A case for constants

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Example: benchmarking with atoms

Assume Łukasiewicz logic (Ł).

$$T_1 := \{\mathrm{Tall}(\mathrm{John}) \equiv \neg\mathrm{Tall}(\mathrm{John})\}$$

The theory T_1 is consistent. In a model of T_1 over $[0,1]_{Ł}$,
the atomic sentence Tall(John) always has the value $^1/_2$.

$$T_2 := \{\mathrm{Tall}(\mathrm{John}) \equiv (\neg\mathrm{Tall}(\mathrm{John}))^{n-1}\}$$

Now Tall(John) always has the value $^1/_n$.

The equation $x \approx (\neg x)^{n-1}$ has a unique solution in $[0,1]_{Ł}$, namely $^1/_n$.

[Torrens: Cyclic Elements in MV-algebras and Post algebras. MLQ, 1994]

$$T_3 := T_2 \cup \{\mathrm{Tall}(\mathrm{Ben}) \equiv k \cdot \mathrm{Tall}(\mathrm{John})\}$$

Now Tall(John) always has the value $^k/_n$.

- implicit definitions of rational numbers
- extralogical assumptions (a theory) left of the turnstile (!)
- recognition (?)

Outline

Logics of importance in this talk:
Łukasiewicz (infinite-valued) logic Ł (mainly); product logic Π (for contrast);
expansions with constants intended for the rationals.

- Expressivity;
- Goguen's *Logic of inexact concepts*;
- Pavelka's results and RPL;
- implicit definitions in Ł;
- complexity;
- admissibility of rules;
- graded syntax;
- conclusions.

Propositional language and axiomatics

Essentially the language of FL_{ew}: $\{\odot, \to, \wedge, \vee, 0, 1\}$.

Definable connectives:

$\neg\alpha$ is $\alpha \to 0$
$\alpha \oplus \beta$ is $\neg(\neg\alpha \vee \neg\beta)$
$\alpha \equiv \beta$ is $\alpha \to \beta \wedge \beta \to \alpha$
etc. Plus definability of FL_{ew}-symbols.

Axioms of Łukasiewicz logic:

(A1) $\varphi \to (\psi \to \varphi)$

(A2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

(A3) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

(A4) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$

The **deduction rule** is modus ponens.

The logic is, formally, the **provability relation** $\vdash_Ł$.

[Łukasiewicz and Tarski: Untersuchungen über den Aussagenkalkül. 1930]

Product logic Π can be obtained as extension of Hájek's BL.

MV-algebras provide the equivalent algebraic semantics of Ł.
[Chang 1958, 1959]

Real-valued interpretation:
the standard MV-algebra $[0,1]_{Ł} := \langle [0,1], \rightarrow^{Ł}, \neg^{Ł} \rangle$, where
- $x \rightarrow^{Ł} y$ is $\min(1, 1-x+y)$ and
- $\neg^{Ł} x$ is $1-x$, for each $x, y \in [0,1]$.

Finite strong standard completeness:

Theorem [Hay 1963]

If $T \cup \{\alpha\}$ is a finite set of formulas, then $T \vdash_{Ł} \alpha$ iff $T \models_{[0,1]_{Ł}} \alpha$.

Łukasiewicz logic – definable functions

For α a propositional formula, let f_α be its interpretation in $[0,1]_Ł$.

A function $f\colon [0,1]^n \to [0,1]$ is a McNaughton function if

- f is continuous
- f is piecewise linear:
  there are finitely many linear polynomials $\{p_i\}_{i\in I}$, with $p_i(\bar{x}) = \Sigma_{j=1}^{n} a_{ij}\, x_j + b_i$, such that for any $\bar{x} \in [0,1]^n$ there is an $i \in I$ with $f(\bar{x}) = p_i(\bar{x})$
- the polynomials p_i have integer coefficients $\bar{a}_i, b_i$.

