On Vopěnka's ultrafinitism

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Outline

- ultrafinitism and feasibility
- semisets and Vopěnka's Alternative Set Theory
- witnessed universes
- reception
- logic

Not necessarily in this order.

Some sources and timeline

[Vopěnka, Hájek: The theory of semisets. North Holland, 1972]

Vopěnka's Alternative Set Theory (AST),
proposed by Vopěnka (ca 1973) and developed in his seminar in Prague (late 1970's).

Vopěnka: Mathematics in the Alternative Set Theory. Teubner, 1979.
Sochor: The alternative set theory and its approach to Cantor's set theory. 1982.
Sochor: Metamathematics of the AST I.–III.,, 1979–1983.
Vopěnka: Úvod do matematiky v alternatívnej teórii množín. Alfa, Bratislava, 1989.

Many papers (not books) available via the Czech Digital Mathematical Library www.dml.cz.

Secondary reference:
[Fletcher: Infinity. Handbook of the Philosophy of Science, vol. 5 (ed. D. Jacquette), Elsevier, pp. 523–585, 2007]

Set-theoretic platonism

Originating with Bolzano and Cantor:

- actual (static) infinity: infinite collections as sets
- rejection of spatial and temporal intuition
- theological justifications for actual infinity

General platonistic position

- abstract objects exist
- truth is independent of our way of knowing it

Cantorian finitism:
Infinite sets behave just as finite sets do;
– Cantor's theory of ordinal and cardinal numbers;
– power-set axiom.

Idealization: divorcing mathematics from experience.

Ultrafinitism and feasibility

Bernays 1935 lecture Platonism in mathematics

"intuitionism as the sole foundation of maths is an "extreme methodological position", contrary to the customary way of doing mathematics, independent of the thinking subject"

"The weakest of the "platonistic" assumptions introduced by arithmetic is that of the totality of integers."

difficulty in capturing the limits of evidence

"From two integers k and l, one passes immediately to k^l. This process leads in a few steps to numbers which are far larger than any occurring in experience, e.g., $67^{257^{729}}$. . . . What does it mean to claim the existence of an Arabic numeral for the foregoing number, since in practice we are not in a position to obtain it? Isn't this rather . . . extending to inaccessible numbers the relations which we can concretely verify for accessible numbers?"

The objection (to founding natural numbers on counting) can be traced back to Cantor: "if generation or construction is all we have to rely on then we would have to abandon even large finite numbers"
[Hallett 1986]

Ultrafinitism and feasibility – cont'd

"In the formalist view of mathematics . . . the finitist statements play the role similar to that played in a scientific theory by experimental data. . . . the other formulae play the role of theoretical or ideal statements whose primary purpose is to smooth things out and to provide short cuts in reasoning."
[Parikh 1971]

Fletcher 2007:
the objection that it is not possible to count up to 10^{100} occurs

- as critique of intuitionism
- as genuine belief that arithmetic ought to be limited to accessible numbers.

Fletcher's "horizon problem"

(No matter which foundational position one takes,)
the totality of all acceptable objects is not acceptable.

In particular,

- (platonist) the totality of all infinite sets is not a set;
- (constructivist) the totality of all finite objects is infinite;
- (strict finitist) the totality of all feasible objects is not feasible.

Vopěnka's natural infinity

Vopěnka rejects actual infinity.
"Efforts of mathematicians to fully grasp actual infinity have been unsuccessful. But this does not diminish the importance of Cantor set theory, which remains a document of human aspiration to surpass the limits of space in a way having no analogy in history."

Cantor's set theory is fully dependent on formal means, without phenomenal content.

"We shall deal with the phenomenon of infinity in accordance with our experience, i.e., as a phenomenon involved in the observation of large, incomprehensible sets. We shall by no means use any ideas of actually infinite sets."

In AST, tries to present a theory of "natural" infinity
which brings back experience and perception to mathematics.

Broadly, this takes place in terms of

- sets: All sets are ZF-finite.
- classes: "properties", possibly not clearly perceived.
- semisets: proper subclasses of sets (forced axiomatically).

Semisets

[Vopěnka, Hájek: The theory of semisets, 1972]

(In a theory with sets and classes,)
a semiset is a subclass of a set.
I.e., classes are not limited to definable collections,
and the intersection of a class and a set is not necessarily a set.

The theory of semisets is conservative over the theory of sets.

[Levy 1984 review]: "TSS does not seem to be a convenient or useful formulation of set theory . . . The wealth of information in it is almost completely inaccessible to the student of set theory."

The book presents the results of Vopěnka set theory seminar
(presented in the semiset paradigm).

Other applications of the concept of semisets can be found in
[Hájek: Why semisets? 1973]

A few remarks on AST

Vopěnka's alternative set theory was first conceived as a system of principles.

