On the complexity of validity degrees in Łukasiewicz logic

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A semantics for Łukasiewicz logic $Ł_\omega$

MV-terms: function symbols $\{\oplus, \neg\}$ on a countably infinite set of variables.

Domain: the interval $[0, 1]$ of the reals.
Interpretation of symbols:

$$x \oplus y = \min(1, x + y)$$
$$\neg x = 1 - x$$

The algebra $[0,1]_Ł = \langle [0,1], \oplus, \neg \rangle$ is an intended semantics of $Ł_\omega$.
The only designated value is 1 (the top element).

The subalgebra on $\{0, 1\}$ is (isomorphic to) the two-element Boolean algebra $\{0, 1\}_\mathrm{B}$.

Definable symbols:

- $x \odot y$ is $\neg(\neg x \oplus \neg y)$;
- $x \vee y$ is $(x \to y) \to y$ and $x \wedge y$ is $\neg(\neg x \vee \neg y)$;
- $x \to y$ is $\neg x \oplus y$;
- $x \equiv y$ is $(x \to y) \odot (y \to x)$.

x^n is $\underbrace{x \odot \cdots \odot x}_{n \text{ times}}$ and nx is $\underbrace{x \oplus \cdots \oplus x}_{n \text{ times}}$.

[Łukasiewicz 1922; Łukasiewicz and Tarski 1930]

McNaughton functions

Denote f_φ the function defined by the term φ in $[0,1]_Ł$.

A function $f\colon [0,1]^n \to [0,1]$ is a McNaughton function if

- f is continuous
- f is piecewise linear: there are finitely many linear polynomials $\{p_i\}_{i\in I}$, with $p_i(\bar{x}) = \Sigma_{j=1}^n a_{ij}\, x_j + b_i$, such that for any $\bar{x} \in [0,1]^n$ there is an $i \in I$ with $f(\bar{x}) = p_i(\bar{x})$
- the polynomials p_i have integer coefficients $\bar{a}_i, b_i$.

Theorem [McNaughton 1951]

Term-definable functions of $[0,1]_Ł$ coincide with McNaughton functions.

Completeness

Axioms for $Ł_\omega$ (with $\neg$ and $\rightarrow$):
$x \rightarrow (y \rightarrow x)$
$(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z))$
$(\neg y \rightarrow \neg x) \rightarrow (x \rightarrow y)$
$((x \rightarrow y) \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x)$

Completeness of $Ł_\omega$ w.r.t. $[0,1]_Ł$:

- tautologies of $[0,1]_Ł$ coincide with theorems
- finite consequence relation coincides with provability from finite theories

[Rose and Rosser 1958; Chang 1958–59]

$[0,1]_Ł$ provides a semantic method of investigating complexity of $Ł_\omega$.

Rational constants

By McNaughton theorem, constant functions (bar 0 and 1) are not term-definable.

One can implicitly define any rational constant in $[0,1]_{Ł}$.

$$x = \neg x$$

has a unique solution $1/2$, so under a theory with the schema $x \equiv \neg x$,
the value of x is fixed at $1/2$.

More generally, $x \equiv (\neg x)^{n-1}$ definex $1/n$; then m/n becomes term definable as mx.

[Torrens 1994; Gispert 2002; Hájek 1998]

Expansion of language with constants

RMV-terms: MV-language expanded with constants for rationals in $[0,1]$.

$[0,1]_{Ł}^{Q} = \langle [0,1], \oplus, \neg, \{r \mid r \in Q \cap [0,1]\} \rangle$.

"Rational Pavelka Logic" (RPL) expands $Ł_\omega$ with some axioms for constants.

The algebra $[0,1]_{Ł}^{Q}$ captures theorems and provability from finite theories in RPL.

[Pavelka 1979; Hájek 1998]

The validity degree

Let $T = \{\psi_1, \dots, \psi_k\}$ be a finite set of terms; write τ for $\psi_1 \odot \cdots \odot \psi_k$.
Let φ be a term.