Theorem [McNaughton 1951]

Term-definable functions of $[0,1]_Ł$ *coincide with McNaughton functions.*

[McNaughton. A theorem about infinite-valued sentential logic. JSL, 1951]

It is nontrivial whether or not a given formula α defines a given function f.
Consider the TAUT problem.

Satisfiability and validity in $[0,1]_Ł$

$$\mathrm{SAT}([0,1]_Ł) := \{\varphi \mid [0,1]_Ł \models \exists \bar{x}(\varphi(\bar{x}) \approx 1)\}$$
$$\mathrm{TAUT}([0,1]_Ł) := \{\varphi \mid [0,1]_Ł \models \forall \bar{x}(\varphi(\bar{x}) \approx 1)\}$$

In particular, if f_φ is subnormal, then $\varphi \notin \mathrm{SAT}([0,1]_Ł)$.

Task: determine $\max f_\varphi$ (for SAT) or $\min f_\varphi$ (for TAUT).
(By McNaughton's theorem, this is well defined.)

The max/min is attained at a vertex of the polyhedral complex given by $\mathrm{dom}(f_\varphi)$).
[Mundici: Satisfiability in many-valued sentential logic is NP-complete. TCS, 1987]

NB. These two ways of rendering satisfiability **coincide in classical logic**.

The validity degree of φ under T (finite or infinite) is

$$\|\varphi\|_T := \inf\{f_\varphi \mid f_\psi = 1\,\forall \psi \in T\},$$

For $T = \{\psi_1, \ldots, \psi_k\}$, write $\tau = \psi_1 \odot \cdots \odot \psi_k$; then $\|\varphi\|_\tau = \min\{f_\varphi \mid f_\tau = 1\}$.

Expressivity

We have learned a rational number ${}^{p}/_{q}$, with q bounded by $\lfloor (\frac{\sharp\varphi + \sharp\tau}{n})^n \rfloor$ where n is the number of variables in φ and $\sharp\varphi$ is the length.

[Aguzzoli Ciabattoni 2002][Aguzzoli 2006]

By McNaughton theorem, unless ${}^{p}/_{q}$ is 0 or 1,

no formula in the language of Łukasiewicz logic defines ${}^{p}/_{q}$.

Consider a set of constants $\mathcal{C} = \{\bar{r} : r \in [0,1] \cap \mathbb{Q}\}$
Expand the language with constants in $\mathcal{C}$.

For the moment, let's fix the interpretation of $\bar{r}$ as r.

Then ${}^{p}/_{q} \leq \min f_\alpha$ if and only if $\overline{{}^{p}/_{q}} \to \alpha \in \mathrm{TAUT}([0,1]_{Ł})$.

And more, based on Goguen's challenge of 1969.

Even more expressivity

The logic $Ł\Pi^1\!/_2$ expands Łukasiewicz logic Ł with
the product connectives $\otimes$ and $\to_\Pi$ and the constant $^1\!/_2$.

By standard completeness of $Ł\Pi^1\!/_2$, this logic extends conservatively
Łukasiewicz logic, Gödel Logic, and product logic.

In the standard $Ł\Pi^1\!/_2$-algebra, all constant rational functions are term-definable.

Let $^k\!/_l \in \mathbb{Q} \cap [0,1]$. Choose $n \in \mathbb{N}$ least s.t. $l < 2^n$. Let $^1\!/_{2^n} := \underbrace{^1\!/_2 \otimes \ldots \otimes {}^1\!/_2}_{n \text{ times}}$.

Then define $^k\!/_{2^n} := \underbrace{^1\!/_{2^n} \oplus \ldots \oplus {}^1\!/_{2^n}}_{k \text{ times}}$ and analogously for $^l\!/_{2^n}$.

Finally, define $^k\!/_l := {}^l\!/_{2^n} \to_\Pi {}^k\!/_{2^n}$.

