Structural completeness and extensions of fuzzy logics with rational constants

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Outline

Setting: Lukasiewicz, Gödel, and product fuzzy logic with rational constants.

Problems:

- structural completions of said logics;
- subquasivariety structure of their equivalent algebraic semantics.

This talk will juxtapose the Łukasiewicz and the product logic with and w/o constants.

Logic, extensions

A (propositional) logic $\vdash$ is a structural consequence relation on the set of terms $Fm_{\mathcal{L}}$ of a language $\mathcal{L}$, i.e., for any $\Gamma \cup \Delta \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$,

- if $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$;
- if $\Gamma \vdash \varphi$ and $\Delta \vdash \gamma$ for each $\gamma \in \Gamma$, then $\Delta \vdash \varphi$;
- $\Gamma \vdash \varphi$ implies $\sigma(\Gamma) \vdash \sigma(\varphi)$ for each substitution σ on $Fm_{\mathcal{L}}$.

All logics $\vdash$ in this work are finitary: if $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$ for some finite $\Delta \subseteq \Gamma$.

For two logics $\vdash$ and $\vdash'$ in $\mathcal{L}$, $\vdash'$ is an extension of $\vdash$ provided that $\Gamma \vdash \varphi$ implies $\Gamma \vdash' \varphi$ for $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$.

The extension is axiomatic if there is a set $\Sigma \subseteq Fm_{\mathcal{L}}$ closed under substitutions such that for all $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$,

$$\Gamma \vdash' \varphi \Longleftrightarrow \Gamma \cup \Sigma \vdash \varphi.$$

Derivable and admissible rules

Let $\Gamma = \{\gamma_1, \dots, \gamma_n\} \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ and let $\vdash$ be a logic in $\mathcal{L}$.

A rule is an expression of the form $\Gamma \rhd \varphi$.

A rule $\Gamma \rhd \varphi$ is derivable in $\vdash$ if $\Gamma \vdash \varphi$.

A term φ is a theorem of $\vdash$ if $\emptyset \vdash \varphi$ (we write $\vdash \varphi$).

A rule $\Gamma \rhd \varphi$ is admissible in $\vdash$ if for each substitution σ on $Fm_{\mathcal{L}}$,
$\vdash \sigma(\gamma)$ for each $\gamma \in \Gamma$ implies $\vdash \sigma(\varphi)$.

A derivable rule of $\vdash$ is admissible in it. The converse is generally not the case.

Structural completeness

A logic $\vdash$ is structurally complete (SC) if all of its admissible rules are derivable.
It is hereditarily structurally complete (HSC) if each extension is SC.

Each logic $\vdash$ admits a unique structural completion $\vdash^+$; i.e.,
a structurally complete extension with the same theorems.

Then a rule is admissible in $\vdash$ if and only if it is derivable in $\vdash^+$.

Admissible rules of $\vdash$ form a standalone structural consequence relation,
and it makes sense to ask

- about its axiomatization;
- whether the relation is decidable.

[Rybakov: Admissibility of logical inference rules. Elsevier, 1997] and references therein.

Active and passive structural completeness

Let $\Gamma = \{\gamma_1, \dots, \gamma_n\} \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ and let $\vdash$ be a logic in $\mathcal{L}$.

A rule $\Gamma \rhd \varphi$ is active in $\vdash$ if there is a substitution σ on $Fm_{\mathcal{L}}$ such that $\vdash \sigma(\gamma)$ for each $\gamma \in \Gamma$. Otherwise the rule is passive.

Any passive rule is admissible in $\vdash$.

Then $\vdash$ is actively structurally complete (ASC) (or almost structurally complete) if each of its active rules that is admissible in it is also derivable in it.

Moreover $\vdash$ is passively structurally complete (PSC) if each of its passive rules is derivable in it.

