P \subseteq \Sigma^*$, find a (nondeterministic) algorithm (i.e., a Turing machine) deciding $P$ with suitable bounds on resources:
- time (number of steps)
- space (number of working tape fields)

as a function of input length.

Thus we investigate the worst-case complexity of problems.
Polynomial-time reducibility

$P_1 \subseteq \Sigma_1^*$, $P_2 \subseteq \Sigma_2^*$ decision problems.

Then $P_1 \preceq_P P_2$ iff there is a polynomial-time-computable function $f : \Sigma_1^* \rightarrow \Sigma_2^*$ such that $P_1 = \{x \in \Sigma_1^* \mid f(x) \in P_2\}$.

$\preceq_P$ is a preorder with equivalence $\approx_P$

If $P_1 \preceq_P P_2$ then

a) complexity of $P_1$ is a lower bound on complexity of $P_2$

b) complexity of $P_2$ is an upper bound on complexity of $P_1$

We know less than we would like to about $\preceq_P$

but we can prove positive results about $\approx_P$
By a logic $L$ we mean a substitution-invariant consequence relation.

We consider semilinear logics: $BL$, $L$, $G$, $\Pi$, $\ldots$, their language fragments and expansions.

Complexity results for these logics are obtained through completeness w.r.t. some classes of algebras whose structure is known.

Problem: how to tackle weaker semilinear logics, such as MTL?
Let $A, C \subseteq \Sigma^*$ be two decision problems. Then the problem $A \cap C$ can be viewed as the $C$-fragment of $A$.

Then, if $A$ is a decision problem and $A'$ is the $C$-fragment of $A$ for some $C \in \mathcal{P}$, then $A' \preceq_P A$.

This definition encompasses:

- ‘language fragments’, given by restricting the propositional language (e.g., the →-fragment)
- fragments of algebraic theories, given by restricting conditions on first order algebraic formulas (e.g., universal fragment)
Some decision problems

We assume the language and strength of FL_{ew}.

**Definition**

Let $\mathcal{L}$ be a language and $\mathcal{L}$ a logic in the language $\mathcal{L}$. We denote

- $\text{THM}(\mathcal{L}) = \{ \varphi \in \text{Fm}_\mathcal{L} \mid \vdash_\mathcal{L} \varphi \}$ (theorems of $\mathcal{L}$)
- $\text{PROV}(\mathcal{L}) = \{ \langle T, \varphi \rangle \in \mathcal{P}_\text{fin}(\text{Fm}_\mathcal{L}) \times \text{Fm}_\mathcal{L} \mid T \vdash_\mathcal{L} \varphi \}$ (finite provability in $\mathcal{L}$)
- $\text{TAUT}(\mathcal{A}) = \{ \varphi \in \text{Fm}_\mathcal{L} \mid \forall e \in \text{Val}(\mathcal{A})(e(\varphi) = 1^\mathcal{A}) \}$ (tautologies of $\mathcal{A}$)
- $\text{CONS}(\mathcal{A}) = \{ \langle T, \varphi \rangle \in \mathcal{P}_\text{fin}(\text{Fm}_\mathcal{L}) \times \text{Fm}_\mathcal{L} \mid T \models^\mathcal{A} \varphi \}$ (finite consequence in $\mathcal{A}$)
- $\text{TAUT}(\mathcal{K}) = \bigcap_{\mathcal{A} \in \mathcal{K}} \text{TAUT}(\mathcal{A})$ (tautologies of $\mathcal{K}$)
- $\text{CONS}(\mathcal{K}) = \bigcap_{\mathcal{A} \in \mathcal{K}} \text{CONS}(\mathcal{A})$ (finite consequence in $\mathcal{K}$)
Some (more) decision problems

Definition

Let \( \mathcal{L} \) be a language, \( \mathbb{K} \) a class of \( \mathcal{L} \)-algebras. We write

1. \( \text{Th}_{\text{Eq}}(\mathbb{K}) \) for the equational theory of \( \mathbb{K} \), i.e., the set of universally quantified \( \mathcal{L} \)-identities valid in \( \mathbb{K} \);

2. \( \text{Th}_{\text{QEq}}(\mathbb{K}) \) for the quasiequational theory of \( \mathbb{K} \), i.e., the set of universally quantified \( \mathcal{L} \)-quasiidentities valid in \( \mathbb{K} \);

3. \( \text{Th}_{\forall}(\mathbb{K}) \) for the universal theory of \( \mathbb{K} \), i.e., the set of universally quantified open \( \mathcal{L} \)-formulas valid in \( \mathbb{K} \);