[Esteva, Godo, Montagna: The ŁΠ and $Ł\Pi^1\!/_2$ logics: Two complete fuzzy systems joining Łukasiewicz and product logics. AML 2001]

Theorem [Marchioni, Montagna 2007]

$\mathrm{TAUT}([0,1]_{Ł\Pi^1\!/_2}) \approx_{\mathrm{P}} \mathrm{Th}_\forall(\mathbb{R})$.

We know $\mathrm{NP} \subseteq \mathrm{Th}_\exists(\mathbb{R}) \subseteq \mathrm{PSPACE}$. Complexity class $\exists\mathbb{R}$.

Goguen's essay

early contribution to philosophy of vagueness

NB. *Synthese* issue on vagueness published 1975 (Fine, Dummett, Wright, *et al.*)

"We approach philosophy as an applied mathematician might approach magnetohydrodynamics or operations research: we do not assume there is some unique best theory, much less that we know it. We give a general (but vague) method for modelling, clarifying and criticizing sufficiently well-codified 'language games'. The models are subject to the process of experimental verification and subsequent modification usual in scientific research."

"A striking feature of the theory which sets it apart from other logics is that the product of a long chain of only slightly unreliable deductions can be very unreliable."

[Goguen: Logic of inexact concepts. Synthese, 1969]

Goguen's essay – cont'd

Goguen's main contribution:

- truth values (of propositions involving inexact concepts) form an algebra
- this algebra is defined by its axioms.

Essentially a complete $\mathrm{FL_w}$-algebra, or a subvariety thereof (adding distributivity) or an expansion (adding an involutive negation), etc. His term 'closg' stands for 'complete-lattice-ordered semigroup'.

"It is not necessary to know particular fuzzy sets as exact mathematical functions to be able to make about them certain assertions of theoretical character which may have philosophical and/or practical significance.
[. . .]
We have used the axiomatic method, in the sense that our underlying assumptions, especially about L, are abstract; it can thus be ascertained to what extent our results apply to some new problem."

[J.A. Goguen: L-fuzzy sets. J. Math. Analysis and Applications, 1967]

Intermezzo I: fuzzy plurivaluationism

[Smith: Fuzzy Logic and Higher Order Vagueness. 2011]

Fuzzy plurivaluationism admits multiple fuzzy models of a discourse without aiming at an intended model.

NB. Smith cites Goguen's essay (on a side issue).

[Běhounek: Comments on Fuzzy Logic and Higher-Order Vagueness by Nicholas J.J. Smith. 2011]

Fuzzy plurivaluationism appears to be semantics for theories developed in fuzzy logic.

[Smith: Reply to Libor Běhounek's Comments on Fuzzy Logic and Higher-Order Vagueness. 2011]

Fuzzy plurivaluations not axiomatizable, not closed under isomorphism, and other distinctions.

Běhounek's ammunition was provided by Goguen.

[Cintula, Fermüller, Godo, Hájek (eds.): Understanding vagueness: logical, philosophical, and lingustic perspectives. College Publications 2011.)

Goguen on sorites

A variant of soritical series:

- 0 is a small number;
- if n is small, then so is $S(n)$;

Yet there exists a number k that is large.

[Dummett: Wang's paradox. Synthese, 1975]

Induction for vague predicates?

Even without induction, one can iterate the inductive step.
Then use speedup (Solovay).

Goguen embraces the truth degree paradigm:

Model "small numbers" as a fuzzy set on the domain of natural numbers.

How true is "if n is small, then so is $S(n)$"?

Does not claim there is a definite choice for the fuzzy set of "small numbers".

Offers a general model — monotone function tending to 0 — avoiding the paradox.

Goguen on sorites – cont'd

The proposition "n is small" gets a value in $[0, 1]$ depending on n.

- First attempt: $v(\text{“}n \text{ is small”}) = 1/2^n$

$$v(x \to y) = \begin{cases} 1 & \text{if } v(x) \leq v(y); \\ {}^{v(y)}/_{v(x)} & \text{if } v(x) > v(y). \end{cases}$$

In particular: $v(\text{“if } n \text{ is small, then so is } S(n)\text{”}) = {}^1/_2$
(Independent of n.)