[Wroński, Overflow rules and a weakening of structural completeness. 2009]
[Dzik, Stronkowski: Almost structural completeness: an algebraic approach. APAL, 2016]

Algebraic characterization

Let $\mathcal{L}$ be a language, $\vdash$ a logic in $\mathcal{L}$, and K a quasivariety of $\mathcal{L}$-algebras that forms the equivalent algebraic semantics for $\vdash$.

Admissibility, structural completeness, etc., introduced for $\vdash$ naturally translate to K: instead of admissible rules, we speak of admissible quasiequations, etc.

We get the following characterization:

- $\vdash$ is SC iff K is generated as a quasivariety by $F_{\mathsf{K}}(\omega)$;
- $\vdash$ is HSC iff K is primitive.

In fact, we have $SC(\mathsf{K}) = \mathbb{Q}(F_{\mathsf{K}}(\omega))$.

[Bergman: Structural completeness in algebra and logic. 1991]

Hájek's basic logic BL

Language: $\{\odot, \to, 0, 1\}$. The following are axioms for BL:

- $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
- $(\varphi \odot (\varphi \to \psi)) \to (\psi \odot (\psi \to \varphi))$
- $(\varphi \to (\psi \to \chi)) \to (\varphi \odot \psi \to \chi)$
- $(\varphi \odot \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
- $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
- $0 \to \varphi$

The deduction rule is modus ponens.

Some more connectives: $\neg\alpha$ is $\alpha \to 0$;
$\alpha \wedge \beta$ is $\alpha \odot (\alpha \to \beta)$;
$\alpha \vee \beta$ is $((\alpha \to \beta) \to \beta) \wedge ((\beta \to \alpha) \to \alpha)$.

BL is algebraized with the variety of BL-algebras.
The subdirectly irreducible algebras are chains (totally ordered).

The variety BL is generated by standard BL-algebras,
i.e., structures given some continuous t-norm $*$ on $[0, 1]$.

[Hájek: Metamathematics of fuzzy logic 1998]
[Cignoli,Esteva,Godo,Torrens: Basic fuzzy logic is the logic of continuous t-norms and their residua, 2000]

Łukasiewicz logic and MV-algebras

Łukasiewicz logic **Ł** extends BL with the axiom $\neg\neg\varphi \to \varphi$.
MV, the variety of MV-algebras, is its equivalent algebraic semantics.

The standard MV-algebra is the structure $R^-_{Ł} = \langle [0,1], \odot^{Ł}, \to^{Ł}, 0, 1 \rangle$ where
$x \odot^{Ł} y = \max(0, x + y - 1)$, and
$x \to^{Ł} y = \min(1, 1 - x + y)$ for each x, y in $[0, 1]$.

The subalgebra of $R^-_{Ł}$ on the rationals is denoted $Q^-_{Ł}$.

MV is generated by $R^-_{Ł}$ as a quasivariety.
Also, MV is generated by $Q^-_{Ł}$ as a quasivariety.

The lattice of subvarieties of MV is countably infinite, and has been described by Komori. (Axiomatization due to Di Nola and Lettieri).

Every rational number in $[0, 1]$ is implicitly definable in $R^-_{Ł}$;
i.e., it is a solution to a finite system of equations.
NB. The algebra $R^-_{Ł}$ has no nontirival automorphisms.

[Chang, Trans. AMS, 1958-59]

Q-universality of MV

For a quasivariety K, denote **L**(K) the lattice of subquasivarieties.

Let K be a quasivariety in a finite language.

K is Q-universal if, for each quasivariety M of algebras in a finite language, **L**(M) is a homomorphic image of a sublattice of **L**(K).

Q-universality propagates to superclasses (in the same language).

Theorem [Adams, Dziobiak 1994]

MV *is Q-universal.*

Prior to that, Dziobiak proved that "certain conditions" (on finite algebras in K) are sufficient for **L**(K) to fail any nontrivial lattice identity.

