Set theory in fuzzy logic: technical and historical considerations

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Czech-Japan joint fuzzy logic workshop
Hakusan 2009

Outline

- An overview of the logical calculus
- FST mimicking ZF
- Strength and non-triviality of FST
- Comparison with similar systems

Target

Formal axiomatic set theory, governed by a fuzzy logic.

The desired theory should:

- generate a *cumulative* universe of sets
- be provably distinct from the classical set theory
- be sufficiently strong
- be consistent and, perhaps, complete

The Logic BL∀Δ

Our calculus BL∀Δ is defined as follows.

Logical symbols: connectives 0, &, $\rightarrow$, Δ, quantifiers $\forall$, $\exists$, object variables, equality $=$.
Particular language $\mathcal{L}$ may have constants, predicate symbols and function symbols. In particular, set theory language has a basic predicate symbol $\in$.
Terms and formulas are defined classically.
A *theory* in a language $\mathcal{L}$ is a set of formulas of $\mathcal{L}$.

Axioms – Propositional BL

- (A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
- (A2) $(\varphi \& \psi) \to \varphi$
- (A3) $(\varphi \& \psi) \to (\psi \& \varphi)$
- (A4) $(\varphi \& (\varphi \to \psi)) \to (\psi \& (\psi \to \varphi))$
- (A5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$
- (A5b) $((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
- (A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
- (A7) $0 \to \varphi$

The deduction rule of BL is modus ponens.

Axioms – Δ Connective

BLΔ has the axioms of BL and the following axioms:

- (Δ1) $\Delta\varphi \vee \neg\Delta\varphi$
- (Δ2) $\Delta(\varphi \vee \psi) \rightarrow (\Delta\varphi \vee \Delta\psi)$
- (Δ3) $\Delta\varphi \rightarrow \varphi$
- (Δ4) $\Delta\varphi \rightarrow \Delta\Delta\varphi$
- (Δ5) $\Delta(\varphi \rightarrow \psi) \rightarrow (\Delta\varphi \rightarrow \Delta\psi)$

The deduction rules are modus ponens and Δ-generalization: $\varphi/\Delta\varphi$.

Axioms – Quantifiers

- (∀1) $\forall x\varphi(x) \to \varphi(t)$ (t substitutable for x in φ)
- (∃1) $\varphi(t) \to \exists x\varphi(x)$ (t substitutable for x in φ)
- (∀2) $\forall x(\chi \to \varphi) \to (\chi \to \forall x\varphi)$ (x not free in χ)
- (∃2) $\forall x(\varphi \to \chi) \to (\exists x\varphi \to \chi)$ (x not free in χ)
- (∀3) $\forall x(\varphi \vee \chi) \to (\forall x\varphi \vee \chi)$ (x not free in χ)

Rule of generalization: $\varphi/\forall x\varphi$.

Axioms – Equality

Equality axioms for set theory:

- reflexivity
- symmetry
- transitivity
- congruence $\forall x, y, z(x = y \& z \in x \rightarrow z \in y)$
- congruence $\forall x, y, z(x = y \& y \in z \rightarrow x \in z)$

We choose to make equality a crisp (0-1) predicate in our set theory, and we add

- $\forall x, y(x = y \vee \neg(x = y))$

BL∀Δ — Semantics

Given a predicate language $\mathcal{L}$, an **L**-structure **M** (where **L** is a BLΔchain) is an interpretation of $\mathcal{L}$ if:
$\mathbf{M} = (M, (r_P)_{P \text{ predicate symbol}}, (m_c)_{c \text{ constant}}, (f_F)_{F \text{ function symbol}})$

- domain $M \neq \emptyset$
- for each n-ary predicate symbol P of $\mathcal{L}$, an **L**-fuzzy n-ary relation $r_P : M^n \to L$
- for each constant c of $\mathcal{L}$, an element $m_c \in M$
- for each n-ary function symbol F of $\mathcal{L}$ a function $f_F : M^n \longrightarrow M$

BL∀Δ — Semantics (Cont.)

An **M**-evaluation of object variables is a mapping $v : \mathcal{V} \longrightarrow M$. For two evaluations v, v', $v \equiv_x v'$ means $v(y) = v'(y)$ for each variable y distinct from x.