The validity degree of φ under τ is

$$\|\varphi\|_\tau = \min\{e(\varphi) \mid e(\tau) = 1\}.$$

In other words, $\|\varphi\|_T$ is the minimum of f_φ on the 1-set of f_τ.

Instance: (R)MV-terms τ and φ.
Output: $\|\varphi\|_\tau$ in $[0, 1]_{Ł}$.

Corresponding syntactic notion is the provability degree: $|\varphi|_\tau = \max\{r \mid \tau \vdash_{\mathrm{RPL}} r \to \varphi\}$ and one has Pavelka completeness:

$$|\varphi|_T = \|\varphi\|_T$$

(also for infinite theories).

[Pavelka 1979; Hájek 1998]

Complexity results for $Ł_\omega$

Consider MV-term $\varphi(x_1, \dots, x_n)$.

f_φ introduces a polyhedral complex C on its domain (i.e., $\bigcup C = [0,1]^n$)
s.t. restriction of f_φ to each (n-dimensional) cell of C is a linear polynomial.

MIN(φ), the minimum value of f_φ on $[0,1]^n$ is attained at a vertex of a cell in C.

Vertices of cells of C occur as solutions of systems of linear equations,
with integer coefficient bounded by $\sharp\varphi$ (the number of occurrences of variables in φ).

For a vertex $\bar{p}$ of an n-dimensional cell of C,

$$\mathrm{den}(\bar{p}) \leq \left(\frac{\sharp\varphi}{n}\right)^n$$

where $\mathrm{den}(\bar{x})$ for a rational vector $\bar{x} = (x_1, \dots, x_n)$ is the least common denominator of $x_1, \dots, x_n$ and $\sharp\varphi$ is the length of φ.

Tautologous terms of the standard MV-algebra are in coNP.

[Mundici 1987; Aguzzoli and Ciabattoni 2000; Aguzzoli 2006]

Non-approximability of minimum value

Theorem

Let $\delta < 1/2$ be a positive real. Suppose ALG is a poly-time algorithm computing, for MV-term φ, a real number ALG(φ) satisfying $|ALG(\varphi) - MIN(\varphi)| \leq \delta$. Then P = NP.

Proof: recognize Boolean non-tautologies using ALG.

Instance: Boolean formula φ, given as $\{\oplus, \wedge\}$-combination of literals.

Then f_φ in $[0,1]_{Ł}$ is a concave function. and MIN(φ) is either 0 or 1.

We have

$$\varphi \notin \mathrm{TAUT}(\{0,1\}_{\mathrm{B}}) \text{ iff } \mathsf{MIN}(\varphi) = 0 \text{ iff } \mathsf{ALG}(\varphi) < 1/2.$$

[H., Savický 2016]

Optimization problems

Consider optimization problems such as the TSP:

given a graph with integer weights on edges,

- which is the roundtrip with a minimal cost? (optimization)
- which is the minimal cost of a roundtrip? (cost)
- given integer k, is there a roundtrip of cost at most k? (decision)

With validity degree, we are looking at a cost version of an optimization problem.

"Usual" binary search using the decision version as an oracle.

$\mathsf{FP}^{\mathsf{NP}}$ is the class of functions computable in poly-time with an NP oracle.

Computing MIN(φ): oracle computation

MIN(φ) is attained at a vertex of a polyhedral decomposition of the domain with rational coordinates $\bar{p}$ with $\mathrm{den}(\bar{p}) \leq (\sharp\varphi/n)^n$.

Oracle: given φ and a rational $r \in [0,1]$, is MIN(φ) $\leq r$?)
This is clearly in NP.
(Actually NP-complete, cf. GenSAT problem in [Mundici and Olivetti, 1998])

Binary search within rationals on $[0,1]$ with denominators up to $N = (\sharp\varphi/n)^n$.
Minimal distance of any two such numbers: $\left|\frac{p_1}{q_1} - \frac{p_2}{q_2}\right| \geq \frac{1}{N^2}$

First, test MIN(φ) ≤ 0; if so, output 0.
Otherwise let $a := 0$ and $b := 1$ and $k := 0$.
Repeat ++k; MIN(φ) $\leq (a+b)/2$? $\begin{cases} \text{Y } b & := (a+b)/2; \\ \text{N } a & := (a+b)/2; \end{cases}$ until $2^k > N^2$.