Adams and Dziobiak prove that said "certain conditions" are sufficient for K to be Q-universal (whence **L**(K) fails any nontrivial lattice identity).

"Certain conditions" hold in many familiar classes, such as (double) Heyting algebras, distributive (double) p-algebras, de Morgan algebras, etc.
Plus, they hold in MV.

[Adams, Dziobiak: Q-universal quasivarieties of algebras, Proc. AMS, 1994]

Product logic

Product logic **P** extends BL with the axiom $\neg\varphi \vee ((\varphi \to \varphi \cdot \psi) \to \psi)$.
PA, the variety of product algebras, is the equivalent algebraic semantics.

The standard product algebra is the structure $\mathsf{R}_P^- = \langle [0,1], \odot^{\mathrm{P}}, \to^{\mathrm{P}}, 0, 1 \rangle$ where
$x \odot^{\mathrm{P}} y = x \cdot y$ (multiplication of reals), and
$x \to^{\mathrm{P}} y = \min(1, y/x$, for each x, y in $[0,1]$.

The subalgebra of R_P^- on the rationals is denoted Q_P^-.

There is a categorical equivalence between product chains and (Abelian) o-groups.
In particular,
– the nonzero elements of a product chain A are the negative cone of some o-group $\Lambda(\mathrm{A})$;
– the negative cone of an o-group G, equipped with a bottom, is a product chain $\Gamma(\mathrm{G})$.

[Gurevich, Kokorin 1963] Each two nontrivial o-groups have the same universal theory.

Hence, each two nontrivial product chains of cardinality at least 3 have the same universal theory, and each such chain generates PA as a quasivariety.

The lattice of subquasivarieties of PA is a three-element chain:
PA, BA (the variety of Boolean algebras), and the trivial variety.

[Hájek,Godo,Esteva: A complete many-valued logic with product conjunction. 1996]
[Cignoli, Torrens: An algebraic analysis of product logic. Multiple-valued logic, 2000]

Gödel(–Dummett) logic

Gödel logic **G** extends BL with the axiom $\varphi \to (\varphi \odot \varphi)$.
It can also be viewed as an extension of intuitionistic logic with Dummett's axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$.

GA, the variety of Gödel algebras, forms the equivalent algebraic semantics.

The standard Gödel algebra is the structure $\mathsf{R}_G^- = \langle [0,1], \odot^{\mathrm{G}}, \to^{\mathrm{G}}, 0, 1 \rangle$ where
$x \odot^{\mathrm{G}} y = \min(x, y)$, and
$x \to^{\mathrm{G}} y = y$ for $y < x$, where x, y in $[0, 1]$.

The subalgebra of R_G^- on the rationals is denoted Q_G^-.

NB. The operations in a Gödel chain are completely determined by its order.
Hence any infinite G-chain generates GA as a quasivariety.

Every extension of **G** is axiomatic; the lattice of extensions is a chain, (dually) ordered by $\omega + 1$.

[Dummett: A propositional calculus with denumerable matrix. JSL,1959]
[Dzik and Wroński, Structural completeness in Gödel's and Dummett's propositional calculi. SL, 1973]

Structural completeness results for some fuzzy logics

G is HSC. [Dzik and Wroński, 1973]

Ł is structurally incomplete. [Dzik 2008]
– admissible quasiequations/clauses are PSPACE-complete [Jeřábek 2010,2013]
– explicit bases (necessarily infinite) provided by [Jeřábek 2010]

Examples in **Ł**:

- $x \leftrightarrow (\neg x)^n \rhd y$ is passive.
- $y \vee (x \leftrightarrow (\neg x)^n) \rhd y$ is active.

P is HSC [Cintula and Metcalfe 2009]

Rational expansions

Let $A = \langle [0,1], \odot^A, \to^A, 0^A, 1^A \rangle$ be the standard MV-algebra $R^-_{Ł}$
(rational Gödel algebra R^-_G, rational product algebra R^-_P respectively).