The value $\|t\|_{\mathbf{M},v}$ of a term t under evaluation v is defined inductively:

- $\|x\|_{\mathbf{M},v} = v(x)$,
- $\|c\|_{\mathbf{M},v} = m_c$,
- $\|F(t_1, \ldots, t_n)\|_{\mathbf{M},v} = f_F(\|t_1\|_{\mathbf{M},v}, \ldots, \|t_n\|_{\mathbf{M},v})$ for n-ary function symbol F and terms $t_1, \ldots, t_n$.

BL∀Δ — Semantics (Cont.)

The value $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ of a formula φ in an **L**-structure **M** and evaluation v in **M** is:

- $\|P(t_1,\ldots,t_n)\|^{\mathbf{L}}_{\mathbf{M},v} = r_P(\|t_1\|_{\mathbf{M},v},\ldots,\|t_n\|_{\mathbf{M},v})$
- $\|t_1 = t_2\|^{\mathbf{L}}_{\mathbf{M},v} = 1^{\mathbf{L}}$ if $\|t_1\|_{\mathbf{M},v} = \|t_2\|_{\mathbf{M},v}$, otherwise $0^{\mathbf{L}}$
- $\|\varphi \,\&\, \psi\|^{\mathbf{L}}_{\mathbf{M},v} = \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} * \|\psi\|^{\mathbf{L}}_{\mathbf{M},v}$
- $\|\varphi \to \psi\|^{\mathbf{L}}_{\mathbf{M},v} = \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v} \Rightarrow \|\psi\|^{\mathbf{L}}_{\mathbf{M},v}$
- $\|0\|^{\mathbf{L}}_{\mathbf{M},v} = 0$
- $\|\Delta\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = \Delta\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$
- $\|\forall x\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = \bigwedge_{v\equiv_x v'} \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v'}$
- $\|\exists x\varphi\|^{\mathbf{L}}_{\mathbf{M},v} = \bigvee_{v\equiv_x v'} \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v'}$

BL∀Δ — Semantics (Cont.)

An **L**-structure **M** is *safe* if $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ is defined for each φ and v.

The truth value of a formula φ of a predicate language $\mathcal{L}$ in a safe **L**-structure **M** for $\mathcal{L}$ is

$$\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = \bigwedge_{v \text{ an } \mathbf{M}-\text{evaluation}} \|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$$

M is an *admissible* **L**-structure for $\mathcal{L}$ if all the axioms for $=$ are 1-true in **M**.

Completeness

Theorem

Let $\mathcal{C}$ be a schematic extension of BL∀Δ, let T be a theory over $\mathcal{C}$, and let φ be a formula of the language of T. Then T proves φ iff for each $\mathcal{C}$-chain $\mathbf{L}$ and each safe admissible $\mathbf{L}$-model $\mathbf{M}$ of T, φ holds in $\mathbf{M}$.

Conservative extension

Let T' in language $\mathcal{L}'$ extend T in language $\mathcal{L} \subseteq \mathcal{L}'$. Then T' is a conservative extension of T whenever

- $P \notin \mathcal{L}$ is an n-ary predicate symbol, $\varphi(x_1, \ldots, x_n)$ a formula of $\mathcal{L}$, and T' results from T by adding the formula $\forall x_1, \ldots, x_n(P(x_1, \ldots, x_n) \equiv \varphi(x_1, \ldots, x_n))$;
- $c \notin \mathcal{L}$ is a constant, $\exists x \Delta \phi(x)$ a closed formula of $\mathcal{L}$ provable in T, and T' results from T by adding the formula $\phi(c)$;
- $F \notin \mathcal{L}$ is an n-ary function symbol, $\varphi(x_1, \ldots, x_n, y)$ a formula of $\mathcal{L}$, such that $T \vdash \forall x_1, \ldots, x_n \exists y \Delta \varphi(x_1, \ldots, x_n, y)$, and T' results from T by adding the formula $\forall x_1, \ldots, x_n\, \varphi(x_1, \ldots, x_n, F(x_1, \ldots, x_n))$ and the congruence axiom
  $x_1 = z_1 \,\&\, \ldots \,\&\, x_n = z_n \rightarrow F(x_1, \ldots, x_n) = F(z_1, \ldots, z_n)$.

Technicalities

BL∀Δ proves the following:

- $\forall x \varphi \,\&\, \psi \rightarrow \forall x(\varphi \,\&\, \psi)$ (x not free in ψ)
- $\forall x \varphi \& \forall x \psi \rightarrow \forall x(\varphi \& \psi)$
- $\exists x \Delta \varphi \rightarrow \Delta \exists x \varphi$
- $\forall x \Delta \varphi \equiv \Delta \forall x \varphi$
- $\Delta(\varphi \vee \neg\varphi) \equiv \Delta(\varphi \rightarrow \Delta\varphi)$.