- Second attempt:

The automorphisms of $[0, 1]_\Pi$ are precisely all the functions x^r (for a positive real r).

Now $v(\text{“if } n \text{ is small, then so is } S(n)\text{”}) = {}^1/_{2^{nr}}$.

If $r > 0$ is small (e.g., $r = 10^{-6}$), then

- $v(\text{“}n \text{ is small”})$ is ${}^1/_{2^{n10^{-6}}}$ (very close to 1);
- in particular, $v(\text{“}10^6 \text{ is small”})$ is ${}^1/_2$;
- $v(\text{“if } n \text{ is small, so is } S(n)\text{”}) = {}^1/_{2^r}$

Goguen's challenge

"A traditional way of develop a logical system is through its tautologies. [...] Tautologies have the advantage of independence of truth set, but no list of tautologies can encompass the entire system because we want to perform calculations with degrees of validity between 0 and 1. In this sense, the logic of inexact concepts does not have a *purely* syntactic form. Semantics, in the form of specific truth values of certain assertions, is sometimes required." [p. 365]

Goguen envisages (but does not develop) graded deduction.

A theorem is a computation such that $[A \to B] \geq a$ (for $a \in L$).

A 'better proof' may have fewer steps or give better approximations.

Pavelka's work (selection)

A fuzzy theory T is a fuzzy set on the domain F of all formulas of a language. The range is a complete lattice L, possibly with other operations.

Then also $\mathrm{Con}(T)$ is a fuzzy set of formulas, namely

$$\mathrm{Con}(T)(\alpha) \coloneqq \bigwedge_v \{v(\alpha) \mid v(\psi) \geq T(\psi)\}$$

Develops L-syntax:

- L-fuzzy set of axioms
- L-rules of inference, such as:

$$\frac{\alpha, (\alpha \to \beta)}{\beta} \left(\frac{p\ ,\ q}{p \odot q}\right) \text{ (graded modus ponens), or } \frac{\alpha}{\overline{r} \to \alpha} \left(\frac{p}{r \to p}\right) \text{ (lifting).}$$

- suitable notion of proof

Identification of suitable class of algebras: $\mathrm{FL_{ew}}$-algebras (already in Goguen).

Pavelka completeness, plus an argument why the only suitable logic is Ł.

[J. Pavelka. On fuzzy logic I, II, III. Zeit. Math. Logik Grundlagen Math., 1979]

Hájek's simplification (between Pavelka and RPL)

Pavelka uses expansion with constants for all elements of L.
Such constants (and formulas) are abstract objects.

Hájek (1995) uses graded formulas $\langle r, \varphi \rangle$ with rational grades
and rational constants (rather than real).
Valuations are still in the real unit interval $[0,1]$.

Hájek's axioms and rules:
instances of the **axioms of** Ł in grade 1;
the constant $\overline{q}$ **in grade** q, for each rational $q \in [0,1]$;
bookkeeping axioms (in grade 1);
all other formulas (of lang. with constants) in grade 0.

graded modus ponens: $\dfrac{\langle q, \varphi \rangle \qquad \langle r, \varphi \to \psi \rangle}{\langle q \odot^{Ł} r, \psi \rangle}$ for all rational $q, r \in [0,1]$

lifting: $\dfrac{\langle q, \varphi \rangle}{\langle r \to^{Ł} q, \overline{r} \to \varphi \rangle}$ for all rational $q, r \in [0,1]$

Proves Pavelka completeness.