Consider a set of propositional constants $\mathcal{C} = \{\boldsymbol{c}_q : q \in \mathbb{Q}\}$.

Note: we use $\mathbb{Q}$ for the rationals in $[0,1]$ and $\mathbb{R}$ for the reals therein.

Rational Lukasiewicz logic **RŁ**, (Rational Gödel logic **RG**, Rational product logic **RP**) is the expansion of Łukasiewicz logic (...) with the constants in $\mathcal{C}$ and the following bookkeeping axioms:

$$(\boldsymbol{c}_r \odot \boldsymbol{c}_s) \leftrightarrow \boldsymbol{c}_{r \odot^A s} \quad \text{and}$$
$$(\boldsymbol{c}_r \to \boldsymbol{c}_s) \leftrightarrow \boldsymbol{c}_{r \to^A s} \quad \text{for any rational numbers } r \text{ and } s \text{ in } [0,1].$$

(And, for convenience, $\boldsymbol{c}_0 \leftrightarrow 0$ and $\boldsymbol{c}_1 \leftrightarrow 1$.)
The logics **RŁ**, **RG**, and **RP** are algebraizable, with equivalent algebraic semantics provided by the varieties RMV, RGA, and RPA respectively.

Moreover, each of the three varieties is generated by its chains.

[Pavelka 1979: On fuzzy logic I,II,III, ZMLGM, 1979]
[Hájek: Metamathematics of Fuzzy Logic, 1998]
[Esteva, Gispert, Godo, Noguera: Adding truth constants to logics ..., FSS, 2007]

On interpretation of constants

If $\mathrm{R}_{Ł}^{-} = \langle [0,1], \odot^{\mathrm{A}}, \to^{\mathrm{A}}, 0, 1 \rangle$ is the standard MV-algebra, then
$\mathrm{R}_{Ł} = \langle [0,1], \odot^{\mathrm{A}}, \to^{\mathrm{A}}, 0, 1, \{q : q \in \mathbb{Q}\} \rangle$ (i.e., $\boldsymbol{c}_q^{\mathrm{A}} = q$)
is the canonical standard RMV-algebra $\mathrm{R}_{Ł}$.

Moreover, $\mathrm{Q}_{Ł}$ denotes the rational subalgebra of $\mathrm{R}_{Ł}$.

Analogously for Gödel and product case, obtaining R_G, R_P, Q_G, Q_P respectively.

Let L be one of **Ł**, **G**, **P** and A a nontrivial RL-chain.
Let $F(\mathrm{A}) = \{q \in \mathbb{Q} : \boldsymbol{c}_q^{\mathrm{A}} = 1^{\mathrm{A}}\}$.
Then $F(\mathrm{A})$ is a filter on Q_L^{-} (one of $\mathrm{Q}_{Ł}^{-}$, Q_G^{-}, Q_P^{-}).
Moreover if $r < s \in \mathbb{Q} \setminus F(A)$, we have $\boldsymbol{c}_r^{\mathrm{A}} < \boldsymbol{c}_s^{\mathrm{A}}$.

In particular,

- $\mathrm{Q}_{Ł}^{-}$ is simple;
- Q_P^{-} has only two proper filters, $\{1\}$ and the one consisting of nonzero elements;
- every nonempty upset in Q_G^{-} is a filter.

Thus we have $\mathrm{Q}_{Ł}^{-} \preceq \mathrm{A}$ for a nontrivial RMV-algebra A.

[Esteva, Gispert, Godo, Noguera: Adding truth constants to logics . . . , FSS, 2007]

RMV is structurally complete

It is known that RMV $= \mathbb{Q}(\mathrm{R}_{Ł})$. [Esteva, Gispert, Godo, Noguera 2007]

Theorem

$\mathrm{R}_{Ł}$ *and* $\mathrm{Q}_{Ł}$ *generate the same quasivariety.*

First, recall that each rational number q is implicitly definable in $\mathrm{R}_{Ł}^{-}$. This means there is a (finite!) set of equations T and a variable x_q occurring in T, such that for each assignment v in $\mathrm{R}_{Ł}^{-}$, $v(x_q) = q$ provided that $v(s) = v(t)$ for each $s \approx t \in T$.