4. \( \text{Th}_{\exists}(\mathbb{K}) \) for the existential theory of \( \mathbb{K} \), i.e., the set of existentially quantified open \( \mathcal{L} \)-formulas valid in some \( \mathbb{A} \in \mathbb{K} \);

5. \( \text{Th}(\mathbb{K}) \) for the first-order theory of \( \mathbb{K} \).
Let $\mathcal{L}$ be a language, $L$ a logic in the language $\mathcal{L}$, and $K$ a class of $\mathcal{L}$-algebras. Then

1. $\text{THM}(L) \preceq_{\text{P}} \text{PROV}(L)$; if $L$ enjoys the deduction theorem, then $\text{THM}(L) \approx_{\text{P}} \text{PROV}(L)$;
2. $\text{Th}_{\text{Eq}}(K) \preceq_{\text{P}} \text{Th}_{\text{Eq}}(K) \preceq_{\text{P}} \text{Th}_{\forall}(K) \preceq_{\text{P}} \text{Th}(K)$ and $\text{Th}_{\exists}(K) \preceq_{\text{P}} \text{Th}(K)$;
3. $\text{TAUT}(K) \approx_{\text{P}} \text{Th}_{\text{Eq}}(K)$ and $\text{CONS}(K) \approx_{\text{P}} \text{Th}_{\text{Eq}}(K)$;
4. $\text{Th}_{\forall}(K) \approx_{\text{P}} \text{Th}_{\exists}(K)$.

Moreover, if $L$ is (finitely strongly) complete w.r.t. $K$, then $\text{THM}(L) = \text{TAUT}(K)$ (and $\text{PROV}(L) = \text{CONS}(K)$).
(Many particular results.) A suitable general result is:

**Theorem**

*For any consistent substructural logic* $L$, $\text{THM}(L)$ *is coNP-hard. If* $L$ *moreover has the disjunction property, then* $\text{THM}(L)$ *is PSPACE-hard.*

This result need not generally hold for fragments. For example, while theoremhood in FL is PSPACE-complete, theoremhood in the $\{., \rightarrow\}$-fragment of FL is NP-complete.

M. Pentus: Lambek calculus is NP-complete. TCS 357, 2006.
Definable connectives

Example:
BL is usually considered in the language \{ \&, \rightarrow, 0 \}.
\( \& \) and \( \lor \) are definable in BL:
\( \varphi \land \psi \) is \( \varphi \& (\varphi \rightarrow \psi) \) and
\( \varphi \lor \psi \) is \( ((\varphi \rightarrow \psi) \rightarrow \psi) \land ((\psi \rightarrow \varphi) \rightarrow \varphi) \)
Routine application of these translations may blow up the formula size.

Lemma

Let \( K \) be a class of \( FL_{ew} \)-algebras. Assume \( c \in \mathcal{L} \) is term-definable in \( K \),
c is not among \{\&, \rightarrow\}, and let \( K^{c_{-}} \) be the class of \( \mathcal{L} \setminus \{c\} \)-reducts of
\( K \). Then \( \text{TAUT}(K) \approx_{P} \text{TAUT}(K^{c_{-}}) \).
Łukasiewicz logic

Łukasiewicz logic $\mathbb{L}$ is the involutive extension of BL. It enjoys finite strong completeness w.r.t. $[0, 1]_\mathbb{L}$. Complexity results are obtained by investigating fragments of the theory of $[0, 1]_\mathbb{L}$.

**Theorem**

$\text{Th}_\forall([0, 1]_\mathbb{L})$ is coNP-complete.

Hence also $\text{THM}(\mathbb{L})$ and $\text{PROV}(\mathbb{L})$ are coNP-complete.

D. Mundici: Satisfiability in many-valued sentential logic is NP-complete. TCS 52(1-2), 1987.
Łukasiewicz logic—membership in coNP

Proof: show $\text{Th}_3([0,1]_L) \leq_P \text{INEQ}$.
where $\text{INEQ}$ is the problem to decide, for a Boolean combination of linear inequalities of the form $\sum_{i=1}^n a_i x_i \leq b$, where $a_i$, $b$ are rationals, whether it holds in the reals.
(The $\text{INEQ}$ problem is NP-complete.)

Lemma

For each $x, y, z \in [0, 1]$ the following hold in the reals:

1. $x \ast_L y = z$ iff
   $((x + y - 1 \geq 0) \land (z = x + y - 1)) \lor ((x + y - 1 < 0) \land (z = 0))$
2. $x \rightarrow_L y = z$ iff $((x \leq y) \land (z = 1)) \lor ((x > y) \land (z = 1 - x + y))$

Given an existential sentence $\Phi$:

- eliminate compound terms in $\Phi$, remove quantifiers
- add boundary conditions
- replace atomic formulas in $\Phi$, using the above lemma
- pass the resulting formula to the algorithm for $\text{INEQ}$.
Implicational fragment of $\text{Th}_\forall([0, 1]_L)$

**Theorem**

$\text{Th}_{\text{Eq}}\{\rightarrow\}([0, 1]_L)$ is coNP-complete.