NB. We have $\langle q, \varphi \rangle \dashv\vdash \langle 1, \overline{q} \to \varphi \rangle$
[Hájek: Fuzzy logic and arithmetical hierarchy. Fuzzy Sets and Systems, 1995]

Rational Pavelka Logic (RPL)

Expands Łukasiewicz logic Ł with

- propositional constants $\{\bar{r} \mid r \in \mathbb{Q} \cap [0,1]\}$
- bookkeeping axioms: for basic connectives $\to$ and $\neg$,

$$\overline{r} \to \overline{s} \equiv \overline{r \to^{Ł} s} \quad \text{and} \quad \neg\overline{r} \equiv \overline{\neg^{Ł} r}$$

Then one derives bookkeeping for all connectives.

Derivable rule in RPL:

$$\frac{\overline{r} \to \varphi \qquad \overline{s} \to (\varphi \to \psi)}{\overline{r \odot s} \to \psi}$$

Hájek (1998): "A graded formula is a pair $\langle r, \varphi \rangle$ where φ is a formula and r a rational element of $[0,1]$; it is just another notation for the formula $\bar{r} \to \varphi$."

[Hájek: Metamathematics of fuzzy logic, 1998]

Rational Pavelka Logic (RPL) – semantics

Standard semantics: $[0,1]_{\text{Ł}}^{Q}$ is an expansion of $[0,1]_{\text{Ł}}$ with canonical interpretation of constants: $v(\overline{r}) = r$.

$Q_{\text{Ł}}^{Q}$ is the subalgebra on the rationals.

The canonical interpretation of constants is the only sound one in $[0,1]_{\text{Ł}}$.

[Esteva, Gispert, Godo, Noguera: Adding truth-constants to logics of continuous t-norms: Axiomatization and completeness results. FSS, 2007]

RPL is algebraizable and its equivalent semantics is the class of RMV-algebras.

Which MV-algebras can be expanded into a RMV-algebra?

- (MV-reduct of) a nontrivial RMV-algebra has a homomorphic image of $Q_{\text{Ł}}$ as subalgebra.
- $Q_{\text{Ł}}$ is simple.

Rational Pavelka Logic (RPL) – completeness

$T \cup \{\varphi\}$ a set of formulas in the language of RPL.

Pavelka completeness:

$$|\varphi|_T = \|\varphi\|_T$$

where

$$|\varphi|_T = \sup\{r \mid T \vdash_{\mathrm{RPL}} \overline{r} \to \varphi\} \qquad \text{(provability degree)}$$

$$\|\varphi\|_T = \inf\{v(\varphi) \mid v \text{ model of } T\} \qquad \text{(validity degree)}$$

Moreover, we have finite strong standard completeness for RPL.
Proved by Hájek using implicit definitions of constants (see further on).

Hence, (prop.) RPL is conservative extension of Ł.

[Hájek: Metamathematics of fuzzy logic, 1998]

Implicit definability in $[0,1]_Ł$

Let $a \in [0,1]$ be a real number, $\varphi(x_1, \ldots, x_n)$ a formula in the language of Ł, and $1 \leq i \leq n$.

The formula φ implicitly defines the value a in variable x_i in $[0,1]_Ł$ iff
- φ satisfiable in $[0,1]_Ł$, and
- $v(\varphi) = 1$ implies $v(x_i) = a$ for each $[0,1]_Ł$-valuation v.

a is implicitly definable in $[0,1]_Ł$ iff there is a formula that defines it.

Toolkits for implicit definability:

- bookkeeping formulas;
- [Hájek:Metamathematics of FL, 1998]
- [Hájek, Paris, Shepherdson: RPL is a conservative extension of Ł, 2000]
- terms for (suitable) McNaughton functions;
- $x \equiv (\neg x)^{n-1}$ defines $1/n$.

Intermezzo II: (Deductive) Beth property

Let L be a logic and $\varphi(x, \bar{y})$ an L-term.

$\varphi(x, \bar{y})$ explicitly defines x in L iff there is a term $\delta(\bar{y})$
such that $\varphi(x, \bar{y}) \vdash_L x \equiv \delta(\bar{y})$.

