Complexity in BL

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Agenda

A decision problem is a $P \subseteq \Sigma^*$

Some basic decision problems in propositional logic:

- theoremhood of a L (tautologousness/satisfiability in a class of algebras $\mathbb{K}$)
- the relation of provability from finite sets of premises in L
- ...

For a given (decidable) decision problem $P \subseteq \Sigma^*$,
find a (nondeterministic) algorithm (i.e., a Turing machine) deciding P with suitable bounds on resources:

- time (number of steps)
- space (number of working tape fields)

as a function of input length.

Thus we investigate the worst-case complexity of problems.

Polynomial-time reducibility

$P_1 \subseteq \Sigma_1^*$, $P_2 \subseteq \Sigma_2^*$ decision problems.

Then $P_1 \preceq_{\mathbf{P}} P_2$ iff there is a polynomial-time-computable function $f\colon \Sigma_1^* \to \Sigma_2^*$ such that $P_1 = \{x \in \Sigma_1^* \mid f(x) \in P_2\}$.

$\preceq_{\mathbf{P}}$ is a preorder with equivalence $\approx_{\mathbf{P}}$

If $P_1 \preceq_{\mathbf{P}} P_2$ then
a) complexity of P_1 is a lower bound on complexity of P_2
b) complexity of P_2 is an upper bound on complexity of P_1

We know less than we would like to about $\preceq_{\mathbf{P}}$
but we can prove positive results about $\approx_{\mathbf{P}}$

Logics

By a logic L we mean a substitution-invariant consequence relation.

We consider semilinear logics:
BL, $\mathrm{Ł}$, G, Π, . . . , their language fragments and expansions

Complexity results for these logics are obtained through completeness w.r.t. some classes of algebras whose structure is known.

Problem: how to tackle weaker semilinear logics, such as MTL?

Fragments

Let $A, C \subseteq \Sigma^*$ be two decision problems.
Then the problem $A \cap C$ can be viewed as the C-fragment of A.

Then, if A is a decision problem and A' is the C-fragment of A for some $C \in \mathbf{P}$, then $A' \preceq_{\mathbf{P}} A$.

This definition encompasses:

- 'language fragments', given by restricting the propositional language (e.g., the $\rightarrow$-fragment)
- fragments of algebraic theories, given by restricting conditions on first order algebraic formulas (e.g., universal fragment)

Some decision problems

We assume the language and strength of $\mathrm{FL_{ew}}$.

Definition

Let $\mathcal{L}$ be a language and L a logic in the language $\mathcal{L}$. We denote

$$\mathrm{THM(L)} = \{\varphi \in \mathrm{Fm}_{\mathcal{L}} \mid \vdash_{\mathrm{L}} \varphi\} \qquad \text{(theorems of L)}$$

$$\mathrm{PROV(L)} = \{\langle T, \varphi\rangle \in \mathcal{P}_{\mathrm{fin}}(\mathrm{Fm}_{\mathcal{L}}) \times \mathrm{Fm}_{\mathcal{L}} \mid T \vdash_{\mathrm{L}} \varphi\} \qquad \text{(finite provability in L)}$$

$$\mathrm{TAUT}(\mathbf{A}) = \{\varphi \in \mathrm{Fm}_{\mathcal{L}} \mid \forall e \in \mathit{Val}(\mathbf{A})(e(\varphi) = 1^{\mathbf{A}})\} \qquad \text{(tautologies of } \mathbf{A}\text{)}$$

$$\mathrm{CONS}(\mathbf{A}) = \{\langle T, \varphi\rangle \in \mathcal{P}_{\mathrm{fin}}(\mathrm{Fm}_{\mathcal{L}}) \times \mathrm{Fm}_{\mathcal{L}} \mid T \models_{\mathbf{A}} \varphi\} \qquad \text{(finite consequence in } \mathbf{A}\text{)}$$

$$\mathrm{TAUT}(\mathbb{K}) = \bigcap_{\mathbf{A} \in \mathbb{K}} \mathrm{TAUT}(\mathbf{A}) \qquad \text{(tautologies of } \mathbb{K}\text{)}$$

$$\mathrm{CONS}(\mathbb{K}) = \bigcap_{\mathbf{A} \in \mathbb{K}} \mathrm{CONS}(\mathbf{A}) \qquad \text{(finite consequence in } \mathbb{K}\text{)}$$

