Set theory in fuzzy logic: technical and historical considerations

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Outline

- An overview of the logical calculus
- FST mimicking ZF
- Strength and non-triviality of FST
- Comparison with similar systems
Formal axiomatic set theory, governed by a fuzzy logic.

The desired theory should:

- generate a *cumulative* universe of sets
- be provably distinct from the classical set theory
- be sufficiently strong
- be consistent and, perhaps, complete
Our calculus $\text{BL} \forall \Delta$ is defined as follows.

Logical symbols: connectives $0$, $\&$, $\rightarrow$, $\Delta$, quantifiers $\forall$, $\exists$, object variables, equality $\equiv$.

Particular language $\mathcal{L}$ may have constants, predicate symbols and function symbols. In particular, set theory language has a basic predicate symbol $\in$.

Terms and formulas are defined classically.

A *theory* in a language $\mathcal{L}$ is a set of formulas of $\mathcal{L}$.
Axioms – Propositional BL

(A1) \((\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))\)

(A2) \((\varphi \& \psi) \rightarrow \varphi\)

(A3) \((\varphi \& \psi) \rightarrow (\psi \& \varphi)\)

(A4) \((\varphi \& (\varphi \rightarrow \psi)) \rightarrow (\psi \& (\psi \rightarrow \varphi))\)

(A5a) \((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \& \psi) \rightarrow \chi)\)

(A5b) \(((\varphi \& \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))\)

(A6) \(((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)\)

(A7) \(0 \rightarrow \varphi\)

The deduction rule of BL is modus ponens.
Axioms – $\Delta$ Connective

$\text{BL} \Delta$ has the axioms of $\text{BL}$ and the following axioms:

1. $\Delta \varphi \lor \neg \Delta \varphi$
2. $\Delta (\varphi \lor \psi) \rightarrow (\Delta \varphi \lor \Delta \psi)$
3. $\Delta \varphi \rightarrow \varphi$
4. $\Delta \varphi \rightarrow \Delta \Delta \varphi$
5. $\Delta (\varphi \rightarrow \psi) \rightarrow (\Delta \varphi \rightarrow \Delta \psi)$

The deduction rules are modus ponens and $\Delta$-generalization: $\varphi/\Delta \varphi$. 
Axioms – Quantifiers

- (∀1) $\forall x \varphi(x) \rightarrow \varphi(t)$ ($t$ substitutable for $x$ in $\varphi$)
- (∃1) $\varphi(t) \rightarrow \exists x \varphi(x)$ ($t$ substitutable for $x$ in $\varphi$)
- (∀2) $\forall x (\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \forall x \varphi)$ ($x$ not free in $\chi$)
- (∃2) $\forall x (\varphi \rightarrow \chi) \rightarrow (\exists x \varphi \rightarrow \chi)$ ($x$ not free in $\chi$)
- (∀3) $\forall x (\varphi \lor \chi) \rightarrow (\forall x \varphi \lor \chi)$ ($x$ not free in $\chi$)

Rule of generalization: $\varphi/\forall x \varphi$. 
Axioms – Equality

Equality axioms for set theory:

- reflexivity
- symmetry
- transitivity
- congruence \( \forall x, y, z (x = y \& z \in x \rightarrow z \in y) \)
- congruence \( \forall x, y, z (x = y \& y \in z \rightarrow x \in z) \)

We choose to make equality a crisp (0-1) predicate in our set theory, and we add

\[ \forall x, y (x = y \lor \neg(x = y)) \]
Given a predicate language $\mathcal{L}$, an $\mathbf{L}$-structure $\mathbf{M}$ (where $\mathbf{L}$ is a $\text{BL}\Delta$-chain) is an interpretation of $\mathcal{L}$ if:

$\mathbf{M} = (M, (r_P)_P \text{ predicate symbol}, (m_c)_c \text{ constant}, (f_F)_F \text{ function symbol})$

- domain $M \neq \emptyset$
- for each $n$-ary predicate symbol $P$ of $\mathcal{L}$, an $\mathbf{L}$-fuzzy $n$-ary relation $r_P : M^n \to L$
- for each constant $c$ of $\mathcal{L}$, an element $m_c \in M$
- for each $n$-ary function symbol $F$ of $\mathcal{L}$ a function $f_F : M^n \to M$
An $M$-evaluation of object variables is a mapping $\nu : \mathcal{V} \rightarrow M$. For two evaluations $\nu$, $\nu'$, $\nu \equiv_x \nu'$ means $\nu(y) = \nu'(y)$ for each variable $y$ distinct from $x$.

