SAT and MaxSAT problems in Łukasiewicz logic

Zuzana Haniková

Institute of Computer Science, Czech Academy of Sciences

Nonclassical Logic Days, Barcelona, Oct 14, 2022

Decision problems & SAT

Fix the basic language of Łukasiewicz logic: $\mathcal{L}$ as $\{\neg, \oplus\}$.
This gives the set $\mathcal{F}$ of (propositional) wff's.
$\mathbb{MV}$ is the class of MV-algebras.

Let $\mathsf{A} \in \mathbb{MV}$.

$$\mathrm{SAT}(\mathsf{A}) = \{\varphi \in \mathcal{F} \mid \exists v_{\mathsf{A}} \, (v_{\mathsf{A}}(\varphi) = 1^{\mathsf{A}})\}$$

Solvability of finite systems of MV-equations in A.

We use $[0,1]_Ł$ for $\Gamma(\mathsf{R}, 1)$ (the "standard MV-algebra").

$\mathrm{SAT}([0,1]_Ł)$ is in NP (in fact, NP-complete).
[Mundici: SAT in many-valued sentential logic is NP-complete, 1987]

Let $\overline{\mathrm{SAT}}(\mathsf{A})$ be the complement of $\mathrm{SAT}(\mathsf{A})$ in $\mathcal{F}$.
Then $\varphi \in \overline{\mathrm{SAT}}(\mathsf{A})$ iff $\mathsf{A} \models \varphi \approx 1 \Rightarrow 0 \approx 1$.
Thus $\overline{\mathrm{SAT}}(\mathsf{A})$ is a fragment of the Q-theory of A.

Optimization tasks & MaxSAT

Let $\varphi_1, \ldots, \varphi_k$ be a list of formulas in $\mathcal{F}$.

$$\text{MaxSAT}(\mathsf{A})(\varphi_1, \ldots, \varphi_k) =$$
$$= \max\{\, m \le k \mid \exists v_A\, \exists I \subseteq \{1, ..., k\}\, [\, |I| = m \,\&\, \forall i \in I(v_{\mathsf{A}}(\varphi_i) = 1^A)\,]\,\&$$
$$\forall v_A \forall I \subseteq \{1, ..., k\}\, [\, |I| > m \Rightarrow \exists i \in I(v_{\mathsf{A}}(\varphi_i) < 1^A)\,]\,\}$$

Maximal number of (simultaneously!) satisfied formulas,
over all assignments.

NB. $\varphi_{i_1}, \ldots, \varphi_{i_n}$ are simultaneously satisfiable in A iff $\wedge_{j=1}^{n} \varphi_{i_j} \in \text{SAT}(\mathsf{A})$.

Geometry and complexity

$\varphi(x_1, \ldots, x_n)$ formula in $\mathcal{F}$; f_φ its interpretation in $[0,1]_{Ł}$;
$\sharp\varphi$ number of occurrences of variables in φ.

The function f_φ introduces a polyhedral complex $C(\varphi)$ on $[0,1]^n$ s.t.

- $[0,1]^n = \bigcup C(\varphi)$ and
- f_φ is linear over each n-dimensional cell of $C(\varphi)$.

The 1-region of φ is union of cells of $C(\varphi)$
such that f_φ attains the value 1 on all points in the cell.
This is a compact subset of $[0,1]^n$.

Vertices of C occur as solutions of systems of linear equations,
with integer coefficient bounded by $\sharp\varphi$.

Theorem [A.–C. 2000]: For a vertex $\bar{p}$ of an n-dimensional cell of C,

$$\mathrm{den}(\bar{p}) \leq \left(\frac{\sharp\varphi}{n}\right)^n$$

where $\mathrm{den}(\bar{p})$ for a rational vector $\bar{p} = (p_1, \ldots, p_n)$ is the
least common denominator of $p_1, \ldots, p_n$.

Corollary: $\mathrm{SAT}([0,1]_{Ł})$ is in NP.

[McNaughton 1951; Mundici 1987; Aguzzoli and Ciabattoni 2000; Aguzzoli 2006]

Geometry and complexity – cont'd

$\varphi_1, \ldots, \varphi_k$ formulas of $\mathcal{L}$, with variables among $x_1, \ldots, x_n$.

The common refinement of $C(\varphi_1)$, ..., $C(\varphi_k)$
(considered n-dimensional) is the polyhedral complex
whose n-dimensional cells are precisely all (n-dimensional) intersections of n-dimensional cells of $C(\varphi_1)$, ..., $C(\varphi_k)$.

