On the complexity of validity degrees in Łukasiewicz logic

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A semantics for Łukasiewicz logic $I_\omega$

MV-terms: function symbols $\{\oplus, \neg\}$ on a countably infinite set of variables.

Domain: the interval $[0, 1]$ of the reals.
Interpretation of symbols:

\[ x \oplus y = \min(1, x + y) \]
\[ \neg x = 1 - x \]

The algebra $[0, 1]_L = \langle [0, 1], \oplus, \neg \rangle$ is an intended semantics of $I_\omega$.
The only designated value is 1 (the top element).

The subalgebra on $\{0, 1\}$ is (isomorphic to) the two-element Boolean algebra $\{0, 1\}_B$.

Definable symbols:

- $x \odot y$ is $\neg (\neg x \oplus \neg y)$;
- $x \lor y$ is $(x \to y) \to y$ and $x \land y$ is $\neg (\neg x \lor \neg y)$;
- $x \to y$ is $\neg x \oplus y$;
- $x \equiv y$ is $(x \to y) \odot (y \to x)$.

$x^n$ is $x \odot \cdots \odot x$ and $nx$ is $x \oplus \cdots \oplus x$.

[Łukasiewicz 1922; Łukasiewicz and Tarski 1930]
McNaughton functions

Denote $f_\varphi$ the function defined by the term $\varphi$ in $[0, 1]_L$.

A function $f : [0, 1]^n \to [0, 1]$ is a McNaughton function if

- $f$ is continuous
- $f$ is piecewise linear: there are finitely many linear polynomials $\{p_i\}_{i \in I}$, with $p_i(\vec{x}) = \sum_{j=1}^n a_{ij} x_j + b_i$, such that for any $\vec{x} \in [0, 1]^n$ there is an $i \in I$ with $f(\vec{x}) = p_i(\vec{x})$
- the polynomials $p_i$ have integer coefficients $\bar{a}_i, b_i$.

**Theorem [McNaughton 1951]**

Term-definable functions of $[0, 1]_L$ coincide with McNaughton functions.
Completeness

Axioms for $\mathcal{L}_\omega$ (with $\neg$ and $\rightarrow$):

\[
x \rightarrow (y \rightarrow x) \\
(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) \\
(\neg y \rightarrow \neg x) \rightarrow (x \rightarrow y) \\
((x \rightarrow y) \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x)
\]

Completeness of $\mathcal{L}_\omega$ w.r.t. $[0, 1]_\mathcal{L}$:

- tautologies of $[0, 1]_\mathcal{L}$ coincide with theorems
- finite consequence relation coincides with provability from finite theories

[Rose and Rosser 1958; Chang 1958–59]

$[0, 1]_\mathcal{L}$ provides a semantic method of investigating complexity of $\mathcal{L}_\omega$. 
Rational constants

By McNaughton theorem, constant functions (bar 0 and 1) are not term-definable.

One can implicitly define any rational constant in \([0, 1]_L\).

\[ x = \neg x \]

has a unique solution 1/2, so under a theory with the schema \(x \equiv \neg x\), the value of \(x\) is fixed at 1/2.

More generally, \(x \equiv (\neg x)^{n-1}\) definex 1/n; then \(m/n\) becomes term definable as \(mx\).

[Torrens 1994; Gispert 2002; Hájek 1998]

Expansion of language with constants

RMV-terms: MV-language expanded with constants for rationals in \([0, 1]\).

\([0, 1]^Q_L = \langle [0, 1], \oplus, \neg, \{ r \mid r \in Q \cap [0, 1] \} \rangle\).

“Rational Pavelka Logic” (RPL) expands \(L_\omega\) with some axioms for constants.

The algebra \([0, 1]^Q_L\) captures theorems and provability from finite theories in RPL.

[Pavelka 1979; Hájek 1998]
The validity degree

Let $T = \{\psi_1, \ldots, \psi_k\}$ be a finite set of terms; write $\tau$ for $\psi_1 \odot \cdots \odot \psi_k$. Let $\varphi$ be a term.

The validity degree of $\varphi$ under $\tau$ is

$$\|\varphi\|_\tau = \min\{e(\varphi) \mid e(\tau) = 1\}.$$ 

In other words, $\|\varphi\|_\tau$ is the minimum of $f_\varphi$ on the 1-set of $f_\tau$.

Instance: (R)MV-terms $\tau$ and $\varphi$.
Output: $\|\varphi\|_\tau$ in $[0, 1]$.