$\varphi(x, \bar{y})$ implicitly defines x in L iff $\varphi(x, \bar{y}), \varphi(z, \bar{y}) \vdash_L x \equiv z$.

L has the (deductive) Beth property iff,
whenever $\varphi(x, \bar{y})$ implicitly defines x, then it also explicitly defines x.

We can see that

- the term $x \equiv (\neg x)^{n-1}$ implicitly defines x;
- Ł does not have the Beth property

[Montagna: Interpolation and Beth's property. APAL, 2006]

Implicit definability via bookkeeping formulas

Bookkeeping formulas, with variables instead of constants.

Example: rationals with denominator 3;
variables $0 \equiv x_0$, $x_{1/3}$, $x_{2/3}$, $x_1 \equiv 1$.
Some bookkeeping formulas involving (only) these variables are

$$x_{2/3} \odot x_{2/3} \equiv x_{1/3}$$
$$x_{1/3} \odot x_{2/3} \equiv x_0$$
$$x_{1/3} \odot x_{1/3} \equiv x_0$$
$$x_{1/3} \to x_0 \equiv x_{2/3}$$

The last line reads $\neg x_{1/3} \equiv x_{2/3}$.
Combined with the first one, it gives $(\neg x_{1/3})^2 \equiv x_{1/3}$.
This implies $v(x_{1/3}) = 1/3$ in $[0,1]_{Ł}$ for all models v of the four axioms.
The second and third formulas are redundant.

If a is implicitly definable in $[0,1]_{Ł}$, then it is a fixed point of any non-trivial automorphism.

Thus $[0,1]_{Ł}$ has no nontrivial automorphisms.

Definability: recognition

Implicit definition of rationals (various options) in $[0,1]_{Ł}$.

Fixed formulas of known behaviour.

What does a random term define?

Implicit definability problem:
Instance: a formula $\varphi(x_1,\ldots,x_n)$, an integer $1 \leq i \leq n$, and a rational number a.
Problem: Does φ define a in x_i?

Theorem [H. 2019]

Implicit definability is both NP*-hard and* coNP*-hard.*

Note:

- In a consistent theory T, no two formulas $\varphi_1, \varphi_2 \in T$ can define values $r_1 \neq r_2$ in the same variable x.
  "Compatibility" of defined values is a necessary condition to consistency.

Instance: a formula τ, representing a finite theory T, and a formula φ.
Problem: $\tau \vdash_{\mathrm{RPL}} \varphi$?

Rational constants are represented as pairs of natural numbers in binary.

Existential theory of the standard RPL-algebra is NP-complete.

Hence, the finite provability relation of RPL are coNP-complete.

- direct proof (from above)
- reduction to Ł, using implicit definitions of rationals
  NB. Polynomial-size definitions!

[Hájek: Computational complexity of t-norm based propositional fuzzy logics with rational truth constants. FSS, 2006]

Computing the validity degree

Instance: formulas τ and φ.
Task: compute $\min(f_\varphi)$ subject to $f_\tau = 1$.

The minimum of f_φ on the (compact) 1-region of f_τ ($\min_\tau(f_\varphi)$) is attained at a vertex of the common refinement of complexes of f_φ and f_τ.

Upper bounds on denominators: $N = ((\sharp\tau + \sharp\varphi)/n)^n$ [Aguzzoli and Ciabattoni, 2006].
Binary search, with oracle, among rationals in $[0,1]$ with denominators at most N.

Oracle: given φ and a rational $r \in [0,1]$, is $\min_\tau(f_\varphi) \leq r$?)
Clearly in NP.

Minimal distance of any two such numbers: $|\frac{p_1}{q_1} - \frac{p_2}{q_2}| \geq \frac{1}{N^2}$

The search yields an interval $[m/2^k, (m+1)/2^k)$ for some m, of length $1/2^k$, with exactly one rational with denominator up to N.