The value $\|t\|_{M,\nu}$ of a term $t$ under evaluation $\nu$ is defined inductively:

1. $\|x\|_{M,\nu} = \nu(x),$
2. $\|c\|_{M,\nu} = m_c,$
3. $\|F(t_1, \ldots, t_n)\|_{M,\nu} = f_F(\|t_1\|_{M,\nu}, \ldots, \|t_n\|_{M,\nu})$ for $n$-ary function symbol $F$ and terms $t_1, \ldots, t_n$. 
The value $\|\varphi\|_{M,\nu}^L$ of a formula $\varphi$ in an $L$-structure $M$ and evaluation $\nu$ in $M$ is:

- $\|P(t_1, \ldots, t_n)\|_{M,\nu}^L = r_P(\|t_1\|_{M,\nu}, \ldots, \|t_n\|_{M,\nu})$
- $\|t_1 = t_2\|_{M,\nu}^L = 1^L$ if $\|t_1\|_{M,\nu} = \|t_2\|_{M,\nu}$, otherwise $0^L$
- $\|\varphi \land \psi\|_{M,\nu}^L = \|\varphi\|_{M,\nu}^L \times \|\psi\|_{M,\nu}^L$
- $\|\varphi \rightarrow \psi\|_{M,\nu}^L = \|\varphi\|_{M,\nu}^L \Rightarrow \|\psi\|_{M,\nu}^L$
- $\|0\|_{M,\nu}^L = 0$
- $\|\bigtriangleup \varphi\|_{M,\nu}^L = \bigtriangleup \|\varphi\|_{M,\nu}^L$
- $\|\forall x \varphi\|_{M,\nu}^L = \land_{\nu \equiv x \nu'} \|\varphi\|_{M,\nu'}^L$
- $\|\exists x \varphi\|_{M,\nu}^L = \lor_{\nu \equiv x \nu'} \|\varphi\|_{M,\nu'}^L$
An \( L \)-structure \( M \) is \textit{safe} if \( \| \varphi \|_{M,v} \) is defined for each \( \varphi \) and \( v \).

The truth value of a formula \( \varphi \) of a predicate language \( L \) in a safe \( L \)-structure \( M \) for \( L \) is

\[
\| \varphi \|_M = \bigwedge_{v \text{ an } M\text{-evaluation}} \| \varphi \|_{M,v}
\]

\( M \) is an \textit{admissible} \( L \)-structure for \( L \) if all the axioms for \( = \) are 1-true in \( M \).
Completeness

Theorem

Let $C$ be a schematic extension of $\text{BL}\forall\Delta$, let $T$ be a theory over $C$, and let $\varphi$ be a formula of the language of $T$. Then $T$ proves $\varphi$ iff for each $C$-chain $L$ and each safe admissible $L$-model $M$ of $T$, $\varphi$ holds in $M$. 
Conservative extension

Let $T'$ in language $\mathcal{L}'$ extend $T$ in language $\mathcal{L} \subseteq \mathcal{L}'$. Then $T'$ is a conservative extension of $T$ whenever

- $P \notin \mathcal{L}$ is an $n$-ary predicate symbol, $\varphi(x_1, \ldots, x_n)$ a formula of $\mathcal{L}$, and $T'$ results from $T$ by adding the formula
  $\forall x_1, \ldots, x_n (P(x_1, \ldots, x_n) \equiv \varphi(x_1, \ldots, x_n))$;

- $c \notin \mathcal{L}$ is a constant, $\exists x \Delta \varphi(x)$ a closed formula of $\mathcal{L}$ provable in $T$, and $T'$ results from $T$ by adding the formula $\varphi(c)$;

- $F \notin \mathcal{L}$ is an $n$-ary function symbol, $\varphi(x_1, \ldots, x_n, y)$ a formula of $\mathcal{L}$, such that $T \vdash \forall x_1, \ldots, x_n \exists y \Delta \varphi(x_1, \ldots, x_n, y)$, and $T'$ results from $T$ by adding the formula
  $\forall x_1, \ldots, x_n \varphi(x_1, \ldots, x_n, F(x_1, \ldots, x_n))$ and the congruence axiom
  $x_1 = z_1 \& \ldots \& x_n = z_n \rightarrow F(x_1, \ldots, x_n) = F(z_1, \ldots, z_n)$. 