Proof: let $\Phi := \Gamma \Longrightarrow \varphi$ be a RMV-quasiidentity, not valid in $\mathrm{R}_{Ł}$.
Let $\boldsymbol{c}_{r_1}, \dots, \boldsymbol{c}_{r_k}$ be the constants in Φ, and let T be a finite set of equations that define all these constants in $\mathrm{R}_{Ł}^{-}$.
let $\Phi^\star$ result from Φ by replacing the rational constants $\boldsymbol{c}_{r_1}, \dots, \boldsymbol{c}_{r_k}$ with $z_1, \dots, z_k$.

Then we get $\mathrm{R}_{Ł}^{-} \not\models T \cup \Gamma^\star \Longrightarrow \varphi^\star$.
Moreover, there is a finite n such that $Ł_n \not\models T \cup \Gamma^\star \Longrightarrow \varphi^\star$.
Now $Ł_n \preceq \mathrm{Q}_{Ł}^{-}$, so $\mathrm{Q}_{Ł}^{-} \not\models T \cup \Gamma^\star \Longrightarrow \varphi^\star$.

Finally, replacing the variable with the constants, we get $\mathrm{Q}_{Ł} \not\models \Phi$.

[Aguzzoli, Ciabattoni: Finiteness in infinite-valued Łukasiewicz logic, 2000]

RMV is structurally complete – cont'd

Hence RMV is structurally complete.
Indeed RMV is minimal, since $\mathrm{Q}_{\text{Ł}} \preceq \mathrm{A}$ for any nontrivial RMV-algebra A,
so for any nontrivial $\mathrm{K} \subseteq \mathrm{RMV}$ we have $\mathrm{RMV} = \mathbb{Q}(\mathrm{Q}_{\text{Ł}}) \subseteq \mathrm{K}$.

Recall conservativity: **RŁ** and **Ł** derive the same rules in language of BL.
(Since they both have finite strong standard completeness.)

Thus, rules admissible in **Ł** are not admissible/derivable in **RŁ**,
unless they are derivable in **Ł**.

Example: recall $x \approx (\neg x)^n$ implicitly defines $^1/_{(n+1)}$.

$x \leftrightarrow (\neg x)^n \rhd y$ is passive in **Ł**.

But not in **RŁ**: substitute $\sigma(x) = \boldsymbol{c}_{1/(n+1)}$

RPA

It is known that RPA $= \mathbb{V}(\mathrm{R}_P)$. [Savický, Cignoli,Esteva,Godo,Noguera, 2006]

Theorem

RPA $= \mathbb{V}(\mathrm{Q}_P)$.

Proof sketch: show that if $t(x_1, \dots, x_k) \approx 1$ fails in R_P, then it fails in Q_P.

To that end, approximate (positive!) irrationals with (sequences of) positive rationals.

There is a two-element congruence $\sim$ on R_P, conflating nonzero elements.
The operations $\odot$ and $\to$ are continuous on the nonzero block.
On the other hand, we can evaluate $a \odot b$ and $a \to b$ if at least one of a, b is 0.

Consider an assignment $\bar{a} = (a_1, \dots, a_k)$ in R_P such that $t(\bar{a}) < 1$.
If all a_i are rational, we are done.
Then for an $i \leq k$, let a_i be irrational (thus positive), and let a_{in} be a sequence of positive rationals tending to a_i.
Let $f_t(x) = t(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_k)$, and analogously for any subterm s of t.
We claim the sequence $f_t(a_{in})$ tends to $f_t(\bar{a})$.
(Note that, since $a_{in} \sim a_i$, we have $f_s(a_{in}) \sim f_s(\bar{a})$ for each subterm s of t.)
So for sufficiently large n, we have $f_t(a_{in}) < 1$.
Replace (irrational) a_i with (rational) a_{in} and repeat for any remaining irrationals.