The existing axiomatization, AST, was intended as an open system
(in the sense of adding new axioms "in the spirit" of the principles)

Focus on development of mathematics and on philosophy,
not axiomatization
("Mathematics in the AST")

The role of the Vopěnka AST seminar was indespensable;
AST exists as seen from today through the efforts of Antonín Sochor, Karel Čuda, Josef Mlček, Kateřina Trlifajová, Alena Vencovská, Jiří Witzany, Pavol Zlatoš, ...

Some materials exists in Czech (Slovak) only
(Vopěnka did not publish in English).

Sets in AST

Consider a language with sets and classes.
AST-axioms for sets:

- (extensionality for sets) $\forall xy(x = y \equiv \forall z(z \in x \equiv z \in y)$;
- (existence of sets) $\exists x \forall Y(Y \notin x)$ and $\forall xy \exists z(z = x \cup \{y\})$;
- (induction) $\varphi(\emptyset) \,\&\, \forall xy(\varphi(x) \to \varphi(x \cup \{y\})) \to \forall x \varphi(x)$;
- (regularity) $\exists x(\varphi(x)) \to \exists x(\varphi(x) \,\&\, \forall y \in x \neg\varphi(y))$,

where φ is a set formula.

This is equivalent to $\mathrm{ZF}_{\mathrm{fin}}$ (i.e., $\mathrm{ZF} \setminus \{\mathrm{Inf}\} \cup \{\neg\mathrm{Inf}\}$).

Thus, i.a.,
– proper subset of x is strictly subvalent to x (set injection);
– $\mathbf{N}$ is a proper class of natural numbers;
– cardinality: for each x, an $\alpha \in N$ s.t. $x \hat{\approx} \alpha$ (set bijection);
– induction for $\mathbf{N}$:
$\varphi(0) \,\&\, \forall \alpha \in \mathbf{N}(\varphi(\alpha) \to \varphi(\mathbf{S}(\alpha))) \to \forall \alpha \in \mathbf{N} \varphi(\alpha)$ for a set formula φ;

$\mathbf{N}$ (with usual op.'s) interprets PA.

Classes and semisets in AST

Class comprehension: for $\Phi(x)$ not containing Y,
$\exists Y \forall x (x \in Y \equiv \Phi(x))$

(properties, without necessarily being able to survey completely the collection)

Set(X) iff $\exists Y (X \in Y)$

Finite classes can be surveyed completely.
Fin(X) iff $\forall Y \subseteq X (Set(Y))$
In particular, finite classes are sets.

"Our infinity is a phenomenon occuring when we observe large sets. It manifests itself as an absence of an easy survey, as our inability to grasp the set in its totality."
[V-AST]

Sms(X) iff $\exists y (X \subseteq y)$

Axiom: proper semisets exist.

Existence of proper semisets implies existence of infinite sets.

E.g. x infinite iff for $y \notin x$ we have $x \approx x \cup \{y\}$,
with $\approx$ a class equivalence (bijection)
("infinite hotel")

Finite natural numbers

$\mathbf{FN} = \{\alpha \in \mathbf{N} : \mathrm{Fin}(\mathbf{n})\}$
($\mathbf{FN}$ is the class of all "finite" or "accessible" numbers)

Since $\forall x \exists \alpha \in \mathbf{N}(\mathbf{x} \hat{\approx} \alpha)$, infinite natural numbers exist and
$\mathbf{FN}$ is a proper initial segment of $\mathbf{N}$, and of any infinite $\alpha \in N$.

$\mathbf{FN}$ is a prototypical semiset in the AST.

Prolongation axiom: any class function on FN is a part of a set function.
Transcending the horizon ("milestones").

$\mathbf{FN}$ interprets PA.

In particular, $\mathbf{FN}$ is closed under addition, multiplication, and exponentiation;
Induction in $\mathbf{FN}$: for any formula Φ,
$\Phi(0)$ and $\forall n \in \mathbf{FN}\, (\boldsymbol{\Phi}(\mathbf{n}) \to \boldsymbol{\Phi}(\mathbf{n} \cup \{\mathbf{n}\})$ implies $\forall n \in \mathbf{FN}\, \boldsymbol{\Phi}(\mathbf{n})$.

Models of the AST in ZF

In ZF, take the set V_ω of hereditarily finite sets.

Let $(V_\omega^*, \in^*)$ be an ultrapower (over non-trivial ultrafilter on ω).
Add each $X \subseteq V^*$ unless there is $x \in V_\omega^*$ s.t. $X = \{y \mid (V_\omega^*, \in^*) \models y \in x$
(Assume CH to cater for the AST-axiom of cardinalities)

This yields a model of AST.
[Pudlák, Sochor 1984]

There is evidence that the intended (mathematical) interpretation of **FN** was the standard natural numbers.