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Technicalities

BL∀∆ proves the following:

- \( \forall x (\varphi \land \psi) \rightarrow \forall x (\varphi \land \psi) \) (\( x \) not free in \( \psi \))
- \( \forall (\varphi \land \forall x \psi) \rightarrow \forall x (\varphi \land \psi) \)
- \( \exists x \Delta \varphi \rightarrow \Delta \exists x \varphi \)
- \( \forall x \Delta \varphi \equiv \Delta \forall x \varphi \)
- \( \Delta (\varphi \lor \neg \varphi) \equiv \Delta (\varphi \rightarrow \Delta \varphi) \).
Fuzzy Set Theory (FST) is a first-order theory over $\text{BL}\forall\Delta$ (or its schematic extension $\mathcal{C}$).

The basic language consists of a predicate symbol $\in$.

We define

- $\forall x \in y(\varphi)$ as $\forall x(x \in y \to \varphi)$
- $\exists x \in y(\varphi)$ as $\exists x(x \in y \& \varphi)$
- $x \subseteq y$ as $\forall z(z \in x \to z \in y)$
Crispness

Definition

- A formula \( \varphi \) is *crisp* in \( T \) iff \( T \vdash \Delta(\varphi \lor \neg \varphi) \).
- We define \( \text{Crisp}(x) \equiv \forall u \Delta(u \in x \lor \neg u \in x) \).

Note that crispness for all basic predicate symbols of the theory means crispness (LEM) for every formula.

\[ \text{Crisp}(x) \rightarrow \Delta \text{Crisp}(x) \] is provable, so crispness is a crisp predicate.
Crisp Equality

**Lemma**

A theory with separation, singletons, and congruence for $\in$ over a logic which proves $(\varphi \rightarrow \varphi \& \varphi) \rightarrow (\varphi \lor \neg \varphi)$ proves $
exists x, y (x = y \lor \neg (x = y))$.

Retaining basic principles like separation or congruence, in Łukasiewicz logic and in product logic we get a crisp $\equiv$.

Thus, **crisp equality** is a universal decision in FST.
Other potential crisp turns

Grishin 1979: In a theory with extensionality, successors, and congruence, crispness of $=$ implies crispness of $\in$.
Remedy (Shirahata 1999): modify the spelling of extensionality
$\forall x, y (x = y \iff (\Delta(x \subseteq y) \& \Delta(y \subseteq x)))$

Powell, Grayson: In a theory with separation (in a weak axiomatic setting), the axiom of foundation
$\forall x (\neg(x = 0) \rightarrow \exists y \in x \neg\exists w (w \in x \& w \in y))$ implies LEM for all formulas.
Remedy (int.): use $\in$-induction instead of foundation.

Intuitionistic logic: adding axiom of choice implies LEM for all formulas.
Remedy: use Zorn’s lemma instead of choice.
Axioms of FST

- (ext.) $\forall x, y (x = y \equiv (\Delta(x \subseteq y) \& \Delta(y \subseteq x)))$
- (empty) $\exists x \Delta \forall y \neg (y \in x)$
- (pair) $\forall x \forall y \exists z \Delta \forall u (u \in z \equiv (u = x \lor u = y))$
- (union) $\forall x \exists z \Delta \forall u (u \in z \equiv \exists y (u \in y \& y \in x))$
- (weak power) $\forall x \exists z \Delta \forall u (u \in z \equiv \Delta(u \subseteq x))$
- (inf.) $\exists z \Delta (\emptyset \in z \& \forall x \in z (x \cup \{x\} \in z))$
- (sep.) $\forall x \exists z \Delta \forall u (u \in z \equiv (u \in x \& \varphi(u, x)))$
  for any $\varphi$ not containing free $z$
- (coll.) $\forall x \exists z \Delta [\forall u \in x \exists v \varphi(u, v) \rightarrow \forall u \in x \exists v \in z \varphi(u, v)]$
  for any $\varphi$ not containing free $z$
- ($\in$-ind.) $\Delta \forall x (\forall y \in x \varphi(y) \rightarrow \varphi(x)) \rightarrow \Delta \forall x \varphi(x)$
  for any $\varphi$
- (support) $\forall x \exists z (\text{Crisp}(z) \& \Delta(x \subseteq z)))$
The Universe $V^L$

The class $V^L$, defined by ordinal induction in ZF.