Structural completions in RPA

A nontrivial quasivariety $\mathsf{K} \subseteq \mathsf{RPA}$ either contains Q_P, or it is termwise-equivalent to PA or to BA.

Theorem

Let K *be a subquasivariety of RPA.*

(i) *If* $\mathsf{Q}_P \in \mathsf{K}$*, the structural completion of* K *is* $\mathbb{Q}(\mathsf{Q}_P)$*.*

(ii) *If* $\mathsf{Q}_P \notin \mathsf{K}$*, then* K *is hereditarily structurally complete.*

Proof sketch (i):
Q_P is a subalgebra of $F_{\mathsf{K}}(\omega)$.
We have $\mathsf{RPA} = \mathbb{V}(\mathsf{Q}_P)$.
Thus the ω-generated free algebras of RPA, K, and $\mathbb{Q}(\mathsf{Q}_P)$ coincide.
Moreover, $\mathbb{Q}(\mathsf{Q}_P) = \mathbb{Q}(F_{\mathbb{Q}(\mathsf{Q}_P)}(\omega))$.

Some quasiidentities in RPA

First, consider the quasiequation $x^2 \approx \frac{1}{2} \Rightarrow 0 \approx 1$;
true in Q_P, but false in R_P.

Second, let P be the set of primes. Now for $p \in P$, consider

$$x^2 \approx \frac{1}{p} \Rightarrow 0 \approx 1 \qquad (\Phi_p)$$

Then $\{\Phi_p : p \in P\}$ has a model, namely Q_P.

The quasiequations Φ_p are independent:
i.e., no set $\{\Phi_r,\ r \in R\}$ with $R \subseteq P$ implies Φ_p unless $p \in R$.

For $R \subseteq P$, let K_R be the quasivariety axiomatized by $\{\Phi_r, r \in R\}$.
For $R, S \subseteq P$, we have $\mathrm{K}_R \subseteq \mathrm{K}_S$ iff $S \subseteq R$.

Hence the lattice of extensions of RPA has the width of the continuum.

Axioms for the universal theory of Q_P

Theorem

Relative to RPA, the universal theory of Q_P is axiomatized by:

(1) $(x \leq y) \curlyvee (y \leq x))$

(2) $(\boldsymbol{c}_q \approx 1) \Longrightarrow (0 \approx 1)$

(3) $(\boldsymbol{c}_p \approx x^n) \Longrightarrow (0 \approx 1)$

for each $q \in [0,1) \cap \mathbb{Q}$ and each $p \in \mathbb{Q}$ and $n \in \omega$ such that $\sqrt[n]{p}$ is irrational.

Proof sketch: we prove that if a RPA-algebra A validates the above axioms, it is partially embeddable in Q_P. By (1) it is enough to address chains. Note also $\mathrm{Q}_P \preceq \mathrm{A}$, thanks to (2).

Let B be a finite partial subalgebra of A on $\{b_1, \dots, b_n\} \subseteq A \setminus \{0\}$. Take the enveloping o-group $\Lambda(\mathrm{A}^-)$ of the product reduct of A. Let $\langle B \rangle$ be the subgroup generated by B in $\Lambda(\mathrm{A}^-)$, and let $\boldsymbol{Q}_B$ be the "rational" subgroup of $\langle B \rangle$ (i.e., with universe $\langle B \rangle \cap \Lambda({\mathrm{Q}_P}^-)$).

Fact 1. Let F be a free abelian group of rank n and G a nontrivial subgroup of F. Then there exists a basis $\{e_1, \dots, e_n\}$ of F, and positive integers $r, d_1, \dots, d_r$ such that $r \leq n$ and G is free abelian with basis $\{e_1^{d_1}, e_2^{d_2}, \dots, e_r^{d_r}\}$.