Introducing FST

Fuzzy Set Theory (FST) is a first-order theory over BL∀Δ (or its schematic extension $\mathcal{C}$).
medskip The basic language consists of a predicate symbol $\in$.

We define

- $\forall x \in y(\varphi)$ as $\forall x(x \in y \rightarrow \varphi)$
- $\exists x \in y(\varphi)$ as $\exists x(x \in y \,\&\, \varphi)$
- $x \subseteq y$ as $\forall z(z \in x \rightarrow z \in y)$

Crispness

Definition

- A formula φ is *crisp* in T iff $T \vdash \Delta(\varphi \vee \neg\varphi)$.
- We define $\mathrm{Crisp}(x) \equiv \forall u \Delta(u \in x \vee \neg u \in x)$.

Note that crispness for all basic predicate symbols of the theory means crispness (LEM) for every formula.

$\mathrm{Crisp}(x) \rightarrow \Delta\mathrm{Crisp}(x)$ is provable, so crispness is a crisp predicate.

Crisp Equality

Lemma

A theory with separation, singletons, and congruence for $\in$ over a logic which proves $(\varphi \to \varphi \& \varphi) \to (\varphi \vee \neg\varphi)$ proves $\forall x, y(x = y \vee \neg(x = y))$.

Retaining basic principles like separation or congruence, in Łukasiewicz logic and in product logic we get a crisp $=$.

Thus, *crisp equality* is a universal decision in FST.

Other potential crisp turns

Grishin 1979: In a theory with extensionality, successors, and congruence, crispness of $=$ implies crispness of $\in$.
Remedy (Shirahata 1999): modify the spelling of extensionality
$\forall x, y(x = y \equiv (\Delta(x \subseteq y) \& \Delta(y \subseteq x)))$

Powell, Grayson: In a theory with separation (in a weak axiomatic setting), the axiom of foundation
$\forall x(\neg(x = 0) \rightarrow \exists y \in x \neg \exists w(w \in x \& w \in y))$ implies LEM for all formulas.
Remedy (int.): use $\in$-induction instead of foundation.

Intuitionistic logic: adding axiom of choice implies LEM for all formulas.
Remedy: use Zorn's lemma instead of choice.

Axioms of FST

- (ext.) $\forall x, y(x = y \equiv (\Delta(x \subseteq y) \& \Delta(y \subseteq x)))$
- (empty) $\exists x \Delta \forall y \neg(y \in x)$
- (pair) $\forall x \forall y \exists z \Delta \forall u(u \in z \equiv (u = x \vee u = y))$
- (union) $\forall x \exists z \Delta \forall u(u \in z \equiv \exists y(u \in y \,\&\, y \in x))$
- (weak power) $\forall x \exists z \Delta \forall u(u \in z \equiv \Delta(u \subseteq x))$
- (inf.) $\exists z \Delta(\emptyset \in z \,\&\, \forall x \in z(x \cup \{x\} \in z))$
- (sep.) $\forall x \exists z \Delta \forall u(u \in z \equiv (u \in x \& \varphi(u, x)))$
  for any φ not containing free z
- (coll.) $\forall x \exists z \Delta[\forall u \in x \exists v\, \varphi(u, v) \rightarrow \forall u \in x \exists v \in z \varphi(u, v)]$
  for any φ not containing free z
- ($\in$-ind.) $\Delta \forall x(\forall y \in x \varphi(y) \rightarrow \varphi(x)) \rightarrow \Delta \forall x \varphi(x)$
  for any φ
- (support) $\forall x \exists z(\mathrm{Crisp}(z) \& \Delta(x \subseteq z)))$

The Universe $V^{\mathbf{L}}$

The class $V^{\mathbf{L}}$, defined by ordinal induction in ZF.
$L^+ = L - \{0\}$.

- $V^{\mathbf{L}}_0 = \{\emptyset\}$
- $V^{\mathbf{L}}_{\alpha+1} = \{f : \mathrm{Fnc}(f) \,\&\, \mathrm{Dom}(f) \subseteq V^{\mathbf{L}}_\alpha \,\&\, \mathrm{Rng}(f) \subseteq L^+\}$ for any ordinal α
- $V^{\mathbf{L}}_\lambda = \bigcup_{\alpha<\lambda} V^{\mathbf{L}}_\alpha$ for limit ordinals λ

$V^{\mathbf{L}} = \bigcup_{\alpha\in\mathrm{Ord}} V^{\mathbf{L}}_\alpha$

The Universe $V^{\mathbf{L}}$ (Cont.)