$L^+ = L - \{0\}$.

- $V^L_0 = \{\emptyset\}$
- $V^L_{\alpha+1} = \{f : \text{Fnc}(f) & \text{Dom}(f) \subseteq V^L_\alpha \& \text{Rng}(f) \subseteq L^+\}$ for any ordinal $\alpha$
- $V^L_\lambda = \bigcup_{\alpha < \lambda} V^L_\alpha$ for limit ordinals $\lambda$
- $V^L = \bigcup_{\alpha \in \text{Ord}} V^L_\alpha$
The Universe $V^L$ (Cont.)

We define two binary functions from $V^L$ into $L$, assigning to any $u, v \in V^L$ the values $\|u \in v\|$ and $\|u = v\|

\[\|u \in v\| = \nu(u) \text{ if } u \in D(v), \text{ otherwise } 0\]
\[\|u = v\| = 1 \text{ if } u = v, \text{ otherwise } 0\]

Using induction on the complexity of formulas, define for any formula $\varphi(x_1, \ldots, x_n)$ an $n$-ary function from $(V^L)^n$ into $L$, assigning to an $n$-tuple $u_1, \ldots, u_n$ the value $\|\varphi(u_1, \ldots, u_n)\|$. 

**Theorem**

Let $\varphi$ be a closed formula provable in FST. Then $\varphi$ is valid in $V^L$, i.e., ZF proves $\|\varphi\| = 1$. 

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An Inner Model of ZF in FST

(Hereditarily crisp transitive set)

\[ HCT(x) \equiv \text{Crisp}(x) \& \forall u \in x(\text{Crisp}(u) \& u \subseteq x) \]

(Hereditarily crisp set)

\[ H(x) \equiv \text{Crisp}(x) \& \exists x' \in HCT(x \subseteq x') \]

Lemma

The class \( H \) is both crisp and transitive in FST:

\[ FST \vdash \forall x(x \in H \lor \neg(x \in H)) \]

\[ FST \vdash \forall x, y(y \in x \& x \in H \rightarrow y \in H) \]
For $\varphi$ a formula in the language of ZF, define $\varphi^H$ inductively:

- $\varphi^H = \varphi$ for $\varphi$ atomic;
- $\varphi^H = \varphi$ for $\varphi = 0$;
- $\varphi^H = \psi^H \& \chi^H$ for $\varphi = \psi \& \chi$;
- $\varphi^H = \psi^H \rightarrow \chi^H$ for $\varphi = \psi \rightarrow \chi$;
- $\varphi^H = (\forall x \in H)\psi^H$ for $\varphi = (\forall x)\psi$.

**Theorem**

Let $\varphi$ be a theorem of ZF. Then $\text{FST} \vdash \varphi^H$.

Thus $H$ is an inner model of ZF in FST and ZF is consistent relative to FST.
Set theory over Intuitionistic Logic


Inner model of ZF in ZF-INT, consisting of stabilized sets, defined with the axiom of double complement $\exists y \forall z (\neg \neg (z \in x) \to (z \in x))$


Builds a Heyting-valued universe, over a complete Heyting algebra, within ZF-INT. Shows relative consistency of ZF to ZF-INT, using a particular Boolean algebra.
Takeuti & Titani’s Set Theory: Logic


Logic given by an algebra of truth values $\mathbb{L}$ on $[0, 1]$, with

- Łukasiewicz connectives,
- Gödel connectives,
- product conjunction,
- globalization $\Box$ (definable),
- the constant $1/2$
- quantifiers $\forall, \exists$
A Gentzen-style axiomatization of the first-order calculus is given. It is complete w. r. t. models over $[0, 1]$, but it includes an infinitary rule.

A similar logic, with also the product implication, is analyzed in P. Hájek’s book and inspired the logic ŁΠ.
The theory FZF uses the connectives of Gödel logic in its axioms, which are a suitable versions of equality, extensionality, pairing, union, power, $\in$-induction, separation, collection, infinity, double complement, and Zorn’s lemma.

Equality admits many-valued interpretations in FZF:

$$\forall u, v \Box (u = v \equiv \forall z (z \in u \equiv z \in v))$$
Work in FZF.
A set $x$ is *stable* iff $\Box(x = \sim \sim x)$.
Define the class $S$ of stable sets.
One can prove that if $\text{ZFC} \vdash \varphi$, then $\text{FZF} \vdash \varphi^S$, so $S$ is an inner model of ZFC in FZF.