Axioms for the universal theory of Q_P – cont'd

We get a basis $\{e_1, \dots, e_m\}$ for $\langle B \rangle$, and a basis $\{e_1^{d_1}, \dots, e_k^{d_k}\}$, with $k \leq m$, for $\boldsymbol{Q}_B$.
Thanks to "irrational roots forbidding" axioms (3),
we can assume wlog $d_i = 1$ for each $i \leq k$.
Thus each e_i is rational in A, i.e., there is a constant $\boldsymbol{c} \in \mathcal{C}$ s.t. $\boldsymbol{c}^{\mathrm{A}} = e_i$.

Now (roughly speaking) let C be a partial subalgebra of A containing $\{b_1, \dots, b_n\}$ and $\{e_1, \dots, e_m\}$, and some elements that show how each b_i is generated from $\{e_1, \dots, e_m\}$.

Fact 2 [Savický et al.]
Each nontrivial chain in RPA with constants interpreted by pairwise distinct elements is partially embeddable in R_P, with canonical intp. of defined rational constants.

We get a copy of C (and of B) in R_P; if $\boldsymbol{c}_q^{\mathrm{A}} \in \mathrm{B}$, then it maps (canonically) to $q \in [0, 1]$; also $\{e_1, \dots, e_k\}$ are rational.

Finally, we replace the irrational elements in the basis by rationals "close enough" s.t. the order of C is not disturbed, and preserving rational independence.
(This does not impact the rational elements of C).
This provides an embedding of C into Q_P; in particular, to the nonzero part.

[Savický, Cignoli, Esteva, Godo, Noguera: On product logic with truth constants. 2006]

Axioms for the quasiequational theory of Q_P

It is well known that RPA has EDP(R)M, witnessed by the equation

$$\nabla(x, y, z, v) = \{(x \leftrightarrow y) \vee (z \leftrightarrow v) \approx 1\}.$$

Consider the following (special case of) theorem.

Theorem [Czelakowski, Dziobiak 1990]

Let K *be a quasivariety with EDPRM witnessed by a single equation* $\alpha(x, y, z, v) \approx 1$. *Let* $\mathrm{M} \subseteq \mathrm{K}_{\mathrm{RFSI}}$. *Furthermore, assume that* $\mathbb{ISP}_{\mathrm{U}}(\mathrm{M})$ *is axiomatized by a set* Σ *of sentences of the form* $\varphi_1 \approx \psi_1 \Longrightarrow (\epsilon_1 \approx \delta_1 \curlyvee \epsilon_2 \approx \delta_2)$. *Then* $\mathbb{Q}(\mathrm{M})$ *is axiomatized relatively to* K *by the set of quasiequations*

$$\varphi_1 \approx \psi_1 \Longrightarrow \alpha(\epsilon_1, \delta_1, \epsilon_2, \delta_2) \approx 1$$

$$\alpha(\varphi_1, \psi_1, x, y) \approx 1 \Longrightarrow \alpha(\epsilon_1, \delta_1, x, y) \approx 1$$

where x, y *are fresh variables and there exists* $\Phi \in \Sigma$ *such that* $\Phi = ((\varphi_1 \approx \psi_1 \Longrightarrow (\epsilon_1 \approx \delta_1 \curlyvee \epsilon_2 \approx \delta_2))$.

Using the above translation, we obtain the following axiomatization (relative to RPA):

- $\boldsymbol{c}_q \vee z \approx 1 \Longrightarrow z \approx 1$ for every rational $q \in [0, 1)$;
- $(\boldsymbol{c}_p \leftrightarrow x^n) \vee z \approx 1 \Longrightarrow z \approx 1$ for every rational $p \in [0, 1]$ and $n \in \omega$ such that $\sqrt[n]{p}$ is irrational.