We define two binary functions from $V^{\mathbf{L}}$ into L, assigning to any $u, v \in V^{\mathbf{L}}$ the values $\|u \in v\|$ and $\|u = v\|$

$$\|u \in v\| = v(u) \text{ if } u \in D(v), \text{ otherwise } 0$$
$$\|u = v\| = 1 \text{ if } u = v, \text{ otherwise } 0$$

Using induction on the complexity of formulas, define for any formula $\varphi(x_1, \ldots, x_n)$ an n-ary function from $(V^{\mathbf{L}})^n$ into L, assigning to an n-tuple $u_1, \ldots, u_n$ the value $\|\varphi(u_1, \ldots, u_n)\|$.

Theorem

Let φ be a closed formula provable in FST. Then φ is valid in $V^{\mathbf{L}}$, i. e., ZF proves $\|\varphi\| = 1$.

An Inner Model of ZF in FST

(Hereditarily crisp transitive set)

$$HCT(x) \equiv \mathrm{Crisp}(x) \& \forall u \in x(\mathrm{Crisp}(u) \& u \subseteq x)$$

(Hereditarily crisp set)

$$\mathrm{H}(x) \equiv \mathrm{Crisp}(x) \,\&\, \exists x' \in \mathrm{HCT}(x \subseteq x')$$

Lemma

The class H is both crisp and transitive in FST:

- $FST \vdash \forall x(x \in H \vee \neg(x \in H))$
- $FST \vdash \forall x, y(y \in x \,\&\, x \in H \rightarrow y \in H)$

An Inner Model of ZF in FST (Cont.)

For φ a formula in the language of ZF, define φ^H inductively:
$\varphi^H = \varphi$ for φ atomic;
$\varphi^H = \varphi$ for $\varphi = 0$;
$\varphi^H = \psi^H \& \chi^H$ for $\varphi = \psi \& \chi$;
$\varphi^H = \psi^H \to \chi^H$ for $\varphi = \psi \to \chi$;
$\varphi^H = (\forall x \in H)\psi^H$ for $\varphi = (\forall x)\psi$.

Theorem

Let φ be a theorem of ZF. Then $\mathrm{FST} \vdash \varphi^{\mathrm{H}}$.

Thus H is an inner model of ZF in FST and ZF is consistent relative to FST.

Set theory over Intuitionistic Logic

William C. Powell: Extending Gödel's negative interpretation to ZF. J. Symb. Logic 40, 1975.

Inner model of ZF in ZF-INT, consisting of stabilized sets, defined with the axiom of *double complement* $\exists y \forall z(\neg\neg(z \in x) \rightarrow (z \in x))$

R. J. Grayson: Heyting-valued models for intuitionistic set theory. Lecture Notes Math. 753, 1979.

Builds a Heyting-valued universe, over a complete Heyting algebra, within ZF-INT. Shows relative consistency of ZF to ZF-INT, using a particular Boolean algebra.

Takeuti & Titani's Set Theory: Logic

G. Takeuti and S. Titani: Fuzzy Logic and Fuzzy Set Theory, Arch. Math. Logic 32, 1992.

Logic given by an algebra of truth values **L** on $[0, 1]$, with

- Łukasiewicz connectives,
- Gödel connectives,
- product conjunction,
- globalization $\Box$ (definable),
- the constant $1/2$
- quantifiers $\forall$, $\exists$

Takeuti & Titani's Set Theory: Logic

A Gentzen-style axiomatization of the first-order calculus is given. It is complete w. r. t. models over $[0, 1]$, but it includes an infinitary rule.

A similar logic, with also the product implication, is analyzed in P. Hájek's book and inspired the logic ŁΠ.

Takeuti & Titani's set theory FZF

The theory FZF uses the connectives of Gödel logic in its axioms, which are a suitable versions of equality, extensionality, pairing, union, power, $\in$-induction, separation, collection, infinity, double complement, and Zorn's lemma.