**Corollary**

*Also all those language fragments of $\text{Th}_\forall([0, 1]_L)$ that contain $\rightarrow$ are coNP-complete.*

Sketch of theorem proof: show

$$\text{Th}_{\text{Eq}}\{\rightarrow, 0\}([0, 1]_L) \preceq_p \text{Th}_{\text{Eq}}\{\rightarrow\}([0, 1]_L)$$

Implicational fragment of \( \text{Th}_\forall([0, 1]_L) \)—cont.

Define translation on \( \{\to, 0\}\)-terms, using a new variable \( p \):

\[
0^\circ \text{ is } p \\
\varphi^\circ \text{ is } (\varphi \to p) \to p \text{ for } \varphi \text{ atomic} \\
(\varphi \to \psi)^\circ \text{ is } \varphi^\circ \to \psi^\circ
\]

**Lemma**

Let \( A \) be an MV-chain, \( a \in A \). Define \( [a, 1] = \{x \in A \mid a \leq x\} \),
\[
x \ast_a y = \max(x \ast y, a), \ x \to_a y = x \to y. \text{ Then} \\
A[a, 1] = \{[a, 1], \ast_a, \to_a, a, 1\} \text{ is an MV-chain.}
\]

Let \( \varphi, \psi \) be \( \{\to, 0\}\)-terms.

If \( \varphi \approx \psi \) does not hold in \([0, 1]_L\), then neither does \( \varphi^\circ \approx \psi^\circ \).
(Clear: consider \( e(p) = 0 \).)

If \( \varphi^\circ \approx \psi^\circ \) does not hold in \([0, 1]_L\), then neither does \( \varphi \approx \psi \).
(Indeed, if \( e(\varphi^\circ) \neq e(\psi^\circ) \), then \( \varphi^\circ \approx \psi^\circ \) is not valid in the MV-algebra \([e(p), 1]\) within \([0, 1]_L\). Use partial embedding into \([0, 1]_L\).)
Hájek’s basic logic

BL is the prelinear, divisible extension of $FL_{ew}$. It is also known as the logic of standard BL-algebras, i.e., algebras $[0, 1]_* = \langle [0, 1], *, \rightarrow, \land, \lor, 0, 1 \rangle$, where $*$ is a continuous t-norm, $\rightarrow$ is the residuum, and $\land, \lor$ are given by natural order of reals.

Theorem

THM(BL) and PROV(BL) are coNP-complete.

Proved using ordinal-sum decomposition of standard BL-algebras into $\mathfrak{L}$, $\mathfrak{G}$, $\Pi$, and trivial components. In particular,

Lemma

If $\varphi(x_1, \ldots, x_k)$ is not a tautology of all standard BL-algebras, then it is not a tautology of a standard algebra with at most $k + 1$ components.

Theorem

Let $A$ be a standard BL-algebra and $L$ the logic of $A$. Then $L$ is coNP-complete.

Proved using the fact that $\text{TAUT}(A)$ can be encoded by a finite word in the (finite) alphabet $\omega L, \omega \Pi, L, G, \Pi,$ based on the ordinal-sum representation for $A$.

Assume $K$ is a nonempty finite class of standard BL-algebras.

$\text{TAUT}(K) = \bigcap_{A \in K} \text{TAUT}(A)$ and $\text{CONS}(K) = \bigcap_{A \in K} \text{CONS}(A)$.

Corollary

If $K$ is a nonempty, finite class of standard BL-algebras and $L$ is the logic of $K$, then $L$ is coNP-complete.

Theorem

Let $K$ be a class of standard BL-algebras. Then there is a finite class $L$ of standard BL-algebras such that $\text{Var}(K) = \text{Var}(L)$.

Proved by (excluding classes $K$ that generate BL or SBL and) analyzing the preorder given by

$$A \leq_{\text{V}} B \text{ iff } \text{Var}(A) \subseteq \text{Var}(B)$$

Corollary

Logics of classes of standard BL-algebras are coNP-complete.

MTL is the prelinear extension of $\text{FL}_{\text{ew}}$ (i.e., ‘BL without divisibility’). MTL is also the logic of left-continuous t-norms and their residua; in particular, MTL is strongly complete w.r.t. this semantics.

MTL is decidable because the variety $\text{MTL}$ enjoys the finite embeddability property.

Problem: what is the complexity of MTL? What is the complexity of $\text{MTL}\{\rightarrow\}$?