Corresponding syntactic notion is the provability degree: $|\varphi|_\tau = \max\{r \mid \tau \vdash_{RPL} r \rightarrow \varphi\}$
and one has Pavelka completeness:

$$|\varphi|_\tau = \|\varphi\|_\tau$$

(also for infinite theories).

[Pavelka 1979; Hájek 1998]
Complexity results for $L_\omega$

Consider MV-term $\varphi(x_1, \ldots, x_n)$.

$f_\varphi$ introduces a polyhedral complex $C$ on its domain (i.e., $\bigcup C = [0, 1]^n$) s.t. restriction of $f_\varphi$ to each ($n$-dimensional) cell of $C$ is a linear polynomial.

$\text{MIN}(\varphi)$, the minimum value of $f_\varphi$ on $[0, 1]^n$ is attained at a vertex of a cell in $C$.

Vertices of cells of $C$ occur as solutions of systems of linear equations, with integer coefficient bounded by $\#\varphi$ (the number of occurrences of variables in $\varphi$).

For a vertex $\bar{p}$ of an $n$-dimensional cell of $C$,

$$\text{den}(\bar{p}) \leq \left(\frac{\#\varphi}{n}\right)^n$$

where $\text{den}(\bar{x})$ for a rational vector $\bar{x} = (x_1, \ldots, x_n)$ is the least common denominator of $x_1, \ldots, x_n$ and $\#\varphi$ is the length of $\varphi$.

Tautologous terms of the standard MV-algebra are in coNP.

[Mundici 1987; Aguzzoli and Ciabattoni 2000; Aguzzoli 2006]
Non-approximability of minimum value

**Theorem**

Let $\delta < 1/2$ be a positive real. Suppose $ALG$ is a poly-time algorithm computing, for $MV$-term $\varphi$, a real number $ALG(\varphi)$ satisfying $|ALG(\varphi) - MIN(\varphi)| \leq \delta$. Then $P = NP$.

Proof: recognize Boolean non-tautologies using $ALG$.

Instance: Boolean formula $\varphi$, given as $\{\oplus, \wedge\}$-combination of literals.

Then $f_{\varphi}$ in $[0, 1]_L$ is a concave function. and $MIN(\varphi)$ is either 0 or 1.

We have

\[ \varphi \notin TAUT(\{0, 1\}_B) \text{ iff } MIN(\varphi) = 0 \text{ iff } ALG(\varphi) < 1/2. \]

[H., Savický 2016]
Consider optimization problems such as the TSP:

given a graph with integer weights on edges,
- which is the roundtrip with a minimal cost? (optimization)
- which is the minimal cost of a roundtrip? (cost)
- given integer $k$, is there a roundtrip of cost at most $k$? (decision)

With validity degree, we are looking at a cost version of an optimization problem.

“Usual” binary search using the decision version as an oracle.

$\text{FP}^{\text{NP}}$ is the class of functions computable in poly-time with an NP oracle.
Computing \( \text{MIN}(\varphi) \): oracle computation

\( \text{MIN}(\varphi) \) is attained at a vertex of a polyhedral decomposition of the domain with rational coordinates \( \vec{p} \) with \( \text{den}(\vec{p}) \leq (\#\varphi/n)^n \).

Oracle: given \( \varphi \) and a rational \( r \in [0, 1] \), is \( \text{MIN}(\varphi) \leq r? \)

This is clearly in NP.

(Actually NP-complete, cf. GenSAT problem in [Mundici and Olivetti, 1998])

Binary search within rationals on \([0, 1]\) with denominators up to \( N = (\#\varphi/n)^n \).

Minimal distance of any two such numbers: \( \left| \frac{p_1}{q_1} - \frac{p_2}{q_2} \right| \geq \frac{1}{N^2} \)

First, test \( \text{MIN}(\varphi) \leq 0 \); if so, output 0.

Otherwise let \( a := 0 \) and \( b := 1 \) and \( k := 0 \).

Repeat ++\( k \); \( \text{MIN}(\varphi) \leq (a + b)/2? \)

\[
\begin{cases} 
Y & b := (a + b)/2; \\
N & a := (a + b)/2;
\end{cases}
\]

until \( 2^k > N^2 \).

This yields an interval \([m/2^k, (m + 1)/2^k]\) for some \( m \), of length \( 1/2^k \), with exactly one rational with denominator up to \( N \).