Theorem [∼A.–C. 2000]: For a vertex $\bar{p}$ of an n-dimensional cell of the common refinement of $C(\varphi_1)$, ..., $C(\varphi_k)$,

$$\mathrm{den}(\bar{p}) \leq \left(\frac{\sum_{i=1}^{k} \sharp\varphi_i}{n}\right)^n$$

[Aguzzoli and Ciabattoni 2000]

A decision version of MaxSAT for $[0,1]_{Ł}$ (D-MaxSAT($[0,1]_{Ł}$)):

$$\{\langle \varphi_1, \ldots, \varphi_k \rangle, m \leq k \mid \mathrm{MaxSAT}([0,1]_{Ł})(\varphi_1, \ldots, \varphi_k) \geq m\}$$

Corollary: D-MaxSAT($[0,1]_{Ł}$) is in NP.

Oracle computation for MaxSAT

Let $\varphi_1, \ldots, \varphi_k$ be given.

Assume an oracle for D-MaxSAT($[0,1]_{Ł}$).

Binary search on the interval of natural numbers $\{0, \ldots, k\}$.

Let $a := 0$ and $b := k$.
While $a < b$ do {
$m := \lfloor {}^{a+b}/_{2} \rfloor$;

$$\text{D-MaxSAT}([0,1]_{Ł})(\langle \varphi_1, \ldots, \varphi_k \rangle, m)? \begin{cases} \text{(yes)} & a := m + 1; \\ \text{(no)} & b := m; \end{cases}$$

}
Return b.

Corollary: MaxSAT($[0,1]_{Ł}$) is in $\mathrm{FP}^{\mathrm{NP}}$.

NB. Any NP-complete problem will do as oracle.

Tableau calculus for MaxSAT[$Ł_3$] [Li, Manyà, Vidal 2020]

Labelled formulas $\{r\} : \alpha$, where $r \in \{0, 1/2, 1\}$ and $\alpha \in \mathcal{F}$.

A labelled formula $\{r\} : \alpha$ is satisfied by an assignment v iff $v(\alpha) = r$.

Given a multiset of labelled formulas $\{1\} : \alpha_1, \ldots, \{1\} : \alpha_k$, with $\alpha_i \in \mathcal{F}$, the calculus computes the minimal number of unsatisfied formulas.

strong disjunction

$$\frac{\{0\}:(\phi_1 \oplus \phi_2)}{\Box \;\Big|\; \begin{array}{c}\{0\}:\phi_1\\ \{0\}:\phi_2\end{array}} \qquad \frac{\{1\}:(\phi_1 \oplus \phi_2)}{\Box \;\Big|\; \{1\}:\phi_1 \;\Big|\; \{1\}:\phi_2 \;\Big|\; \begin{array}{c}\{\frac{1}{2}\}:\phi_1\\ \{\frac{1}{2}\}:\phi_2\end{array}}$$

$$\frac{\{\frac{1}{2}\}:(\phi_1 \oplus \phi_2)}{\Box \;\Big|\; \begin{array}{c}\{0\}:\phi_1\\ \{\frac{1}{2}\}:\phi_2\end{array} \;\Big|\; \begin{array}{c}\{\frac{1}{2}\}:\phi_1\\ \{0\}:\phi_2\end{array}}$$

negation

$$\frac{\{0\}:\neg\phi}{\{1\}:\phi} \qquad \frac{\{\frac{1}{2}\}:\neg\phi}{\{\frac{1}{2}\}:\phi} \qquad \frac{\{1\}:\neg\phi}{\{0\}:\phi}$$

....

For a variable x_i, list its labels $(r_{i,1}, \ldots, r_{i,m})$ in decreasing frequency.
Append $\Box$ for each occurrence of a label distinct from $r_{i,1}$.

Finite-valued reduction for MaxSAT$[0,1]_{Ł}$

Work in $[0,1]_{Ł}$.

Consider $\varphi_1, \ldots, \varphi_k \in \mathcal{F}$.

Assume that the intersection of 1-regions of $\{\varphi_i\}_{i\in I}$, for some $I \subseteq \{1, \ldots, k\}$, is nonempty.

Then there is at least one vertex p of a cell in the intersection, namely, of the common refinement of $C(\varphi_i)$, $i \in I$.

We have

$$\mathrm{den}(\bar{p}) \leq \left(\frac{\Sigma_{i\in I} \sharp\varphi_i}{n}\right)^n \leq \left(\frac{\Sigma_{i=1}^{k} \sharp\varphi_i}{n}\right)$$

Hence, the coordinates of $\bar{p}$ belong to $Ł_N$ with

$$N = \mathrm{lcm}\{1, 2, \ldots, \left(\frac{\Sigma_{i=1}^{k} \sharp\varphi_i}{n}\right)^n\} \approx \exp(\left(\frac{\Sigma_{i=1}^{k} \sharp\varphi_i}{n}\right)^n)$$