Some (more) decision problems

Definition

Let $\mathcal{L}$ be a language, $\mathbb{K}$ a class of $\mathcal{L}$-algebras. We write

1. $\mathrm{Th}_{\mathrm{Eq}}(\mathbb{K})$ for the equational theory of $\mathbb{K}$, i.e., the set of universally quantified $\mathcal{L}$-identities valid in $\mathbb{K}$;
2. $\mathrm{Th}_{\mathrm{QEq}}(\mathbb{K})$ for the quasiequational theory of $\mathbb{K}$, i.e., the set of universally quantified $\mathcal{L}$-quasiidentities valid in $\mathbb{K}$;
3. $\mathrm{Th}_{\forall}(\mathbb{K})$ for the universal theory of $\mathbb{K}$, i.e., the set of universally quantified open $\mathcal{L}$-formulas valid in $\mathbb{K}$;
4. $\mathrm{Th}_{\exists}(\mathbb{K})$ for the existential theory of $\mathbb{K}$, i.e., he set of existentially quantified open $\mathcal{L}$-formulas valid in some $\mathbf{A} \in \mathbb{K}$;
5. $\mathrm{Th}(\mathbb{K})$ for the first-order theory of $\mathbb{K}$.

Some reductions

Let $\mathcal{L}$ be a language, L a logic in the language $\mathcal{L}$, and $\mathbb{K}$ a class of $\mathcal{L}$-algebras. Then

1. $\mathrm{THM}(\mathrm{L}) \preceq_{\mathbf{P}} \mathrm{PROV}(\mathrm{L})$; if L enjoys the deduction theorem, then $\mathrm{THM}(\mathrm{L}) \approx_{\mathbf{P}} \mathrm{PROV}(\mathrm{L})$;
2. $\mathrm{Th}_{\mathrm{Eq}}(\mathbb{K}) \preceq_{\mathbf{P}} \mathrm{Th}_{\mathrm{QEq}}(\mathbb{K}) \preceq_{\mathbf{P}} \mathrm{Th}_{\forall}(\mathbb{K}) \preceq_{\mathbf{P}} \mathrm{Th}(\mathbb{K})$ and $\mathrm{Th}_{\exists}(\mathbb{K}) \preceq_{\mathbf{P}} \mathrm{Th}(\mathbb{K})$;
3. $\mathrm{TAUT}(\mathbb{K}) \approx_{\mathbf{P}} \mathrm{Th}_{\mathrm{Eq}}(\mathbb{K})$ and $\mathrm{CONS}(\mathbb{K}) \approx_{\mathbf{P}} \mathrm{Th}_{\mathrm{QEq}}(\mathbb{K})$;
4. $\mathrm{Th}_{\forall}(\mathbb{K}) \approx_{\mathbf{P}} \overline{\mathrm{Th}_{\exists}}(\mathbb{K})$.

Moreover, if L is (finitely strongly) complete w.r.t. $\mathbb{K}$, then $\mathrm{THM}(\mathrm{L}) = \mathrm{TAUT}(\mathbb{K})$ (and $\mathrm{PROV}(\mathrm{L}) = \mathrm{CONS}(\mathbb{K})$).

Lower bounds

(Many particular results.) A suitable general result is:

Theorem

For any consistent substructural logic L, $\mathrm{THM}(\mathrm{L})$ *is* **coNP**-*hard. If* L *moreover has the disjunction property, then* $\mathrm{THM}(\mathrm{L})$ *is* **PSPACE**-*hard.*

This result need not generally hold for fragments.
For example, while theoremhood in FL is **PSPACE**-complete, theoremhood in the $\{., \rightarrow\}$-fragment of FL is **NP**-complete.

R. Horčík, K. Terui: Disjunction property and complexity of substructural logics. TCS 412(31), 2011.
M. Pentus: Lambek calculus is NP-complete. TCS 357, 2006.

Definable connectives

Example:
BL is usually considered in the language $\{\,\&\,, \to, 0\}$.
$\wedge$ and $\vee$ are definable in BL:
$\varphi \wedge \psi$ is $\varphi \,\&\, (\varphi \to \psi)$ and
$\varphi \vee \psi$ is $((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$
Routine application of these translations may blow up the formula size.