Then take $I = [0, 1]^S$ and construct the universe $S^I$ in $S$.
It holds in $S$ that $\langle S^I, \|\| \rangle$ is an $\mathbb{L}$-valued model of FZF.
For a sentence $\varphi$, let $\varphi^\sim = \{ x : x \in \{ \emptyset \} \land \varphi \}$. Thus each subset of $\{ \emptyset \}$ is the truth value of a sentence, and we have ($\emptyset \in \varphi^\sim$) $\equiv \varphi$.

Moreover, there is an order-preserving isomorphism $\rho$ between $[0, 1]^S$ and the sets $A$ s. t. $\Box (A \subseteq \{ \emptyset \})$.

For every formula $\varphi$, one can take $|\varphi| = \rho^{-1}(\varphi^\sim)$ as the inner truth value of $\varphi$ in $[0, 1]^S$. 
TT — completeness

Theorem

There exists a function $F(x)$ such that
1. for every formula $\varphi(x_1, \ldots, x_n)$
   
   $$\forall x_1, \ldots, x_n \in S^l \cap S (||\varphi(x_1, \ldots, x_n)|| = |\varphi(F(x_1), \ldots, F(x_n))|)$$

2. $\forall u \exists x \in S^l \cap S \Box (u = F(x))$.

Hence $F$ is a truth-value preserving isomorphism between $S^l$ and
the universe of FZF.
Moreover, if $FZF \vdash ||\varphi|| = 1$ in $S^l$, then $FZF \vdash \varphi$. 

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Set theory in fuzzy logic: technical and historical considerations

Logic:
language $\land, \lor, \to, 0, \exists, \forall$
evaluated on a complete bounded lattice
with $e(x \to y) = \begin{cases} 1 & \text{if } e(x) \leq e(y) \\ 0 & \text{otherwise} \end{cases}$ for any evaluation $e$.
and a definable $\square$ s. t. $\square \varphi$ is $1 \to \varphi$.
Thus, $e(\square \varphi) = \begin{cases} 1 & \text{if } e(\varphi) = 1 \\ 0 & \text{otherwise} \end{cases}$ for any evaluation $e$.
Quantifiers evaluated as usual.
Lattice-valued logic vs. $\text{BL}\forall\Delta$

In a $\text{BL}\Delta$-chain, one can interpret Titani’s semantics as follows:

- $(\varphi \land_T \psi)'$ is $\varphi' \land \psi'$
- $(\varphi \lor_T \psi)'$ is $\varphi' \lor \psi'$
- $(\varphi \rightarrow_T \psi)'$ is $\Delta(\varphi \rightarrow \psi)$
- $(\Box \varphi)'$ is $\Delta \varphi'$

Then all the rules of lattice-valued logic are sound in any $\text{BL}\Delta$-chain. Thus if $\varphi$ is provable in lattice-valued logic, then $\varphi'$ is true in $\text{BL}\Delta$-chains, and hence provable in $\text{BL}\forall\Delta$ by completeness.

(However, the interpretation is not faithful. For example, $\alpha \land (\beta \lor \gamma) \equiv (\alpha \land \beta) \lor (\alpha \land \gamma)$ is not a theorem of lattice-valued logic.)
It follows from Titani’s axiom for equality, namely,

\[(x = y \land T \varphi(x)) \rightarrow T \varphi(y)\]

that \(=\) is “crisp” (i.e., two-valued) in lattice-valued set theory, as the formula \((x = y \land \Box(x = x) \rightarrow T \Box(x = y))\) is an instance of this axiom.
Set-theoretic axioms

For the following set-theoretic axioms of lattice-valued set theory, their ‘'-translations are provable in FST: extensionality, pairing, union, (weak) power, separation, collection, support, infinity. As for $\in$-induction, it spells

$$\Delta \forall x (\Delta \forall y (y \in x \to \varphi(y)) \to \varphi(x)) \to \forall x \varphi(x)$$

It is unclear whether this formula is provable in FST. In order to fully interpret lattice-valued set theory in FST, one can adopt the stronger version of $\in$-induction (which is valid in $\mathcal{V}^L$) instead of the original axiom. Further, one would have to adopt Zorn’s lemma, which is an axiom of lattice-valued set theory.