Pick a value in $(m/2^k, (m+1)/2^k)$ and compute best rational approximations.

MAX value is in $\mathrm{FP}^{\mathrm{NP}}$; actually $\mathrm{FP}^{\mathrm{NP}}$-complete (under appropriate reductions).

[H: On the complexity of validity degrees in Łukasiewicz logic. CiE 2020]

Non-approximability of minimum value (and validity degree)

We assume the language of Ł and T empty (i.e., τ is 1).

Theorem

Let $\delta < 1/2$ be a positive real.
Suppose $\texttt{alg}$ *is a polynomial time algorithm computing, for each formula φ, a real number* $\texttt{alg}(\varphi)$ *satisfying* $|\texttt{alg}(\varphi) - \min(\varphi)| \leq \delta$.
Then $\mathrm{P} = \mathrm{NP}$.

Proof: recognize Boolean non-tautologies using ALG.

Instance: Boolean formula φ, given as $\{\oplus, \wedge\}$-combination of literals.

Then f_φ in $[0,1]_{\text{Ł}}$ is a concave function. And $\min(\varphi)$ is either 0 or 1.

We have

$$\varphi \notin \mathrm{TAUT}(\{0,1\}_{\mathrm{B}}) \;\text{ iff }\; \min(\varphi) = 0 \;\text{ iff }\; \texttt{alg}(\varphi) < 1/2.$$

[H., Savický 2016]

$\mathbb{RMV}$ is (hereditarily) structurally complete

Lemma: $\mathbb{RMV} = \mathbf{QVar}([0,1]^Q_{Ł})$.
[Esteva, Gispert, Godo, Noguera: Adding truth-constants to logics of continuous t-norms: Axiomatization and completeness results. FSS, 2007]

Theorem [Gispert, H., Moraschini, Stronkowski 2022(?)]

$[0,1]^Q_{Ł}$ *and* $Q^Q_{Ł}$ *generate the same quasivariety.*

Proof:
let $\Phi := s_1 \approx t_1 \,\&\, \dots \,\&\, s_n \approx t_n \Rightarrow s \approx t$ be a quasiequation in the language of RMV.
Assume $[0,1]^Q_{Ł} \not\models \Phi$.

Let $\bar{r_1}, \dots, \bar{r_k}$ be the constants in Φ.
T (conjunction of equations) implicitly defines these constants in $[0,1]_{Ł}$ in some variables $z_1, \dots, z_k$.

Replace each $\bar{r_i}$ with z_i in Φ, obtaining a $\Phi^\star$.

Then $[0,1]_{Ł} \not\models T \Rightarrow \Phi^\star$.
Since $[0,1]_{Ł}$ and $Q_{Ł}$ have the same universal theory, [Gispert 2002]
we have $Q_{Ł} \not\models T \Rightarrow \Phi^\star$.

Replacing each z_i with $\bar{r_i}$ and removing T, we get $Q^Q_{Ł} \not\models \Phi$.

Hence $\mathbb{RMV}$ is minimal: we know $Q^Q_{Ł} \preceq \mathcal{A}$ for any nontrivial RMV-algebra $\mathcal{A}$;
so for any nontrivial $\mathbb{K} \subseteq \mathbb{RMV}$ we have $\mathbb{RMV} = \mathbf{QVar}(Q^Q_{Ł}) \subseteq \mathbb{K}$.

On admissible rules in Ł

Ł is structurally incomplete.
Basis of admissible rules has been described by
[Jeřábek: Admissible rules of Łukasiewicz logic. JLC, 2010]
Recall conservativity: RPL and Ł derive the same rules in language of Ł.
(Since they both have finite strong standard completeness.)

Thus, rules admissible in Ł are not admissible (=derivable) in RPL,
unless they are derivable in Ł.

Example: recall $x \approx (\neg x)^n$ implicitly defines ${}^1/_{(n+1)}$.

$x \leftrightarrow (\neg x)^n \rhd y$ is passive in Ł.