Lemma

Let $\mathbb{K}$ be a class of $\mathrm{FL_{ew}}$-algebras. Assume $c \in \mathcal{L}$ is term-definable in $\mathbb{K}$, c is not among $\{\&, \to\}$, and let $\mathbb{K}^{c^-}$ be the class of $\mathcal{L} \setminus \{c\}$-reducts of $\mathbb{K}$. Then $\mathrm{TAUT}(\mathbb{K}) \approx_{\mathbf{P}} \mathrm{TAUT}(\mathbb{K}^{c^-})$.

Łukasiewicz logic

Lukasiewicz logic Ł is the involutive extension of BL.
It enjoys finite strong completeness w.r.t. $[0,1]_{\text{Ł}}$.
Complexity results are obtained by investigating fragments of the theory of $[0,1]_{\text{Ł}}$.

Theorem

$\mathrm{Th}_{\forall}([0,1]_{\text{Ł}})$ *is* **coNP**-*complete.*

Hence also $\mathrm{THM}(\text{Ł})$ and $\mathrm{PROV}(\text{Ł})$ are **coNP**-complete.

D. Mundici: Satisfiability in many-valued sentential logic is NP-complete. TCS 52(1-2), 1987.
R. Hähnle: Many-valued logic and mixed integer programming. Annals of Mathematics and Artificial Intelligence 12(3-4), 1994.

Łukasiewicz logic—membership in coNP

Proof: show $\mathrm{Th}_\exists([0,1]_Ł) \preceq_{\mathbf{P}} \mathrm{INEQ}$.
where INEQ is the problem to decide, for a Boolean combination of linear inequalities of the form $\Sigma_{i=1}^{n} a_i x_i \leq b$, where a_i, b are rationals, whether it holds in the reals.
(The INEQ problem is NP-complete.)

Lemma

For each $x, y, z \in [0,1]$ the following hold in the reals:

1. $x *_Ł y = z$ *iff*
   $((x + y - 1 \geq 0) \wedge (z = x + y - 1)) \vee ((x + y - 1 < 0) \wedge (z = 0))$
2. $x \rightarrow_Ł y = z$ *iff* $((x \leq y) \wedge (z = 1)) \vee ((x > y) \wedge (z = 1 - x + y))$

Given an existential sentence Φ:

- eliminate compound terms in Φ, remove quantifiers
- add boundary conditions
- replace atomic formulas in Φ, using the above lemma
- pass the resulting formula to the algorithm for INEQ.

Implicational fragment of $\mathrm{Th}_\forall([0,1]_Ł)$

Theorem

$\mathrm{Th}_{\mathrm{Eq}}{}^{\{\to\}}([0,1]_Ł)$ *is* **coNP**-*complete.*

Corollary

Also all those language fragments of $\mathrm{Th}_\forall([0,1]_Ł)$ *that contain* $\to$ *are* **coNP**-*complete.*

Sketch of theorem proof: show

$$\mathrm{Th}_{\mathrm{Eq}}{}^{\{\to,0\}}([0,1]_Ł) \preceq_{\mathbf{P}} \mathrm{Th}_{\mathrm{Eq}}{}^{\{\to\}}([0,1]_Ł)$$

F. Esteva, L. Godo, P. Hájek and F. Montagna: Hoops and Fuzzy Logic, J. Logic Comput. 13(4), 2003.

Implicational fragment of $\mathrm{Th}_\forall([0,1]_Ł)$—cont.

Define translation on $\{\rightarrow, 0\}$-terms, using a new variable p:

$$\begin{aligned} 0^\circ &\text{ is } p \\ \varphi^\circ &\text{ is } (\varphi \rightarrow p) \rightarrow p \text{ for } \varphi \text{ atomic} \\ (\varphi \rightarrow \psi)^\circ &\text{ is } \varphi^\circ \rightarrow \psi^\circ \end{aligned}$$

Lemma

Let $\mathbf{A}$ *be an MV-chain,* $a \in \mathbf{A}$*. Define* $[a,1] = \{x \in \mathbf{A} \mid a \leq x\}$*,* $x *_a y = \max(x * y, a)$*,* $x \rightarrow_a y = x \rightarrow y$*. Then* $A[a,1] = \{[a,1], *_a, \rightarrow_a, a, 1\}$ *is an MV-chain.*

Let φ, ψ be $\{\rightarrow, 0\}$-terms.

If $\varphi \approx \psi$ does not hold in $[0,1]_Ł$, then neither does $\varphi^\circ \approx \psi^\circ$.
(Clear: consider $e(p) = 0$.)