Decidability of RPA-admissible rules

Theorem

Admissibility in RPA *is decidable.*

Proof: quasiequational theory of Q_P is recursively axiomatized, hence r.e.

On the other hand, one can generate all tuples $\vec{a}$ of rationals
such that

$$\gamma_1^{\mathrm{Q}_P}(\vec{a}) = \cdots = \gamma_n^{\mathrm{Q}_P}(\vec{a}) = 1 \text{ and } \varphi^{\mathrm{Q}_P}(\vec{a}) \neq 1$$

Hence also the complement of the quasiequational theory of Q_P is r.e.

Let K be a quasivariety ~~in a finite language~~.

K is Q-universal if, for each quasivariety M of algebras in a finite language, **L**(M) is a homomorphic image of a sublattice of **L**(K).

Theorem

RPA *is Q-universal.*

Proof notes:

Ψ ... set of prime numbers. Denote

$$\mathcal{S}(\mathcal{P}_{\mathrm{fin}}(\Psi))$$

the lattice of subalgebras (sub-join-semilattices) of $\langle \mathcal{P}_{\mathrm{fin}}(\Psi), \cup, \emptyset \rangle$. We have

Theorem: [Adams and Dziobiak] K quasivariety and $\mathsf{M} \in \mathbb{L}(\mathsf{K})$. If there is a surjective homomorphism of complete lattices $h \colon \mathbb{L}(\mathsf{M}) \to \mathcal{S}(\mathcal{P}_{\mathrm{fin}}(\Psi))$, then K is Q-universal.

1. For $X = \{p_1, \dots, p_n\} \in \mathcal{P}_{\rm fin}(\Psi)$, set

$$\mathrm{A}_X \coloneqq \text{the subalgebra of } \underbrace{\mathrm{R}_P \times \cdots \times \mathrm{R}_P}_{n\text{-times}} \text{ generated by } \langle {}^1/\sqrt{p_1}, \dots, {}^1/\sqrt{p_n} \rangle.$$

2. Let

$$\mathrm{M} \coloneqq \mathbb{Q}\{\mathrm{A}_X : X \in \mathcal{P}_{\rm fin}(\Psi)\}.$$

Then $\mathrm{M} \subseteq \mathbb{Q}(\mathrm{R}_P)$, since each A_X embeds into a direct power of R_P.

3.

$$h \colon \mathbf{L}(\mathrm{M}) \to \mathcal{S}(\mathcal{P}_{\rm fin}(\Psi))$$

defined for every $\mathrm{K} \in \mathbf{L}(\mathrm{M})$ as

$$h(\mathrm{K}) \coloneqq \{X \in \mathcal{P}_{\rm fin}(\Psi) : \mathrm{A}_X \in \mathrm{K}\}.$$

4. The map h is a well-defined surjective homomorphism of complete lattices.

Resume of our results

- **RŁ** is SC;
- **RP** is not SC. We axiomatize SC(**RP**) and show it is decidable. Moreover, SC(RPA) and the three proper subvarieties are the only (H)SC extensions of RPA;
- we characterize PSC in **RP**;
- the extensions of **RG** forms an uncountable chain; for arbitrary extensions, there is also an uncountable antichain;
- for extensions of **RG**, ASC, SC, and HSC coincide, and occur if and only if the quasivariety algebraizing the extension is singly generated by a chain;
- extensions of **RG** are PSC iff the quasivariety algebraizing the extension has the JEP.

Concluding remarks

Fuzzy logics with constants (rational, or what have you) are philosophically defensible.

They have a great pedigree.
[Goguen, Pavelka, Hájek, Esteva, Godo, Cignoli, Noguera, Paris, Cintula, Shepherdson] and many others

They also give rise to some nice maths.

Adding rational constants may seem innocuous. (In particular, they are implicitly definable in Łukasiewicz logic.)

However constants impact structural completeness and subquasivariety lattice.

The end

Many thanks for your attention.