AST – overview

Inspiration

Vopěnka embraces Robinson's nonstandard analysis as a major influence.

"When I familiarised myself with Skolem's construction [i.e., Skolem 1933], towards the end of 1960 I was able to construct a nonstandard model of set theory (more precisely, Gödel-Bernays axiomatic theory)."
[Vopěnka NMI 2014]

Aspiration

"It is not evident that Cantor's set theory is the best possible description of our comprehension of the real world . . . It is by no means obvious that Cantor's set theory is the best possible framework for mathematics . . . The alternative set theory, which was created by Vopěnka, is an attempt to construct a theory that could serve as an alternative to Cantor's set theory."
[Sochor 1982]

Reception

"Does AST live up to its name? . . . few mathematicians are willing to do mathematics in several set theories with different properties. No mathematician can be expected to forsake Cantor's set theory and to adopt AST as his or her only set theory without having reasons much more compelling than those given by the author."
[Levy 1984]

The shifting horizon

Can 10 be considered an “infinite” number, under some circumstances?

Tally marks

With effort, the horizon of clarity of perception shifts away from the observer.

Increasingly larger numbers present themselves as “infinite”
in increasing number of contexts.

Example 1:
Although $10^{10} \in \mathbf{FN}$ can be proved in relatively few steps
(**FN** closed under exponentiation!),
this string needs to be viewed as a pointer to a number.
The number itself (numeral) is beyond the horizon
and we cannot prove (in a “finite” number of steps) equality of the two representations.

Example 2:
A chessboard is easier to survey than a linear arrangement of 64 strokes.

Witnessed universes

A theory of a witnessed universe imposes
existence of a semiset within a concrete set.

Vopěnka uses "concrete" broadly:
for example closed PA-terms, even with exponentiation,
describe concrete sets.

"entirely concrete set (say, of all natural numbers less than $67^{293^{159}}$)"

Reasons for not accepting such axioms in the AST:

- philosophical: accessibility (of sets) depends on context
- mathematical: AST with $\{\neg(t \in \mathbf{FN})\}$ for a closed PA-term t is inconsistent

[Sochor 1982]

Idealization: AST is a theory of limit universes

NB. The theory of witnessed universes remains undeveloped.

Logic

Quantification over feasible domains:
"It is fine to use predicate calculus when studying the semiset **FN**, its properties and their mutual relationships. However, extra care needs to be taken with the quantifiers, as the semiset **FN** is not sharply delimited towards the horizon that bounds the size of finite natural numbers. On the other hand, predicate calculus can be applied to the study of a set γ, with **FN** $\subseteq \gamma$, without worry."
[Vopěnka NMI 2014]

"Graded" provability:
Proofs obeying laws of logic [esp. inductive proofs] have decreasing convincingness as their length increases. From this we could conclude that even finitary mathematical stetements can have quite complicated truth valueds, not just two."
[V-AST]

Echoes of AST

"The most comprehensive attempt at providing a foundation for mathematics along these lines [i.e., identifying 'feasible' with 'standard'] is *alternative set theory* (AST), developed by Vopěnka (1979) and Sochor (1984). ... Feasibility theories of this type may be considered unsatisfying, as they say nothing about the size of infeasible numbers."

On the other hand, adding $\neg F(\theta)$, for PR-term θ, to a theory with $F(n) \rightarrow F(n+1)$, leads to inconsistency.
The idea of axiomatizing feasibility with a single unary predicate is too simple-minded.
[Fletcher 2007]

"The origin of such theories [using non-standard arithmetic] in mathematical logic can be traced to attempts to come to terms with strict finitism as a foundational standpoint. But I will also suggest below that they are applicable as part of a theory of vague predicates within natural language. In this setting, the use of fromal arithmetic and nonstandard methods more generally is almost completely foreign. ... The only exception to this of which I am aware is work on vagueness in the tradition of Vopěnka's alternative set theory."
[Dean 2018]

NB. Almost no intersections with paraconsistent treatment of ultrafinitism.

Conclusions

The AST presented a coherent development of infinity ("lack of an easy survey").
The claim to "natural" derives from this new development.

Neither the theory of semisets nor the AST became a widely-used paradigm, contrary to the authors' (explicit) expectations.

The AST is provably consistent in ZF. (Not an issue for Vopěnka.)
Claim to feasibility, in the sense of standardness.

Witnessed universes, as well as variations in metatheory, are an undeveloped (possibly murky) area.

The AST can be viewed as an idealization ("limit case") of individual, context-dependent (possibly classically inconsistent) theories.

Vopěnka did not choose to compare his theory to other "feasibilist" systems, or more broadly, to other alternative foundational attempts.

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