If $\varphi^\circ \approx \psi^\circ$ does not hold in $[0,1]_Ł$, then neither does $\varphi \approx \psi$.
(Indeed, if $e(\varphi^\circ) \neq e(\psi^\circ)$, then $\varphi^\circ \approx \psi^\circ$ is not valid in the MV-algebra $[e(p), 1]$ within $[0,1]_Ł$. Use partial embedding into $[0,1]_Ł$.)

Hájek's basic logic

BL is the prelinear, divisible extension of $\mathrm{FL_{ew}}$.
It is also known as the logic of standard BL-algebras, i.e., algebras $[0,1]_* = \langle [0,1], *, \rightarrow, \wedge, \vee, 0, 1 \rangle$, where $*$ is a continuous t-norm, $\rightarrow$ is the residuum, and $\wedge$, $\vee$ are given by natural order of reals.

Theorem

$\mathrm{THM(BL)}$ *and* $\mathrm{PROV(BL)}$ *are* **coNP**-*complete.*

Proved using ordinal-sum decomposition of standard BL-algebras into $\mathrm{Ł}$, G, Π, and trivial components. In particular,

Lemma

If $\varphi(x_1, \ldots, x_k)$ is not a tautology of all standard BL-algebras, then it is not a tautology of a standard algebra with at most $k+1$ components.

P. Hájek: Metamathematics of Fuzzy Logic. Kluwer AP, 1998.
M. Baaz, P. Hájek, F. Montagna, H. Veith: Complexity of t-tautologies. APAL 113(1-3), 2002.

Logics of standard BL-algebras

Theorem

Let **A** *be a standard BL-algebra and* L *the logic of* **A**. *Then* L *is* **coNP**-*complete.*

Proved using the fact that $\mathrm{TAUT}(A)$
can be encoded by a finite word in the (finite) alphabet
ωŁ, $\omega\Pi$, Ł, G, Π,
based on the ordinal-sum representation for **A**.

Assume $\mathbb{K}$ is a nonempty finite class of standard BL-algebras.
$\mathrm{TAUT}(\mathbb{K}) = \bigcap_{\mathbf{A}\in\mathbb{K}} \mathrm{TAUT}(\mathbf{A})$ and
$\mathrm{CONS}(\mathbb{K}) = \bigcap_{\mathbf{A}\in\mathbb{K}} \mathrm{CONS}(\mathbf{A})$.

Corollary

If $\mathbb{K}$ *is a nonempty, finite class of standard* BL-*algebras and* L *is the logic of* $\mathbb{K}$, *then* L *is* **coNP**-*complete.*

ZH: A note on the complexity of propositional tautologies of individual t-algebras. Neural Network World 12(5),2002.

Classes of standard BL-algebras

Theorem

Let $\mathbb{K}$ be a class of standard BL-algebras. Then there is a finite class $\mathbb{L}$ of standard BL-algebras such that $\mathbf{Var}(\mathbb{K}) = \mathbf{Var}(\mathbb{L})$.

Proved by (excluding classes $\mathbb{K}$ that generate $\mathbb{BL}$ or $\mathbb{SBL}$ and) analyzing the preorder given by

$$\mathbf{A} \leq_V \mathbf{B} \text{ iff } \mathbf{Var}(\mathbf{A}) \subseteq \mathbf{Var}(\mathbf{B})$$

Corollary

Logics of classes of standard BL-algebras are **coNP**-*complete.*

ZH: Varieties generated by standard BL-algebras. Preprint.

Monoidal t-norm logic

MTL is the prelinear extension of $\mathrm{FL_{ew}}$ (i.e., 'BL without divisibility').
MTL is also the logic of left-continuous t-norms and their residua;
in particular, MTL is strongly complete w.r.t. this semantics.

MTL is decidable because the variety $\mathbb{MTL}$ enjoys the finite embeddability property.

Problem: what is the complexity of MTL? What is the complexity of $\mathrm{MTL}^{\{\rightarrow\}}$?

F. Esteva, L. Godo: Monoidal t-norm based logic. FSS 124(3), 2001.
S. Jenei, F. Montagna: A proof of standard completeness for Esteva and Godo's logic MTL. Studia Logica 70(2), 2002.
W. J. Blok, C. J. van Alten: The finite embeddability property for residuated lattices, pocrims and BCK-algebras. Algebra Univers. 48, 2000.