Equality admits many-valued interpretations in FZF:
$\forall u, v \Box(u = v \equiv \forall z(z \in u \equiv z \in v))$

TT — V^L in FZF

Work in FZF.
A set x is *stable* iff $\Box(x = \sim\sim x)$.
Define the class S of stable sets.
One can prove that if $ZFC \vdash \varphi$, then $FZF \vdash \varphi^S$, so S is an inner model of ZFC in FZF.

Then take $I = [0,1]^S$ and construct the universe S^I in S.
It holds in S that $\langle S^I, ||.|| \rangle$ is an **L**-valued model of FZF.

Takeuti and Titani — inner truth values

For a sentence φ, let $\varphi^{\sim} = \{x : x \in \{\emptyset\} \wedge \varphi\}$. Thus each subset of $\{\emptyset\}$ is the truth value of a sentence, and we have $(\emptyset \in \varphi^{\sim}) \equiv \varphi$.

Moreover, there is an order-preserving isomorphism ρ between $[0,1]^S$ and the sets A s. t. $\Box(A \subseteq \{\emptyset\})$.

For every formula φ, one can take $|\varphi| = \rho^{-1}(\varphi^{\sim})$ as the inner truth value of φ in $[0,1]^S$.

TT — completeness

Theorem

There exists a function $F(x)$ such that

1. for every formula $\varphi(x_1, \ldots, x_n)$

$$\forall x_1, \ldots, x_n \in^{\Box} S^I \cap S\,(||\varphi(x_1, \ldots, x_n)|| = |\varphi(F(x_1), \ldots, F(x_n))|)$$

2. $\forall u \exists x \in^{\Box} S^I \cap S\, \Box(u = F(x))$.

Hence F is a truth-value preserving isomorphism between S^I and the universe of FZF.
Moreover, if $FZF \vdash ||\varphi|| = 1$ in S^I, then $FZF \vdash \varphi$.

Titani's Lattice-Valued Set Theory

S. Titani: A lattice-valued set theory, Arch. Math. Logic 38, 1999.

Logic:
language $\wedge$, $\vee$, $\rightarrow$, 0, $\exists$, $\forall$
evaluated on a complete bounded lattice
with $e(x \rightarrow y) = \begin{cases} 1 & \text{if } e(x) \leq e(y) \\ 0 & \text{otherwise} \end{cases}$ for any evaluation e.
and a definable $\Box$ s. t. $\Box\varphi$ is $1 \rightarrow \varphi$.
Thus, $e(\Box\varphi) = \begin{cases} 1 & \text{if } e(\varphi) = 1 \\ 0 & \text{otherwise} \end{cases}$ for any evaluation e.
Quantifiers evaluated as usual.

Lattice-valued logic vs. BL∀Δ

In a BLΔ-chain, one can interpret Titani's semantics as follows:

- $(\varphi \wedge_T \psi)'$ is $\varphi' \wedge \psi'$
- $(\varphi \vee_T \psi)'$ is $\varphi' \vee \psi'$
- $(\varphi \rightarrow_T \psi)'$ is $\Delta(\varphi \rightarrow \psi)$
- $(\Box\varphi)'$ is $\Delta\varphi'$

Then all the rules of lattice-valued logic are sound in any BLΔ-chain. Thus if φ is provable in lattice-valued logic, then φ' is true in BLΔ-chains, and hence provable in BL∀Δ by completeness.

(However, the interpretation is not faithful. For example, $\alpha \wedge (\beta \vee \gamma) \equiv (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ is not a theorem of lattice-valued logic.)

Crisp equality

It follows from Titani's axiom for equality, namely,

$$(x = y \wedge_T \varphi(x)) \rightarrow_T \varphi(y)$$

that $=$ is "crisp" (i. e., two-valued) in lattice-valued set theory, as the formula $(x = y \wedge \Box(x = x) \rightarrow_T \Box(x = y)$ is an instance of this axiom.

Set-theoretic axioms

For the following set-theoretic axioms of lattice-valued set theory, their $'$-translations are provable in FST: extensionality, pairing, union, (weak) power, separation, collection, support, infinity.
As for $\in$-induction, it spells
$\Delta\forall x(\Delta\forall y(y \in x \to \varphi(y)) \to \varphi(x)) \to \forall x\varphi(x)$
It is unclear whether this formula is provable in FST.
In order to fully interpret lattice-valued set theory in FST, one can adopt the stronger version of $\in$-induction (which is valid in $V^{\mathbf{L}}$) instead of the original axiom. Further, one would have to adopt Zorn's lemma, which is an axiom of lattice-valued set theory.