Pick a value in \((m/2^k, (m + 1)/2^k)\) and compute best rational approximations.

\( \text{MAX} \) value is in \( \text{FP}^\text{NP} \).
Computing the Validity Degree: oracle computation

Instance: (R)MV-terms $\tau$ and $\varphi$.
Output: $\|\varphi\|_{\tau}$.

To obtain upper bound for binary search, get rid of constants. (Implicit definability.)

The minimum of $f_\varphi$ on the (compact) 1-region of $f_\tau$ is attained at a vertex of the common refinement of complexes of $f_\varphi$ and $f_\tau$.

Upper bounds on denominators: $N = (((\#_\tau + \#_\varphi)/n)^n$ from [Aguzzoli and Ciabattoni, 2006].

Validity Degree in $\text{FP}^\text{NP}$.
Let \( f, g : \Sigma^* \rightarrow N \).

A metric reduction of \( f \) to \( g \) is a pair \((h_1, h_2)\) of p-time functions (with \( h_1 : \Sigma^* \rightarrow \Sigma^* \) and \( h_2 : \Sigma^* \times N \rightarrow N \)) such that \( f(x) = h_2(x, g(h_1(x))) \) for each \( x \in \Sigma^* \).

Let \( z : N \rightarrow N \). \( \text{FP}^{\text{NP}}[z(n)] \) is the class of functions computable in P-time with NP oracle with at most \( z(|x|) \) oracle calls for input \( x \). (So \( \text{FP}^{\text{NP}} = \text{FP}^{\text{NP}}[n^{O(1)}] \).)

**Theorem [Krentel 1988]**

Assume \( P \neq NP \). Then \( \text{FP}^{\text{NP}}[O(\log \log n)] \neq \text{FP}^{\text{NP}}[O(\log n)] \neq \text{FP}^{\text{NP}}[n^{O(1)}] \).

In particular, there are no metric reductions from \( \text{FP}^{\text{NP}} \)-complete problems to problems in \( \text{FP}^{\text{NP}}[O(\log n)] \).

[Krentel: Complexity of optimization problems, 1988]
Weighted MaxSAT problem

- **Weighted MaxSAT**
  
  Instance: Boolean CNF formula $C_1 \land \cdots \land C_n$ ($k$ variables) with weights $w_1, \ldots, w_n$.
  
  Output: $\max_e \sum_i w_i e(C_i)$ (max sum of weights of true clauses over all assignments to $\varphi$).

**Theorem [Krentel 1988]**

*Weighted MaxSAT is complete in $FP^{NP}$ (under metric reductions).*
Validity Degree: lower bound

**Theorem**

*Validity degree is \( \text{FP}^{\text{NP}} \)-complete under metric reductions.*

Proof: reduce weighted MaxSAT to Validity Degree.

Maximize \( \sum_i w_i e(C_i) \) over all assignments \( e \).

\begin{itemize}
  \item Switch min and max (using \( \neg \));
  \item scale weights: take \( w = \sum_i w_i \) and replace \( w_i \) with \( w'_i = w_i / w \) (and de-scale again);
\end{itemize}

Build a theory \( T \) (or \( \tau \)) to

\begin{itemize}
  \item make assignments Boolean ( adding \( x_i \lor \neg x_i \) for each \( i \in \{1, \ldots, k\} \))
  \item implicitly condition each \( w'_i \) with \( C_i \) under \( v \):
    \begin{itemize}
      \item \( b \equiv (\neg b)^w - 1 \) (implicitly defines \( 1/w \ ));
      \item \( y_i \rightarrow b \) and \( wy_i \equiv C_i \) for each \( i \in \{1, \ldots, n\} \); then
        \begin{itemize}
          \item \( v(C_i) = 0 \) implies \( v(y_i) = 0 \)
          \item \( v(C_i) = 1 \) implies \( v(y_i) \geq 1/w \)
        \end{itemize}
        and so \( v(y_i) = v(C_i)1/w \) for any model \( e \) of \( T \);
      \item \( z_i \equiv w_i y_i \);
    \end{itemize}
\end{itemize}

which yields \( v(z_i) = v(C_i)w'_i \) for any model \( v \) of \( T \) and any \( i \).

Finally, let \( \Phi \) be \( \neg(z_1 \oplus z_2 \oplus \cdots \oplus z_n) \). Compute \( m = \|\Phi\|_T \) and return \( (1 - m)w \).
Thank you!