This yields an interval $[m/2^k, (m+1)/2^k)$ for some m, of length $1/2^k$, with exactly one rational with denominator up to N.

Pick a value in $(m/2^k, (m+1)/2^k)$ and compute best rational approximations.

MAX value is in $\mathrm{FP}^{\mathrm{NP}}$.

Computing the Validity Degree: oracle computation

Instance: (R)MV-terms τ and φ.
Output: $\|\varphi\|_\tau$.

To obtain upper bound for binary search, get rid of constants. (Implicit definability.)

The minimum of f_φ on the (compact) 1-region of f_τ
is attained at a vertex of the common refinement of complexes
of f_φ and f_τ.

Upper bounds on denominators: $N = ((\sharp\tau + \sharp\varphi)/n)^n$
from [Aguzzoli and Ciabattoni, 2006].

Validity Degree in $\mathrm{FP}^{\mathrm{NP}}$.

Metric reductions, and a separation

[Krentel 1998]
Let $f, g : \Sigma^* \to N$.
A metric reduction of f to g is a pair (h_1, h_2) of p-time functions
(with $h_1 : \Sigma^* \to \Sigma^*$ and $h_2 : \Sigma^* \times N \to N$)
such that $f(x) = h_2(x, g(h_1(x)))$ for each $x \in \Sigma^*$.

Let $z : N \to N$. $\mathrm{FP}^{\mathrm{NP}}[z(n)]$ is the class of functions
computable in P-time with NP oracle with at most $z(|x|)$ oracle calls for input x.
(So $\mathrm{FP}^{\mathrm{NP}} = \mathrm{FP}^{\mathrm{NP}}[n^{O(1)}]$.)

Theorem [Krentel 1988]

Assume $P \neq NP$. Then $FP^{NP}[O(\log\log n)] \neq FP^{NP}[O(\log n)] \neq FP^{NP}[n^{O(1)}]$.

In particular, there are no metric reductions from $\mathrm{FP}^{\mathrm{NP}}$-complete problems
to problems in $\mathrm{FP}^{\mathrm{NP}}[O(\log n)]$.

[Krentel: Complexity of optimization problems, 1988]

Weighted MaxSAT problem

- Weighted MaxSAT

Instance: Boolean CNF formula $C_1 \wedge \cdots \wedge C_n$ (k variables) with weights $w_1, \ldots, w_n$.
Output: $\max_e \Sigma_i w_i e(C_i)$(max sum of weights of true clauses over all assignments to φ).

Theorem [Krentel 1988]

Weighted MaxSAT is complete in FP^{NP} (under metric reductions).

Validity Degree: lower bound

Theorem

Validity degree is FP^{NP}-complete under metric reductions.

Proof: reduce weighted MaxSAT to Validity Degree.
Maximize $\Sigma_i w_i e(C_i)$ over all assigments e.

– Switch min and max (using $\neg$);
– scale weights: take $w = \Sigma_i w_i$ and replace w_i with $w_i' = w_i/w$ (and de-scale again);

Build a theory T (or τ) to
– make assignments Boolean (adding $x_i \vee \neg x_i$ for each $i \in \{1, \ldots, k\}$)
– implicitly condition each w_i' with C_i under v:

- $b \equiv (\neg b)^{w-1}$ (implicitly defines $1/w$);
- $y_i \to b$ and $wy_i \equiv C_i$ for each $i \in \{1, \ldots, n\}$; then
  - $v(C_i) = 0$ implies $v(y_i) = 0$
  - $v(C_i) = 1$ implies $v(y_i) \geq 1/w$

  and so $v(y_i) = v(C_i)1/w$ for any model e of T;
- $z_i \equiv w_i y_i$;

which yields $v(z_i) = v(C_i)w_i'$ for any model v of T and any i.

Finally, let Φ be $\neg(z_1 \oplus z_2 \oplus \cdots \oplus z_n)$. Compute $m = \|\Phi\|_T$ and return $(1 - m)w$.

Thank you!