But not in RPL: substitute $\sigma(x) = {}^1/_{(n+1)}$

Structural completions in $\mathbb{RPA}$

A nontrivial quasivariety $\mathbb{K} \subseteq \mathbb{RPA}$ either contains Q_Π^Q,
or it is termwise-equivalent to extensions of Pi.

Theorem [Gispert, H., Moraschini, Stronkowski 2022(?)]

Let $\mathbb{K}$ be a subquasivariety of $R\Pi$.

(i) *If $Q_\Pi^Q \in \mathbb{K}$, the structural completion of $\mathbb{K}$ is $\mathbf{QVar}(Q_\Pi^Q)$.*

(ii) *If $Q_\Pi^Q \notin \mathbb{K}$, then $\mathbb{K}$ is hereditarily structurally complete.*

Proof sketch (i):
Q_Π^Q is a subalgebra of $F_{\mathbb{K}}(\omega)$.
We have $\mathbb{RPA} = \mathbf{Var}(Q_\Pi^Q)$.
Thus the ω-generated free algebras of Π, $\mathbb{K}$, and $\mathbf{QVar}(Q_\Pi^Q)$ coincide.
Moreover, $\mathbf{QVar}(Q_\Pi^Q) = \mathbf{QVar}(F_{\mathbf{QVar}(Q_\Pi^Q)}(\omega))$.

[Gispert, H., Moraschini, Stronkowski: Structural completeness in many-valued logics with rational constants. Accepted, NDJFL]

Logics with graded syntax

Recall the logic GRPL, Hájek's "graded version" of RPL:

- constants $\mathcal{C}$ indexed by the rationals
- all formulas are graded (it is not possible to derive a non-graded formula)
- axioms of Ł and bookkeeping in grade 1, $\bar{r}$ in grade r, other formulas in grade 0
- graded modus ponens and lifting.

A. GRPL embeds in RPL:
$T \vdash_{\mathrm{GRPL}} \langle q, \varphi \rangle$ if and only if $\{\overline{r} \to \psi \mid \langle r, \psi \rangle \in T\} \vdash_{\mathrm{RPL}} \overline{q} \to \varphi$.
Hájek: "$\langle r, \varphi \rangle$ is just another notation for $\bar{r} \to \varphi$"

B. RPL embeds in GRPL:
$T \vdash_{\mathrm{RPL}} \varphi$ if and only if $\{\langle 1, \psi \rangle \mid \psi \in T\} \vdash_{\mathrm{GRPL}} \langle 1, \varphi \rangle$.

C. GRPL embeds in GRPL:
$T \vdash_{\mathrm{GRPL}} \langle q, \varphi \rangle$ if and only if $\{\langle 1, r \to \psi \rangle \mid \langle r, \psi \rangle \in T\} \vdash_{\mathrm{GRPL}} \langle 1, q \to \varphi \rangle$.
Moreover, (on the right) there is a proof s.t. all graded formulas have the grade 1.

Conclusion

Fuzzy logics with constants have a great pedigree.

[Goguen, Pavelka, Hájek, Esteva, Godo, Gispert, Cignoli, Noguera, Paris, Cintula, Shepherdson]

They give rise to interesting mathematics.

In a well defined sense, RPL is not stronger or more expressive than Ł.

Rather, RPL can be a useful shorthand for Ł.
No graded syntax, and no constants, are needed for reasoning about graded truth.

So-called "logic with graded syntax" (with rational grades)
are a proper syntactic fragment of RPL.

Artificial precision? (Which side of the turnstile?)

"He is ten times the mathematician I am."

[Marra: The problem of Artificial Precision in Theories of Vagueness: A Note on the Role of Maximal Consistency. Erkenntnis, 2014]
[Běhounek: In Which Sense is Fuzzy Logic a Logic for Vagueness? PRUV, 2014]

(Disclaimer: applies mainly to Łukasiewicz logic.)